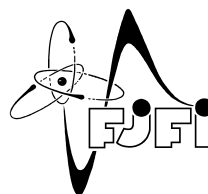




CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering
Department of Physics



Discrete Hardy Inequalities of Higher Order

Diskrétní Hardyho nerovnosti vyšších řádů

Research Project

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I. OSOBNÍ A STUDIJNÍ ÚDAJE

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Smyslem výzkumného úkolu je prozkoumat existující výsledky a otevřené problémy ve vybraných partiích teorie Hardyho nerovností vyjmenovaných níže v návaznosti na studentovu BP, ref. [9]. Detailnější průzkum jednoho bodu na úkor ostatních je možný.

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2. Diskrétní Hardyho nerovnosti ve vyšší dimenzi, ref. [5,6].
3. Teorie kritičnosti pro obecnou mocninu diskrétního Laplaceova na polopřímce, ref. [3,7].
4. Hardyho nerovnosti pro operátory derivací z teorie ortogonálních polynomů, ref. [8].
5. Vážené diskrétní Hardyho nerovnosti, ref. [4].

Seznam doporučené literatury:

- [1] Das, Keller, Pinchover; Preprint (2024) arXiv:2407.02116.
- [2] Fischer, Keller, Pogorzelski; Integral Equations Operator Theory 95 (2023).
- [3] Gerhat, Krejčířík, Štampach; Preprint (2023) arXiv:2307.09919.
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- [8] Kostenko; Lett. Math. Phys. 111 (2021).
- [9] Štampach, Wacławek; Preprint (2024) arXiv:2405.07742.

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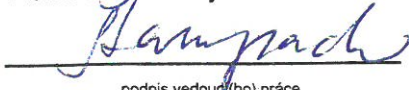
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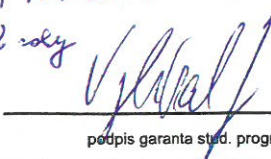
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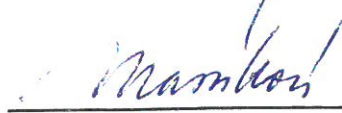
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podpis garanta stud. programu



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III. PŘEVZETÍ ZADÁNÍ

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Podpis studenta

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Author's declaration:

I declare that this Research Project is entirely my own work, and I have listed all the used sources in the bibliography.

Prague, April 30, 2025

Jakub Waclawek

Název práce:

Diskrétní Hardyho nerovnosti vyšších řádů

Autor: Jakub Waclawek

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Abstrakt: Tento projekt se zabývá diskrétními p -Hardyho nerovnostmi vyššího řádu $\ell \in \mathbb{N}$ na polo-přímce. Dosavadní výsledky pro nerovnosti prvního řádu $\ell = 1$ a lineární případ, kdy $p = 2$, rozšíříme pro obecnou kombinaci $\ell \in \mathbb{N}$ a $p > 1$. Odvodíme, jak systematicky konstruovat abstraktní p -Hardyho váhy libovolného řádu v závislosti na dané parametrické posloupnosti $\mathbf{g}$. Konkrétní volbou $\mathbf{g}$ potom dostaneme explicitní kandidáty a pro jisté kombinace ℓ a p dokážeme, že tyto váhy splňují příslušné Hardyho nerovnosti. Nakonec ukážeme, že takto získaná váha prvního řádu odpovídá Herglotzově–Nevanlinnově funkci, čímž potvrdíme pozměněnou hypotézu týkající se jejího asymptotického rozvoje od autorů Fisher–Keller–Pogorzelski publikovanou v [Integr. Equ. Oper. Theory, 95(24), 2023].

Klíčová slova: diskrétní p -Birmanova nerovnost, diskrétní p -Hardyho nerovnost, diskrétní p -Rellichova nerovnost, Herglotzovy–Nevanlinnovy funkce, optimální váhy, úplná monotonie

Title:

Discrete Hardy Inequalities of Higher Order

Author: Jakub Waclawek

Abstract: This project investigates discrete p -Hardy inequalities of higher order $\ell \in \mathbb{N}$ on the half-line, extending previously known results for the first-order case $\ell = 1$ and the linear case $p = 2$. We develop a systematic method for deriving abstract p -Hardy weights of arbitrary order given suitable parameter sequence $\mathbf{g}$. Furthermore, explicit candidate weights are constructed via a specific choice of the parameter sequences. While the general setting remains unsolved, we show that these candidates satisfy the Hardy inequality for various combinations of $\ell \in \mathbb{N}$ and $p > 1$. Finally, by establishing that the first-order explicit p -Hardy weight corresponds to a Herglotz–Nevanlinna function, we confirm a modified version of a conjecture concerning its asymptotic expansion, originally proposed by Fischer–Keller–Pogorzelski in [Integr. Equ. Oper. Theory, 95(24), 2023].

Key words: discrete p -Birman inequality, discrete p -Hardy inequality, discrete p -Rellich inequality, Herglotz–Nevanlinna functions, optimal weights, complete monotonicity

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Introduction

In 1915, when trying to find an elementary proof of the *Hilbert inequality*, Hardy stumbled upon an intriguing truth, that the convergence of the series $\sum_{n=1}^{\infty} a_n^2$ implies the convergence of the series of cumulative averages $\sum_{n=1}^{\infty} (A_n/n)^2$, where $A_n := \sum_{k=1}^n a_k$. Now, for more than a century, this theorem has been famously known as the weak form of the *discrete Hardy inequality*. Since then, many mathematicians have contributed to the vast history of the Hardy inequality, enriching the initial claim with many improvements and generalizations (we refer the reader to [16] for a remarkable historical survey). Amongst the first, in 1919, M. Riesz proved that the inequality

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n} \right)^p \leq C_p \sum_{n=1}^{\infty} a_n^p \quad (1)$$

holds for all non-negative sequences $a \in C(\mathbb{N})$, where $C(\mathbb{N})$ denotes the space of all complex sequences indexed by $\mathbb{N}$, and for all $p > 1$, with $C_p = (p^2/(p-1))^p$. Thereby arose a natural question, what is the optimal constant C_p on the right-hand side of (1). Hardy immediately pointed out that the estimates in Riesz' proof were rather rough and that the constant could be improved to $(p\zeta(p))^p$, where ζ denotes the Riemann zeta function, which was confirmed by I. Schur in 1920. Finally, a year later, E. Landau sent a letter to Hardy proving that inequality (1) holds with $C_p = (p/(p-1))^p$ together with the optimality of the constant.

Denoting $u_n := A_n$, or equivalently $a_n =: u_n - u_{n-1}$, the *classical discrete p -Hardy inequality* can be written in its more renowned form as

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}|^p \geq \left(\frac{p-1}{p} \right)^p \sum_{n=1}^{\infty} \frac{|u_n|^p}{n^p} \quad (2)$$

for all finitely supported sequences $u \in C_0(\mathbb{N}_0)$ with $u_0 = 0$. Let us also mention that it was mainly the continuous version of (2) that found various applications, e.g. in PDEs, spectral theory or mathematical physics. The *continuous p -Hardy inequality* asserts that

$$\int_0^{\infty} |\varphi'(x)|^p dx \geq \left(\frac{p-1}{p} \right)^p \int_0^{\infty} \frac{|\varphi(x)|^p}{x^p} dx \quad (3)$$

for any smooth function $\varphi \in C_0^{\infty}(0, \infty)$ with a compact support. The analogies between (2) and (3) are obvious. In fact, in 1921, after proving the inequality (2), E. Landau pointed out that the discrete version of the Hardy inequality can be deduced from the continuous one, assuming a suitable step function. Moreover, it can be shown that the inequalities are equivalent, meaning that not only the continuous inequality implies the discrete one, but that the converse is also true; see [12].

After a dormant century regarding improvements of the classical Hardy inequality, in [13], M. Keller, Y. Pinchover, and F. Pogorzelski discovered that, in the case $p = 2$, even though the constant $1/4$ in (2) is the biggest possible, the classical Hardy weight $1/(4n^2)$ can still be improved to

$$\rho_n^{\text{KPP}} := 2 - \sqrt{\frac{n-1}{n}} - \sqrt{\frac{n+1}{n}} > \frac{1}{4n^2}$$

for all $n \in \mathbb{N}$, i.e. that

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}|^2 \geq \sum_{n=1}^{\infty} \rho_n^{\text{KPP}} |u_n|^2 \quad (4)$$

holds for all $u \in C_0(\mathbb{N}_0)$ with $u_0 = 0$. Additionally, the authors proved in [14] that the weight ρ^{KPP} is *optimal*. Optimality is a rather strong property (see Definition (1.1) below), which, in particular, implies that the weight is *critical*, meaning that ρ^{KPP} cannot be further improved by a pointwise bigger sequence, such that the analogue of (4) would still be true. In contrast, it is a well-known fact that the classical continuous Hardy weight $1/(4x^2)$ in (3), for $p = 2$, is indeed critical, meaning that if (3) holds with $1/(4x^2)$ replaced by a measurable function $\rho(x) \geq 1/(4x^2)$ for a.e. $x > 0$, then $\rho(x) = 1/(4x^2)$ for a.e. $x > 0$.

It is reasonable to expect, that the weight $((p-1)/(pn))^p$ in (2) admits an improvement even for general $p > 1$. Indeed, F. Fisher, M. Keller, and F. Pogorzelski showed in [9], that the classical discrete p -Hardy inequality (2) can be improved to

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}|^p \geq \sum_{n=1}^{\infty} \rho_n^{\text{FKP}} |u_n|^p \quad (5)$$

for all $u \in C_0(\mathbb{N}_0)$ with $u_0 = 0$, where the weight ρ^{FKP} is defined as

$$\rho_n^{\text{FKP}} := \left(1 - \left(1 - \frac{1}{n}\right)^{1/q}\right)^{p-1} - \left(\left(1 + \frac{1}{n}\right)^{1/q} - 1\right)^{p-1}, \quad (6)$$

where the constant q is given by the relation $1/p + 1/q = 1$. Moreover, the authors also proved that the weight ρ^{FKP} is optimal and satisfies

$$\rho_n^{\text{FKP}} > \left(\frac{p-1}{p}\right)^p \frac{1}{n^p} \quad (7)$$

for all $n \in \mathbb{N}$. Note, that for $p = 2$ the weight ρ^{FKP} coincides with ρ^{KPP} .

Another interesting branch of theory of the Hardy inequalities is seeking for analogues of (2) and (3) with differences, resp. derivatives, of higher order. In 1961, M. Š. Birman [3] derived the Hardy inequality of an arbitrary order $\ell \in \mathbb{N}$ (also called the *Birman inequality*), which reads

$$\int_0^\infty \left|\varphi^{(\ell)}(x)\right|^2 dx \geq \left(\frac{1}{2}\right)_\ell^2 \int_0^\infty \frac{|\varphi(x)|^2}{x^{2\ell}} dx \quad (8)$$

for all $\varphi \in C_0^\infty(0, \infty)$, where we used the combinatorial notation for the *Pochhammer symbol* defined by formula

$$(\nu)_n := \nu(\nu+1) \dots (\nu+n-1) \quad (9)$$

for all $\nu \in \mathbb{R}$ and $n \in \mathbb{N}$. Let us remark, that for $\ell = 2$ the inequality (8) was discovered by F. Rellich in 1954, published posthumously in [19], and is commonly known as the *Rellich inequality*. The discrete version of (8) was discovered only recently in [12]. For concise notation, let us first introduce certain difference operators acting on the following spaces. It turns out, that it is beneficial to consider sequences indexed by $\mathbb{Z}$, with zero entries up to a certain index. Namely, we denote $C(\mathbb{Z})$ the space of all complex sequences indexed by $\mathbb{Z}$ and its subspaces

$$H^\ell := \{u \in C(\mathbb{Z}) \mid u_n = 0 \text{ for all } n < \ell\} \quad \text{and} \quad \mathcal{H}_0^\ell := H^\ell \cap C_0(\mathbb{Z}) \quad (10)$$

for all $\ell \in \mathbb{N}$, where $C_0(\mathbb{Z})$ denotes all finitely supported sequences in $C(\mathbb{Z})$. Next, we define *discrete gradient* and *divergence* as

$$(\nabla u)_n := u_n - u_{n-1} \quad \text{and} \quad (\text{div} u)_n := u_{n+1} - u_n \quad (11)$$

for all $u \in C(\mathbb{Z})$ and $n \in \mathbb{Z}$. The *discrete Laplace operator* on $C(\mathbb{Z})$ is then given by the familiar formula $\Delta := \text{div} \circ \nabla$, i.e.

$$\Delta u_n = u_{n-1} + 2u_n - u_{n+1} \quad (12)$$

for all $n \in \mathbb{Z}$. The reasoning behind considering these spaces of sequences can be found in [26]. The discrete Hardy inequality of order $\ell \in \mathbb{N}$, i.e. the *discrete Birman inequality*, found in [12] can now be expressed as

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left| (-\Delta)^{\ell/2} u_n \right|^2 \geq \left(\frac{1}{2} \right)^2 \sum_{n=\ell}^{\infty} \frac{|u_n|^2}{n^{2\ell}} \quad (13)$$

for all $u \in \mathcal{H}_0^\ell$, where $\lceil x \rceil$ stands for the lowest integer greater or equal to $x \in \mathbb{R}$ and the half-integer powers of the discrete Laplacian (12) are defined as

$$(-\Delta)^{\ell/2} := \begin{cases} (-\Delta)^m & \text{if } \ell = 2m \in 2\mathbb{N}, \\ \nabla \circ (-\Delta)^m & \text{if } \ell = 2m + 1 \in 2\mathbb{N}_0 + 1. \end{cases} \quad (14)$$

Being aware of the improvement of the classical discrete Hardy inequality (4), it is natural to expect that the inequality (13) also admits a refinement. This was shown in 2024 in [26], where the authors asserted the inequality

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left| (-\Delta)^{\ell/2} u_n \right|^2 \geq \sum_{n=\ell}^{\infty} \rho_n^{(\ell)} |u_n|^2 \quad (15)$$

for all $u \in \mathcal{H}_0^\ell$, together with the optimality of the weight $\rho^{(\ell)}$, defined as

$$\rho_n^{(\ell)} := \frac{(-\Delta)^\ell \mathfrak{g}_n^{(\ell)}}{\mathfrak{g}_n^{(\ell)}} \quad \text{with} \quad \mathfrak{g}_n^{(\ell)} := \sqrt{n} \prod_{j=1}^{\ell-1} (n-j) \quad (16)$$

for all $n \in \mathbb{N}_0$.

The ultimate aim of this project is to establish analogous inequalities to (13) and (15) in the general case of $p > 1$. While this appears to be a particularly demanding problem and remains unresolved in full generality, we have obtained numerous partial results in this direction. So far, this is a rather unexplored area of research and, to our best knowledge, no results are available regarding discrete p -Hardy inequalities of higher order on the half-line; see [4] for some results in the continuous setting and [18] for the continuous p -Rellich inequality

$$\int_0^\infty |\varphi''(x)|^p \, dx \geq \left(\frac{p-1}{p} \right)^p \int_0^\infty \frac{|\varphi(x)|^p}{x^{2p}} \, dx,$$

which holds true for all $\varphi \in C_0^\infty(0, \infty)$.

The project is structured as follows. In Chapter 1, we extend the results of [26] to the general case $p > 1$. Via Theorems 1.3, 1.8, and 1.11, we develop a method for constructing abstract p -Hardy weights in terms of a parameter sequence $\mathfrak{g}$ subject to appropriate monotonicity and growth conditions. Chapters 2 and 3 focus on explicit p -Hardy weights corresponding to a specific choice of the parameter sequence $\mathfrak{g}^{(\ell,p)}$, as defined in (1.12). Chapter 2 analyzes the properties of $\mathfrak{g}^{(\ell,p)}$, exploiting the theory of *completely monotonic* functions. The main result of this chapter is Theorem 2.4, where we obtain explicit p -Hardy weights $\rho^{(\ell,p)}$, given by formula (1.13), for certain values of $\ell \in \mathbb{N}$ and $p > 1$. We also formulate Conjecture 2.5, which posits that the claim of Theorem 2.4 remains valid for all admissible parameters. Finally, in Chapter 3, which is based on [28], we address a modified version of a conjecture from [9], establishing an asymptotic expansion of the weight $\rho^{(1,p)}$ in Theorem 3.3. We do so by utilizing the tools of complex analysis. In particular, we show that the function f_p , defined by (3.6) and closely related to the weight $\rho^{(1,p)}$, is a *Herglotz–Nevanlinna* function.

Chapter 1

Abstract p -Hardy inequalities

In this chapter, we explain how to construct p -Hardy weights of any order $\ell \in \mathbb{N}$ given a *parameter sequence* $\mathbf{g}$ satisfying certain conditions. In fact, we slightly modify and extend the strategy from [26], where the authors considered the linear case $p = 2$, to the more complex general case with an arbitrary $p > 1$. Furthermore, we will discuss optimality of the acquired weights under stronger assumptions on $\mathbf{g}$.

First, let us recall the definition of p -Hardy weights of higher order and their optimality, conferring with [8] and [26].

Definition 1.1. Let $\ell \in \mathbb{N}$ and $p > 1$. A non-negative sequence $\{\rho_n\}_{n=\ell}^\infty$ is called a *discrete p -Hardy weight of order ℓ* if and only if the *discrete p -Hardy inequality of order ℓ*

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left| (-\Delta)^{\ell/2} u_n \right|^p \geq \sum_{n=\ell}^{\infty} \rho_n |u_n|^p \quad (1.1)$$

holds for any $u \in \mathcal{H}_0^\ell$. In addition, the weight ρ is said to be *optimal* if it exhibits the following three properties:

- (i) *Criticality*: The weight ρ is called *critical* if for any p -Hardy weight $\tilde{\rho}$ of order ℓ such that $\tilde{\rho}_n \geq \rho_n$ for all $n \geq \ell$ it follows that $\tilde{\rho} = \rho$.
- (ii) *Non-attainability*: The weight ρ is called *non-attainable* if whenever the equality in (1.1) is attained for $u \in \mathcal{H}^\ell$ such that the right-hand side is finite of (1.1), necessarily $u \equiv 0$.
- (iii) *Optimality near infinity*: The weight ρ is called *optimal near infinity* if for any $M \geq \ell$ and $\varepsilon > 0$ there exists $u \in \mathcal{H}_0^M$ such that

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left| (-\Delta)^{\ell/2} u_n \right|^p < (1 + \varepsilon) \sum_{n=\ell}^{\infty} \rho_n |u_n|^p.$$

In this project, we mainly focus on criticality and optimality near infinity, as these properties gained the most attention in the past. Criticality means that the p -Hardy weight enjoying this property cannot be further improved by a pointwise bigger sequence. Optimality near infinity means that the constant 1 on the right-hand side of (1.1) cannot be further improved, even if we restrict ourselves only to sequences from $\mathcal{H}_0^M \subset \mathcal{H}_0^\ell$ for any $M \geq \ell$. Let us remark that optimality does not characterize the weight uniquely as was observed in [26], where the authors found infinitely many optimal Hardy weights of any order ℓ for $p = 2$.

Remark 1.2. In the linear case $p = 2$, we have the identity $\nabla^* = -\text{div}$, where $*$ is the adjoint in $\ell^2(\mathbb{Z})$, from which it follows that the Hardy inequality can be interpreted in the sense of quadratic forms in $\ell^2(\mathbb{N})$ as a lower bound $-\Delta \geq \rho$ for the discrete Laplace operator (12), where ρ stands for the operator of multiplication by the Hardy weight ρ . Similarly, for $\ell \in \mathbb{N}$, we have the polarization identity

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left| (-\Delta)^{\ell/2} u_n \right|^2 = \sum_{n=\ell}^{\infty} \overline{u_n} (-\Delta)^{\ell} u_n = \left\langle u, (-\Delta)^{\ell} u \right\rangle$$

for any $u \in \mathcal{H}_0^{\ell}$, where we consider $\mathcal{H}_0^{\ell}$ to be endowed with the Euclidean scalar product, hence we may rewrite (1.1) as $(-\Delta)^{\ell} \geq \rho$. The absence of scalar product in the space $\ell^p(\mathbb{N})$ for $p \neq 2$ results in a need for a slightly different approach for deriving p -Hardy inequalities of higher order. Nonetheless, it is still possible to *polarize* the ℓ^p -norm of ∇u , where $u \in \mathcal{H}_0^1$ as

$$\sum_{n=1}^{\infty} |\nabla u_n|^p = \sum_{n=1}^{\infty} \overline{u_n} (-\Delta_p) u_n,$$

where the p -Laplace operator is defined as $\Delta_p := \text{div} \circ (\nabla \cdot)^{(p-1)}$ (we omit $\circ$ when composing difference operators to simplify the notation below). Moreover, for streamline notation, we denoted $\nu^{(\alpha)} := \nu |\nu|^{\alpha-1}$ for all $\nu \in \mathbb{R}$ and $\alpha > 0$ with the convention $0^{(\alpha)} = 0$.

Our starting point shall be a weighted generalization of an abstract p -Hardy inequality of order 1 analogous to [7, Theorem 3.1]. In [7], the authors study p -Hardy inequalities in abstract graph setting. For our purposes, we conciser their results for the standard line graph on $\mathbb{N}_0$. The proof of the weighted generalization is reminiscent to the one from [7] or [9, Proposition 4], but for the reader's convenience it will be revisited in Section 1.1. If not stated otherwise, we always assume $p > 1$ and $\ell \in \mathbb{N}$ in the following. Furthermore, for two numbers $\alpha, \beta \in \mathbb{R}$, we write $\alpha \lesssim \beta$, if and only if there exists positive constant $C_p > 0$ such that $\alpha \leq C_p \beta$. If not stated otherwise the constant may only depend on p .

Theorem 1.3. *Let $V \in C(\mathbb{Z})$ such that $V_n \geq 0$ for all $n \geq 1$ and $\mathbf{g} \in H^1$ non-decreasing such that $\mathbf{g}_n > 0$ for all $n \geq 1$. Then for any $u \in \mathcal{H}_0^1$, we have the inequality*

$$\sum_{n=1}^{\infty} V_n |\nabla u_n|^p + \sum_{n=1}^{\infty} \frac{\text{div}(V(\nabla \mathbf{g})^{p-1})_n}{\mathbf{g}_n^{p-1}} |u_n|^p \geq 0. \quad (1.2)$$

If, in addition, u is a real-valued sequence, we have the reverse inequality

$$\sum_{n=1}^{\infty} V_n |\nabla u_n|^p + \sum_{n=1}^{\infty} \frac{\text{div}(V(\nabla \mathbf{g})^{p-1})_n}{\mathbf{g}_n^{p-1}} |u_n|^p \lesssim \sum_{n=1}^{\infty} V_{n+1} \mathcal{R}(p, \mathbf{g}; u)_n, \quad (1.3)$$

where the remainder term on the right-hand side is given for all $n \geq 1$ by

$$\mathcal{R}(p, \mathbf{g}; u)_n := \begin{cases} \left| \sqrt{\frac{\mathbf{g}_n}{\mathbf{g}_{n+1}}} u_{n+1} - \sqrt{\frac{\mathbf{g}_{n+1}}{\mathbf{g}_n}} u_n \right|^p & \text{if } p \in (1, 2], \\ \left| \sqrt{\frac{\mathbf{g}_n}{\mathbf{g}_{n+1}}} u_{n+1} - \sqrt{\frac{\mathbf{g}_{n+1}}{\mathbf{g}_n}} u_n \right|^2 \left(|\text{div} u_n| + \frac{|u_n|}{\mathbf{g}_n} \text{div} \mathbf{g}_n \right)^{p-2} & \text{if } p \in (2, \infty). \end{cases} \quad (1.4)$$

Remark 1.4. The lower estimate (1.2) asserts an abstract weighed p -Hardy inequality for any complex sequence $u \in \mathcal{H}_0^{\ell}$. On the other hand, the upper estimate (1.3) indicates how tight the inequality is, which will be essential later for deriving optimality of particular p -Hardy weights. However, we only claim that the upper estimate holds for real-valued sequences $u \in \mathcal{H}_0^{\ell}$. This is due to the fact, that these inequalities are rooted in the fundamental inequalities given by Lemma 1.12, where the upper

estimate (1.15), resp. (1.16), also claims the result only for $\alpha \in \mathbb{R}$. Unfortunately, we are unaware whether the inequality (1.15), and therefore inequality (1.3), can be extended to all complex numbers. Note that the formula (1.5) for the remainder $\mathcal{R}(p, \mathbf{g}; u)$ applies for all $p > 1$, since the same is true for the inequality (1.15). Nonetheless, the formula (1.4), which is a consequence of inequality (1.16), will be important for deducing asymptotic behavior of the remainder for $p \in (1, 2]$ in Section 1.1.5.

Remark 1.5. If the parameter sequence $\mathbf{g} \in H^1$ does not enjoy the monotonicity assumption, a slight modification of Theorem 1.3 still holds. In particular, for any $V \in C(\mathbb{Z})$ such that $V_n \geq 0$ for all $n \geq 1$ and $\mathbf{g} \in H^1$ such that $\mathbf{g}_n > 0$ for all $n \geq 1$, we have the inequalities

$$0 \leq \sum_{n=1}^{\infty} V_n |\nabla u_n|^p + \sum_{n=1}^{\infty} \frac{\operatorname{div}(V(\nabla \mathbf{g})^{(p-1)})_n}{\mathbf{g}_n^{p-1}} |u_n|^p \lesssim \sum_{n=1}^{\infty} V_{n+1} \mathcal{R}(p, \mathbf{g}; u)_n,$$

for all real-valued $u \in \mathcal{H}_0^1$ with

$$\mathcal{R}(p, \mathbf{g}; u)_n := \left| \sqrt{\frac{\mathbf{g}_n}{\mathbf{g}_{n+1}}} u_{n+1} - \sqrt{\frac{\mathbf{g}_{n+1}}{\mathbf{g}_n}} u_n \right|^2 \left(\left| \sqrt{\frac{\mathbf{g}_n}{\mathbf{g}_{n+1}}} u_{n+1} - \sqrt{\frac{\mathbf{g}_{n+1}}{\mathbf{g}_n}} u_n \right| + M \left(\frac{|u_n|}{\mathbf{g}_n} \right) |\operatorname{div} \mathbf{g}_n| \right)^{p-2}$$

for all $p > 1$, where we made use of the *middle operator* acting on $C(\mathbb{Z})$ defined as

$$Mu_n := \frac{u_{n+1} + u_n}{2} \quad \text{for all } n \in \mathbb{Z}.$$

Note that the lower estimate holds even for complex sequences and that the formula (1.4) can be considered instead if $p \in (1, 2]$. The proof is analogous to the proof of Theorem 1.3 and is therefore omitted; see [7, Lemma 3.8] for corresponding underlying inequality.

In [26], the authors studied the linear case $p = 2$ via equalities similar to the one-sided estimate (1.3). As was suggested in [7], we may infer an analogous result in the general case with $p > 1$. In fact, an equivalence can be established between both sides of (1.3) in the following sense. We say that the numbers $\alpha, \beta \in \mathbb{R}$ are equivalent and we denote this as $\alpha \simeq \beta$, if and only if $\beta \lesssim \alpha \lesssim \beta$. Since the underlying equivalence (1.20) holds only for $\alpha \in \mathbb{R}$, we must restrict ourselves to real-valued sequences in the following proposition as well.

Proposition 1.6. *Let $V \in C(\mathbb{Z})$ such that $V_n \geq 0$ for all $n \geq 1$ and $\mathbf{g} \in H^1$ non-decreasing such that $\mathbf{g}_n > 0$ for all $n \geq 1$. Then for any real-valued sequence $u \in \mathcal{H}_0^1$, we have the equivalence*

$$\sum_{n=1}^{\infty} V_n |\nabla u_n|^p + \sum_{n=1}^{\infty} \frac{\operatorname{div}(V(\nabla \mathbf{g})^{p-1})_n}{\mathbf{g}_n^{p-1}} |u_n|^p \simeq \sum_{n=1}^{\infty} V_{n+1} \mathcal{R}(p, \mathbf{g}; u)_n, \quad (1.6)$$

where the remainder on the right-hand side is given by formula (1.5) for all $p > 1$.

Remark 1.7. Note that for the linear case $p = 2$, the equivalence becomes an equality and holds even for complex-valued sequences $u \in \mathcal{H}_0^1$; see [12, 26].

The weight V in formulas (1.2), (1.3), or (1.6), allows us to iterate it in order to obtain analogous inequalities for higher order difference operators on the left-hand side. The key concept for deriving such inequalities relies on a suitable choice of the parameter sequence $\mathbf{g}$ in each step of the iteration. This way, we obtain convenient representations of weights V in terms of $\mathbf{g}$, because of which we may ensure their non-negativity under additional assumptions on $\mathbf{g}$.

Motivated by Remark 1.2, we define *discrete p -Laplace operator of order $\ell \in \mathbb{N}$* as

$$-\Delta_p^{(\ell)} := (-1)^\ell \operatorname{div} \nabla^{\ell-1} (\operatorname{div}^{\ell-1} \nabla \cdot)^{(p-1)} \quad \text{and} \quad -\Delta_p^{(0)} := \operatorname{I}. \quad (1.7)$$

Note, that for $\ell = 1$, the definition (1.7) coincides with standard discrete p -Laplace operator and that for any $\ell \in \mathbb{N}$, we have the identity

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left| (-\Delta)^{\ell/2} u_n \right|^p = \sum_{n=\ell}^{\infty} \overline{u_n} (-\Delta_p^{(\ell)}) u_n,$$

which holds true for any $u \in \mathcal{H}_0^\ell$.

Theorem 1.8. *Suppose that $\mathbf{g} \in H^\ell$ satisfies assumptions*

$$\operatorname{div}^k \mathbf{g}_n > 0 \text{ for all } n \geq \ell - k \text{ and } k \in \{0, \dots, \ell - 1\}, \quad (\text{A1})$$

$$-\Delta_p^{(\ell-k)} \operatorname{div}^k \mathbf{g}_n \geq 0 \text{ for all } n \geq \ell - k \text{ and } k \in \{1, \dots, \ell\}. \quad (\text{A2})$$

Let us denote $\rho_n(\mathbf{g}) := -\Delta_p^{(\ell)} \mathbf{g}_n / \mathbf{g}_n^{p-1}$ for all $n \geq \ell$. Then for all $u \in \mathcal{H}_0^\ell$, we have the inequality

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left| (-\Delta)^{\ell/2} u_n \right|^p - \sum_{n=\ell}^{\infty} \rho_n(\mathbf{g}) |u_n|^p \geq 0. \quad (1.8)$$

If, in addition, u is a real-valued sequence, we have the reverse inequality

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left| (-\Delta)^{\ell/2} u_n \right|^p - \sum_{n=\ell}^{\infty} \rho_n(\mathbf{g}) |u_n|^p \lesssim \sum_{k=0}^{\ell-1} \mathcal{R}_k^{(\ell)}(p, \mathbf{g}; u), \quad (1.9)$$

where the remainders on the right-hand side are given by

$$\mathcal{R}_k^{(\ell)}(p, \mathbf{g}; u) := \sum_{n=\ell-k}^{\infty} \frac{-\Delta_p^{(\ell-1-k)} \operatorname{div}^{k+1} \mathbf{g}_n}{(\operatorname{div}^{k+1} \mathbf{g}_n)^{p-1}} \mathcal{R}(p, \operatorname{div}^k \mathbf{g}; \operatorname{div}^k u)_n \quad (1.10)$$

with $\mathcal{R}(p, \mathbf{g}; u)$ defined by (1.4) for $p \in (1, 2]$, resp. (1.5) for $p \in (2, \infty)$. For $k = \ell - 1$, the coefficient in front of the remainder term $\mathcal{R}(p, \operatorname{div}^k \mathbf{g}; \operatorname{div}^k u)$ is to be understood as 1.

In view of Proposition 1.6, for real-valued sequences, we may strengthen inequality (1.9) to equivalence with remainders corresponding to (1.5) for all $p > 1$.

Proposition 1.9. *Suppose that $\mathbf{g} \in H^\ell$ satisfies assumptions (A1) and (A2). Let us denote $\rho_n(\mathbf{g}) := -\Delta_p^{(\ell)} \mathbf{g}_n / \mathbf{g}_n^{p-1}$ for all $n \geq \ell$. Then for all real-valued sequences $u \in \mathcal{H}_0^\ell$, we have the equivalence*

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left| (-\Delta)^{\ell/2} u_n \right|^p - \sum_{n=\ell}^{\infty} \rho_n(\mathbf{g}) |u_n|^p \simeq \sum_{k=0}^{\ell-1} \left[\sum_{n=\ell-k}^{\infty} \frac{-\Delta_p^{(\ell-1-k)} \operatorname{div}^{k+1} \mathbf{g}_n}{(\operatorname{div}^{k+1} \mathbf{g}_n)^{p-1}} \mathcal{R}(p, \operatorname{div}^k \mathbf{g}; \operatorname{div}^k u)_n \right], \quad (1.11)$$

where the remainders $\mathcal{R}$ on the right-hand side are given by (1.5) for all $p > 1$.

Let us further suppose the inequality in assumption (A2) for $k = 0$, i.e. $-\Delta_p^{(\ell)} \mathbf{g}_n \geq 0$ for all $n \geq \ell$, then $\{\rho_n(\mathbf{g})\}_{n=\ell}^\infty$ is a non-negative sequence, hence the following corollary.

Corollary 1.10. *Suppose that $\mathbf{g} \in H^\ell$ satisfies assumptions (A1), (A2), and, in addition,*

$$-\Delta_p^{(\ell)} \mathbf{g}_n \geq 0 \text{ for all } n \geq \ell. \quad (\text{A3})$$

Then $\rho(\mathbf{g}) = -\Delta_p^{(\ell)} \mathbf{g} / \mathbf{g}^{p-1}$ is a discrete p -Hardy weight of order ℓ .

By imposing additional requirements on the asymptotic behavior of $\mathbf{g}_n$, as $n \rightarrow \infty$, we obtain sufficient conditions for criticality and optimality near infinity of the discrete p -Hardy weights $\rho(\mathbf{g})$. Unfortunately, we were not able to assert non-attainability, which still remains an open problem. In order to do so, one may try to extend the equivalence (1.11) from real sequences in $\mathcal{H}_0^\ell$ to all complex sequences for which the left-hand side of (1.1) is finite and then use the same methods as in [26]. We consider the following statement to be our first main result.

Theorem 1.11. *Suppose that $\mathbf{g} \in H^\ell$ satisfies (A1), (A2), (A3), and, in addition, suppose that $\mathbf{g}$ admits the asymptotic expansion*

$$\mathbf{g}_n = \sum_{j=0}^{2\ell} \alpha_j n^{\ell-j-1/p} + \mathcal{O}\left(n^{-\ell-1-1/p}\right) \text{ for some } \alpha_j \in \mathbb{R} \text{ with } \alpha_0 \neq 0, \quad (\text{A4})$$

as $n \rightarrow \infty$. Then the discrete p -Hardy weight $\rho(\mathbf{g}) = -\Delta_p^{(\ell)} \mathbf{g} / \mathbf{g}^{p-1}$ of order ℓ is critical and optimal near infinity.

The final piece of the puzzle is to find a concrete parameter sequence $\mathbf{g}$, which satisfies all the assumptions (A1)–(A4). In this regard, motivated by the choice of $\mathbf{g}$ for $p = 2$ in (16) and for $p > 1$ and $\ell = 1$ in [9], we define our candidate for such sequence as

$$\mathbf{g}_n^{(\ell,p)} := n^{1-1/p} \prod_{j=1}^{\ell-1} (n-j) \quad (1.12)$$

for all $n \geq 0$ and $\mathbf{g}_n^{(\ell,p)} := 0$ for all $n < 0$. Even though it is fairly easy to verify assumptions (A1) and (A4) for the sequence (1.12), the non-linearity of the discrete p -Laplace operator $-\Delta_p^{(k)}$ of order $k \in \mathbb{N}$ makes validating assumptions (A2) and (A3) a very non-trivial problem. Regrettably, we were unable to develop an approach for proving these assumptions in a general setting given by a combination of $p > 1$ and $\ell \in \mathbb{N}$. Numerous partial results will be given in Section 2, where we discuss properties of the sequences given by formula (1.12) in greater detail.

Bearing Corollary 1.10 in mind, we denote the supposed p -Hardy weight given by the parameter sequence (1.12) as

$$\rho_n^{(\ell,p)} := \frac{-\Delta_p^{(\ell)} \mathbf{g}_n^{(\ell,p)}}{\left(\mathbf{g}_n^{(\ell,p)}\right)^{p-1}} \quad (1.13)$$

for all $n \geq \ell$. For $p = 2$, a remarkable property of the weights $\rho^{(\ell,p)} = \rho^{(\ell)}$ given by (16), was found in [26]. The authors showed that $\rho_n^{(\ell)}$ has a convergent series representation in negative powers of n with all coefficients being non-negative for $\ell = 1$ and positive for $\ell \geq 2$. This is useful for obtaining tighter Hardy inequalities of relatively simple form considering more terms of the truncated series representation. Unsurprisingly, the leading term of the expansion was shown to coincide with the classical Hardy weight of order ℓ in (13), which, in turn, demonstrates that $\rho^{(\ell)}$ provides an improvement over this weight. Similar result was observed to hold true for p -Hardy inequalities of order 1 with integer $p \geq 2$ in [9]. On the other hand, even though the authors successfully proved that the weight $\rho^{(1,p)} = \rho^{\text{FKP}}$ defined by (6) improves upon the classical p -Hardy weight (7) for general $p > 1$, the non-negativity of all coefficients in the incriminated expansion was only conjectured. Analysis of the sequence $\rho_n^{(1,p)}$ poses an intriguing problem, which is to be tackled in Section 3, where we answer a slight modification of the above mentioned conjecture from [9] in the affirmative.

1.1 Proofs

In this section, we gradually work out the proofs of Theorems 1.3, 1.8, and 1.11. The proofs of Propositions 1.6 and 1.9 will be indicated only briefly as they follow similar ideas as their counterparts, i.e. Theorems 1.3 and 1.8.

1.1.1 Proof of Theorem 1.3

The fundamental inequalities underlying (1.2) and (1.3) are given by the following lemma. Inequality (1.14) can be found in [10, Lemma 2.6], but for reader's convenience, we restate the inequality together with its short proof. Conversely, we refer the reader to [7, Lemma 3.8] for the non-trivial and lengthy proofs of inequalities (1.15) and (1.16).

Lemma 1.12. *Let $t \in [0, 1]$ and $p > 1$. Then it is true that*

$$0 \leq |\alpha - t|^p - (1 - t)^{p-1}(|\alpha|^p - t) \quad \text{for any } \alpha \in \mathbb{C}, \quad (1.14)$$

$$0 \leq |\alpha - t|^p - (1 - t)^{p-1}(|\alpha|^p - t) \lesssim t |\alpha - 1|^2 (|\alpha - t| + 1 - t)^{p-2} \quad \text{for any } \alpha \in \mathbb{R}. \quad (1.15)$$

If, moreover, $p \in (1, 2]$ then

$$0 \leq |\alpha - t|^p - (1 - t)^{p-1}(|\alpha|^p - t) \lesssim t^{p/2} |\alpha - 1|^p \quad \text{for any } \alpha \in \mathbb{R}. \quad (1.16)$$

Proof (of inequality 1.14). The inequality is trivially true for $t \in \{0, 1\}$, hence we may assume $t \in (0, 1)$. Additionally, for fixed $|\alpha|$, the minimum of the right-hand side of (1.14) is evidently attained for $\alpha > 0$. Since the right-hand side is also non-negative for $\alpha \leq t^{1/p}$, it remains to check the inequality for $\alpha > t^{1/p}$. Dividing by $\alpha^p - t > 0$, we obtain

$$\frac{(\alpha - t)^p}{\alpha^p - t} \geq (1 - t)^{p-1}.$$

Taking the derivative of the left-hand side reveals, that it has minimum $(1 - t)^{p-1}$ at $\alpha = 1$, completing the proof. $\square$

Now, we are ready to prove Theorem 1.3. We will validate inequalities (1.2) and (1.3) simultaneously, not differentiating between the real and complex cases, in order to be more concise. Hence, for simplicity, we assume a real-valued sequence $u \in \mathcal{H}_0^1$. The proof of the lower estimate (1.2) for complex sequences can be verified similarly, realizing the fundamental inequality (1.14) admits $\alpha \in \mathbb{C}$.

Suppose $\mathbf{g} \in H^1$ is a non-decreasing sequence such that $\mathbf{g}_n > 0$ for all $n \geq 1$ and $u \in \mathcal{H}_0^1$. We will show that for all $n \geq 2$, we have the inequalities

$$0 \leq |\nabla u_n|^p - \left(\frac{|u_n|^p}{\mathbf{g}_n^{p-1}} - \frac{|u_{n-1}|^p}{\mathbf{g}_{n-1}^{p-1}} \right) (\nabla \mathbf{g}_n)^{p-1} \lesssim \mathcal{R}(p, \mathbf{g}; u)_{n-1}, \quad (1.17)$$

where the right-hand side is given by (1.4) for $p \in (1, 2]$, resp. by (1.5) for $p \in (2, \infty)$, and the unspecified constant is independent of n . Next, consider $V_n \geq 0$ for all $n \geq 1$. Multiplying both sides of (1.17) by V_n and summing over n from 2 to ∞ , we arrive at

$$0 \leq \sum_{n=2}^{\infty} V_n |\nabla u_n|^p + \sum_{n=1}^{\infty} V_{n+1} (\nabla \mathbf{g}_{n+1})^{p-1} \frac{|u_n|^p}{\mathbf{g}_n^{p-1}} - \sum_{n=2}^{\infty} V_n (\nabla \mathbf{g}_n)^{p-1} \frac{|u_n|^p}{\mathbf{g}_n^{p-1}} \lesssim \sum_{n=2}^{\infty} V_n \mathcal{R}(p, \mathbf{g}; u)_{n-1}.$$

Moreover, a shift of the index on the the right-hand side reveals

$$0 \leq \sum_{n=2}^{\infty} V_n |\nabla u_n|^p + \sum_{n=2}^{\infty} \frac{\operatorname{div}(V(\nabla \mathbf{g})^{p-1})_n}{\mathbf{g}_n^{p-1}} |u_n|^p + V_2 (\nabla \mathbf{g}_2)^{p-1} \frac{|u_1|^p}{\mathbf{g}_1^{p-1}} \lesssim \sum_{n=1}^{\infty} V_{n+1} \mathcal{R}(p, \mathbf{g}; u)_n.$$

Recalling $u_0 = \mathfrak{g}_0 = 0$ and $\mathfrak{g}_1 > 0$, we infer that

$$V_1 |\nabla u_1|^p = V_1 (\nabla \mathfrak{g}_1)^{p-1} \frac{|u_1|^p}{\mathfrak{g}_1^{p-1}}.$$

Adding and subtracting this term to the middle part of the previous inequality yields Theorem 1.3.

In order to finish the proof, we have to establish inequalities in (1.17), which will be done in two separate steps:

- a) We verify inequalities in (1.17) for $p \in (1, 2]$ with the remainder given by (1.4).
- b) We verify inequalities in (1.17) for $p \in (2, \infty)$ with the remainder given by (1.5).
- a) Suppose $p \in (1, 2]$. We will verify that for any $n \geq 2$, we have inequalities

$$0 \leq |u_n - u_{n-1}|^p - \left(\frac{|u_n|^p}{\mathfrak{g}_n^{p-1}} - \frac{|u_{n-1}|^p}{\mathfrak{g}_{n-1}^{p-1}} \right) (\mathfrak{g}_n - \mathfrak{g}_{n-1})^{p-1} \lesssim \left| \sqrt{\frac{\mathfrak{g}_{n-1}}{\mathfrak{g}_n}} u_n - \sqrt{\frac{\mathfrak{g}_n}{\mathfrak{g}_{n-1}}} u_{n-1} \right|^p. \quad (1.18)$$

Let us first discuss the case $u_{n-1} = 0$. If also $u_n = 0$, then (1.18) is trivial. If $u_n \neq 0$, then (1.18) stems from the inequalities $0 \leq 1 - (1-t)^{p-1} \leq t \leq t^{p/2}$, which hold true for any $t \in [0, 1]$, after substituting $t = \mathfrak{g}_{n-1}/\mathfrak{g}_n \in (0, 1]$ by the monotonicity of $\mathfrak{g}$.

For $u_{n-1} \neq 0$, we set $\varphi := u/\mathfrak{g}$ and take $\alpha = \varphi_n/\varphi_{n-1} \in \mathbb{R}$ and $t = \mathfrak{g}_{n-1}/\mathfrak{g}_n \in (0, 1]$. Inequalities (1.16) reveal

$$0 \leq \left| \frac{\varphi_n}{\varphi_{n-1}} - \frac{\mathfrak{g}_{n-1}}{\mathfrak{g}_n} \right|^p - \left(1 - \frac{\mathfrak{g}_{n-1}}{\mathfrak{g}_n} \right)^{p-1} \left(\left| \frac{\varphi_n}{\varphi_{n-1}} \right|^p - \frac{\mathfrak{g}_{n-1}}{\mathfrak{g}_n} \right) \lesssim \left(\frac{\mathfrak{g}_{n-1}}{\mathfrak{g}_n} \right)^{p/2} \left| \frac{\varphi_n}{\varphi_{n-1}} - 1 \right|^p.$$

Finally, multiplying the latter inequality by $\mathfrak{g}_n^p |\varphi_{n-1}|^p$ yields (1.18).

b) Suppose $p \in (2, \infty)$. The lower estimate in (1.17) is identical to the corresponding estimate in part a). Recalling definition (1.5), the upper estimate rewrites to

$$|\nabla u_n|^p - \left(\frac{|u_n|^p}{\mathfrak{g}_n^{p-1}} - \frac{|u_{n-1}|^p}{\mathfrak{g}_{n-1}^{p-1}} \right) (\nabla \mathfrak{g}_n)^{p-1} \lesssim \left| \sqrt{\frac{\mathfrak{g}_{n-1}}{\mathfrak{g}_n}} u_n - \sqrt{\frac{\mathfrak{g}_n}{\mathfrak{g}_{n-1}}} u_{n-1} \right|^2 \left(|\nabla u_n| + \frac{|u_{n-1}|}{\mathfrak{g}_{n-1}} \nabla \mathfrak{g}_n \right)^{p-2}.$$

If, $u_{n-1} \neq 0$, the proof of the latter inequality proceeds similarly as in part a) after substituting $\alpha = \varphi_n/\varphi_{n-1}$, where $\varphi = u/\mathfrak{g}$, and $t = \mathfrak{g}_{n-1}/\mathfrak{g}_n$ to the inequality (1.15). Finally, if $u_{n-1} = 0$, the upper estimate is a consequence of the Bernoulli inequality $1 - (1-t)^{p-1} \leq (p-1)t$ being true for any $t \in [0, 1]$. The proof of Theorem 1.3 is complete. $\square$

1.1.2 Proof of Proposition 1.6

The strategy of the proof is reminiscent to Section 1.1.1. Suppose $\mathfrak{g} \in H^\ell$ is a non-decreasing function such that $\mathfrak{g}_n > 0$ for all $n \geq 1$ and a real sequence $u \in \mathcal{H}_0^1$. Here, we establish the equivalence

$$|\nabla u_n|^p - \left(\frac{|u_n|^p}{\mathfrak{g}_n^{p-1}} - \frac{|u_{n-1}|^p}{\mathfrak{g}_{n-1}^{p-1}} \right) (\nabla \mathfrak{g}_n)^{p-1} \simeq \left| \sqrt{\frac{\mathfrak{g}_{n-1}}{\mathfrak{g}_n}} u_n - \sqrt{\frac{\mathfrak{g}_n}{\mathfrak{g}_{n-1}}} u_{n-1} \right|^2 \left(|\nabla u_n| + \frac{|u_{n-1}|}{\mathfrak{g}_{n-1}} \nabla \mathfrak{g}_n \right)^{p-2}. \quad (1.19)$$

for any $n \geq 2$ and $p > 1$. Again, summing over n from 2 to ∞ and the same manipulations as in Section 1.1.1 yield the claim of Proposition 1.6.

To conclude the proof, we verify (1.19). In [7], the authors found that inequality (1.15) holds as an equivalence, i.e. for any $\alpha \in \mathbb{R}$, $t \in [0, 1]$, and $p > 1$, we have

$$|\alpha - t|^p - (1-t)^{p-1} (|\alpha|^p - t) \simeq t |\alpha - 1|^2 (|\alpha - t| + 1 - t)^{p-2}. \quad (1.20)$$

If $u_{n-1} \neq 0$, we may, once again, consider $\alpha = \varphi_n/\varphi_{n-1}$, where $\varphi = u/\mathbf{g}$, and $t = \mathbf{g}_{n-1}/\mathbf{g}_n$ in the equivalence (1.20) and multiply by $\mathbf{g}_n^p |\varphi_{n-1}|^p$. This amounts to the desired equivalence. If, on the other hand, $u_{n-1} = 0$, we obtain (1.19) if we show the equivalence $1 - (1-t)^{p-1} \simeq t$. This, however, is an immediate consequence of the elementary inequalities

$$(1-t)^{p-1} \leq \begin{cases} 1 - (p-1)t & \text{if } p \in (1, 2], \\ 1 - t & \text{if } p \in (2, \infty), \end{cases} \quad \text{and} \quad (1-t)^{p-1} \geq \begin{cases} 1 - t & \text{if } p \in (1, 2], \\ 1 - (p-1)t & \text{if } p \in (2, \infty). \end{cases}$$

The proof of Proposition 1.6 is complete. $\square$

1.1.3 Proof of Theorem 1.8

Throughout the proof, we will frequently make use of the *forward shift operator* S acting on $C(\mathbb{Z})$ as

$$Su_n := u_{n+1} \tag{1.21}$$

for all $n \in \mathbb{Z}$. Evidently, for all $k \in \mathbb{Z}$ and $n \in \mathbb{Z}$, we have $S^k u_n = u_{n+k}$. Recalling definition (11), we observe the elementary operator formulas $\text{div} = S - I$ and $\nabla = I - S^{-1}$, hence $\text{div} = S\nabla = \nabla S$. The fact that $SS^{-1} = S^{-1}S = I$ is the main reason for considering sequences $u \in C(\mathbb{Z})$ indexed by $\mathbb{Z}$, since analogous operators acting on $C(\mathbb{N})$ differ by a rank one operator $Tu = (u_1, 0, 0, \dots)$. Note also, that the spaces H^ℓ defined by (10) are not preserved under the action of S . Rather S , and therefore div , maps H^ℓ onto $H^{\ell-1}$.

We shall proceed by induction in $\ell \in \mathbb{N}$, where the induction step will be separated into two parts corresponding to the parity of ℓ :

- a) We verify Theorem 1.8 for $\ell = 1$.
- b) Assuming Theorem 1.8 holds for all $\ell \leq 2m - 1$, where $m \in \mathbb{N}$, we prove it for $\ell = 2m$.
- c) Assuming Theorem 1.8 holds for all $\ell \leq 2m$, where $m \in \mathbb{N}$, we prove it for $\ell = 2m + 1$.

This is necessary, because when lowering a half-integer power of the discrete Laplacian (14), ∇ or div appear depending on the parity of ℓ . In particular, we have

$$(-\Delta)^{\ell/2} = \begin{cases} \nabla(-\Delta)^{(\ell-1)/2} & \text{if } \ell \in 2\mathbb{N}_0 + 1, \\ -\text{div}(-\Delta)^{(\ell-1)/2} & \text{if } \ell \in 2\mathbb{N}. \end{cases}$$

For simplicity, we assume a real-valued sequence $u \in \mathcal{H}_0^\ell$ and verify inequalities (1.8) and (1.9) simultaneously. The proof of the lower estimate (1.8) for complex sequences is identical.

a) For $\ell = 1$, Theorem 1.8 coincides with Theorem 1.3 with $V \equiv 1$. Note that by definition of the discrete divergence (11), assumption (A2) ensures the monotonicity of $\mathbf{g}$.

b) Suppose Theorem 1.8 holds for all $\ell \leq 2m - 1$. For any $u \in \mathcal{H}_0^{2m}$, we have

$$\sum_{n=\lceil \frac{2m}{2} \rceil}^{\infty} \left| (-\Delta)^{2m/2} u_n \right|^p = \sum_{n=m}^{\infty} |(-\Delta)^m u_n|^p = \sum_{n=m}^{\infty} \left| (-\Delta)^{(2m-1)/2} \text{div} u_n \right|^p.$$

Given $\mathbf{g} \in H^{2m}$ such that assumptions (A1) and (A2) are fulfilled with $\ell = 2m$, we infer that $\text{div} \mathbf{g} \in H^{2m-1}$ comply these assumptions for $\ell = 2m - 1$. By induction hypothesis applied to $\text{div} u \in \mathcal{H}_0^{2m-1}$ with $\text{div} \mathbf{g}$, we obtain

$$0 \leq \sum_{n=m}^{\infty} |(-\Delta)^m u_n|^p + \sum_{n=2m-1}^{\infty} \frac{\Delta_p^{(2m-1)} \text{div} \mathbf{g}_n}{(\text{div} \mathbf{g}_n)^{p-1}} |\text{div} u_n|^p \lesssim \sum_{k=0}^{2m-2} \mathcal{R}_k^{(2m-1)}(p, \text{div} \mathbf{g}; \text{div} u), \tag{1.22}$$

where the remainders on the right-hand side are defined by (1.10). Obviously, we have the identity

$$\mathcal{R}_k^{(2m-1)}(p, \operatorname{div} \mathbf{g}; \operatorname{div} u) = \mathcal{R}_{k+1}^{(2m)}(p, \mathbf{g}; u)$$

for all $k \in \{0, \dots, 2m-2\}$. Next, we shift the index n in the second sum arriving at

$$\sum_{n=2m-1}^{\infty} \frac{\Delta_p^{(2m-1)} \operatorname{div} \mathbf{g}_n}{(\operatorname{div} \mathbf{g}_n)^{p-1}} |\operatorname{div} u_n|^p = \sum_{n=1}^{\infty} \frac{S^{2m-2} \Delta_p^{(2m-1)} \operatorname{div} \mathbf{g}_n}{(S^{2m-2} \operatorname{div} \mathbf{g}_n)^{p-1}} |\nabla S^{2m-1} u_n|^p$$

and we apply Theorem 1.3 with $S^{2m-1} u \in \mathcal{H}_0^1$, non-decreasing sequence $S^{2m-1} \mathbf{g} \in H^1$, and the weight

$$V_n := \frac{S^{2m-2} (-\Delta_p^{(2m-1)}) \operatorname{div} \mathbf{g}_n}{(S^{2m-2} \operatorname{div} \mathbf{g}_n)^{p-1}},$$

which is non-negative for all $n \geq 1$ by assumption (A2), obtaining

$$0 \leq \sum_{n=1}^{\infty} V_n |\nabla S^{2m-1} u_n|^p + \sum_{n=1}^{\infty} \frac{\operatorname{div}(V(\nabla S^{2m-1} \mathbf{g})^{p-1})_n}{(S^{2m-1} \mathbf{g}_n)^{p-1}} |S^{2m-1} u_n|^p \lesssim \sum_{n=1}^{\infty} V_{n+1} \mathcal{R}(p, S^{2m-1} \mathbf{g}; S^{2m-1} u)_n.$$

Let us examine the prefactor in the second sum in greater detail. We observe that

$$\operatorname{div}(V(\nabla S^{2m-1} \mathbf{g})^{p-1}) = \operatorname{div} \left(\frac{S^{2m-2} (-\Delta_p^{(2m-1)}) \operatorname{div} \mathbf{g}_n}{(S^{2m-2} \operatorname{div} \mathbf{g}_n)^{p-1}} (S^{2m-2} \operatorname{div} \mathbf{g}_n)^{p-1} \right) = S^{2m-1} \Delta_p^{(2m)} \mathbf{g}.$$

Recalling also the definition of the remainders (1.10), the latter inequality rewrites to

$$0 \leq - \sum_{n=2m-1}^{\infty} \frac{\Delta_p^{(2m-1)} \operatorname{div} \mathbf{g}_n}{(\operatorname{div} \mathbf{g}_n)^{p-1}} |\operatorname{div} u_n|^p + \sum_{n=2m}^{\infty} \frac{\Delta_p^{(2m)} \mathbf{g}_n}{\mathbf{g}_n^{p-1}} |u_n|^p \lesssim \mathcal{R}_0^{(2m)}(p, \mathbf{g}; u). \quad (1.23)$$

Note that all the constants defining the upper estimates are the same and are determined by the fundamental inequality (1.15), resp. (1.16). Altogether, inequalities (1.22) and (1.23) combine to

$$0 \leq \sum_{n=m}^{\infty} |(-\Delta)^m u_n|^p + \sum_{n=2m}^{\infty} \frac{\Delta_p^{(2m)} \mathbf{g}_n}{\mathbf{g}_n^{p-1}} |u_n|^p \lesssim \sum_{k=0}^{2m-1} \mathcal{R}_k^{(2m)}(p, \mathbf{g}; u),$$

which concludes part b).

c) Finally, we assume Theorem 1.8 holds for all $\ell \leq 2m$ and we verify it for $\ell = 2m+1$. Suppose $u \in \mathcal{H}_0^{2m+1}$ and $\mathbf{g} \in H^{2m+1}$ satisfying assumptions (A1) and (A2). This time, we have

$$\sum_{n=\lceil \frac{2m+1}{2} \rceil}^{\infty} \left| (-\Delta)^{(2m+1)/2} u_n \right|^2 = \sum_{n=m+1}^{\infty} |\nabla (-\Delta)^m u_n|^2 = \sum_{n=m}^{\infty} |(-\Delta)^m \operatorname{div} u_n|^2$$

and we may apply the induction hypothesis on $\operatorname{div} u \in \mathcal{H}_0^{2m}$ and the parameter sequence $\operatorname{div} \mathbf{g} \in H^{2m}$, which meets assumptions (A1) and (A2) with $\ell = 2m$. The rest of the proof can be done similarly as in part b), therefore the technical details are left as an exercise. The proof of Theorem 1.8 is complete. $\square$

1.1.4 Proof of Proposition 1.9

The proof can be done similarly as in Section 1.1.3, i.e. by induction in $\ell \in \mathbb{N}$. Here, the corresponding base case is given by Proposition 1.6. The induction step is reminiscent to the previous proof Section 1.1.3 with a few subtle differences, hence the details are omitted once again. The proof of Proposition 1.9 is complete. $\square$

1.1.5 Proof of Theorem 1.11

The proof will be done in two separate steps, in which we show that the weight $\rho(\mathfrak{g}) = -\Delta_p^{(\ell)} \mathfrak{g} / \mathfrak{g}^{p-1}$ exhibits the two desired properties from Definition 1.1: criticality and optimality near infinity. The proof will be analogous to the linear case $p = 2$ in [26]. The key idea of both parts is to substitute specific convenient real sequences $\{u^N\}_{N \geq 2} \subset \mathcal{H}_0^\ell$ into the inequality (1.9) and inspect the asymptotic behavior of the remainders $\mathcal{R}_k^{(\ell)}(p, \mathfrak{g}; u^N)$ on the right-hand side, as $N \rightarrow \infty$. For this purpose, we restate the following lemmas from [26]; the proofs can be found therein.

Lemma 1.13. *Let $k \in \mathbb{N}$, $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, and g be a sequence admitting the asymptotic expansion*

$$g_n = \sum_{j=0}^k a_j n^{\alpha-j} + \mathcal{O}(n^{\alpha-k-1}), \quad n \rightarrow \infty, \quad (1.24)$$

for some $a_j \in \mathbb{R}$ and $a_0 \neq 0$. Then the following claims hold true.

(i) For all $m \in \mathbb{Z}$, there exist coefficients $a_j^{(m)} \in \mathbb{R}$ such that

$$S^m g_n = a_0 n^\alpha + \sum_{j=1}^k a_j^{(m)} n^{\alpha-j} + \mathcal{O}(n^{\alpha-k-1}), \quad n \rightarrow \infty.$$

(ii) For all $m \in \mathbb{N}_0$, there exist coefficients $b_j^{(m)} \in \mathbb{R}$ with $b_0^{(m)} \neq 0$ such that

$$M^m g_n = \sum_{j=0}^k b_j^{(m)} n^{\alpha-j} + \mathcal{O}(n^{\alpha-k-1}), \quad n \rightarrow \infty.$$

(iii) For all $m \in \{0, \dots, k\}$, there exist coefficients $c_j^{(m)} \in \mathbb{R}$ with $c_m^{(m)} \neq 0$ such that

$$\text{div}^m g_n = \sum_{j=m}^k c_j^{(m)} n^{\alpha-j} + \mathcal{O}(n^{\alpha-k-1}), \quad n \rightarrow \infty.$$

Claim (iii) of Lemma 1.13 asserts that the discrete divergence $\text{div} \mathfrak{g}$, for $\mathfrak{g} \in C(\mathbb{Z})$ of the form (1.24), may be treated analogously as the derivative $\mathfrak{g}'$, where we naturally extend $\mathfrak{g}$ to a function on $(0, \infty)$, as far as the asymptotic behavior is concerned. The following auxiliary claim, a generalization of the Mean Value Theorem, unfolds further connections between $\text{div} \mathfrak{g}$ and $\mathfrak{g}'$.

Lemma 1.14. *Let $n \in \mathbb{Z}$, $N \in \mathbb{N}$ and $g \in C([n, n+N])$ such that $g \in C^N((n, n+N))$. Then there exists $\xi \in (n, n+N)$ such that*

$$\text{div}^N g_n = g^{(N)}(\xi),$$

where we denoted $g_n := g(n)$.

Finally, we will make use of the Leibnitz formula for the discrete divergence.

Lemma 1.15. *For all $m \in \mathbb{N}_0$ and sequences $f, g \in C(\mathbb{Z})$, we have*

$$\text{div}^m(fg) = \sum_{j=0}^m \binom{m}{j} (\text{div}^j M^{m-j} f) (\text{div}^{m-j} M^j g),$$

where the multiplication of sequences is to be understood point-wise.

a) *Proof of criticality:* Suppose $\tilde{\rho}$ is a p -Hardy weight of order ℓ such that $\tilde{\rho}_n \geq \rho_n(\mathbf{g})$ for all $n \geq \ell$. Employing inequality (1.9) for the weight $\rho(\mathbf{g})$ and the discrete p -Hardy inequality (1.1) of order ℓ for the weight $\tilde{\rho}$, we find that

$$0 \leq \sum_{n=\ell}^{\infty} (\tilde{\rho}_n - \rho_n(\mathbf{g})) |u_n|^p \lesssim \sum_{k=0}^{\ell-1} \mathcal{R}_k^{(\ell)}(p, \mathbf{g}; u) \quad (1.25)$$

for all real sequences $u \in \mathcal{H}_0^\ell$.

Quick observation of the remainders determined by formula (1.10) reveals that they are all simultaneously annihilated if we set $u = \mathbf{g}$, i.e. $\mathcal{R}_k^{(\ell)}(p, \mathbf{g}; \mathbf{g}) \equiv 0$ for all $k \in \{0, \dots, \ell-1\}$ and $p > 1$. Unfortunately, we cannot directly substitute $u = \mathbf{g}$ to (1.25) since $\mathbf{g} \notin \mathcal{H}_0^\ell$. To this end, we introduce the following regularization of $\mathbf{g}$. First, let us define a smooth cut-off function ξ^N , for any $N \geq 2$, as

$$\xi^N(x) := \begin{cases} 1 & \text{if } x \leq N, \\ \eta\left(\frac{2 \ln N - \ln x}{\ln N}\right) & \text{if } N < x \leq N^2, \\ 0 & \text{if } x > N^2, \end{cases} \quad (1.26)$$

where η is a smooth function, such that $\eta \equiv 0$ on $(-\infty, \varepsilon)$ and $\eta \equiv 1$ on $(1 - \varepsilon, \infty)$ for an arbitrary fixed $\varepsilon \in (0, 1/2)$. We set $u^N := \xi^N \mathbf{g}$ for all $N \geq 2$, where $\xi_n^N := \xi^N(n)$ for all $n \geq 1$.

Notice that by definition (1.26), we have $\xi^N \rightarrow 1$, and therefore $u^N \rightarrow \mathbf{g}$, point-wise as $N \rightarrow \infty$. We will show that this regularization admits the estimate

$$\mathcal{R}_k^{(\ell)}(p, \mathbf{g}; u^N) \lesssim \frac{1}{\ln^{p-1} N} \quad \text{if } p \in (1, 2] \quad \text{and} \quad \mathcal{R}_k^{(\ell)}(p, \mathbf{g}; u^N) \lesssim \frac{1}{\ln N} \quad \text{if } p \in (2, \infty) \quad (1.27)$$

for all $k \in \{0, \dots, \ell-1\}$, where $\lesssim$ means the inequality $\leq$ up to a N -independent constant, which may depend on k or p . Fatou's lemma, together with the inequality (1.25), then implies

$$\sum_{n=\ell}^{\infty} (\tilde{\rho}_n - \rho_n(\mathbf{g})) \mathbf{g}_n^p = \sum_{n=\ell}^{\infty} (\tilde{\rho}_n - \rho_n(\mathbf{g})) \lim_{N \rightarrow \infty} |u_n^N|^p \leq \liminf_{N \rightarrow \infty} \sum_{n=\ell}^{\infty} (\tilde{\rho}_n - \rho_n(\mathbf{g})) |u_n^N|^p = 0.$$

Bearing in mind, that all the terms in the sum on the left are non-negative and $\mathbf{g}_n > 0$ for all $n \geq \ell$ by (A1), we may conclude that $\tilde{\rho}_n = \rho_n(\mathbf{g})$ for all $n \geq \ell$, which proves the criticality of the p -Hardy weight $\rho(\mathbf{g})$.

It remains to establish the estimate (1.27) which is to be done in the rest of part a). Let us recall that the remainders are defined by the formula

$$\mathcal{R}_k^{(\ell)}(p, \mathbf{g}; u^N) = \sum_{n=\ell-k}^{\infty} \frac{-\Delta_p^{(\ell-1-k)} \operatorname{div}^{k+1} \mathbf{g}_n}{(\operatorname{div}^{k+1} \mathbf{g}_n)^{p-1}} \mathcal{R}(p, \operatorname{div}^k \mathbf{g}; \operatorname{div}^k u^N)_n$$

and inspect the asymptotic behavior of the summands as $n \rightarrow \infty$, with $N \geq 2$ fixed.

We begin by deducing an upper bound of the first factor

$$\frac{-\Delta_p^{(\ell-1-k)} \operatorname{div}^{k+1} \mathbf{g}_n}{(\operatorname{div}^{k+1} \mathbf{g}_n)^{p-1}} \lesssim n^{p\left(\frac{1}{p} + k - \ell + 1\right) - 1} \quad (1.28)$$

for any $n \geq \ell - k$ and $k \in \{0, \dots, \ell-1\}$, where the unspecified constant is n -independent. By assumption (A4) and claim (iii) of Lemma 1.13, we observe that

$$\operatorname{div}^m \mathbf{g}_n = c_0 n^{\ell-m-1/p} \left(1 + \sum_{j=m+1}^{2\ell} \frac{c_j}{c_0} n^{m-j} + \mathcal{O}\left(n^{m-2\ell-1}\right) \right) \quad \text{as } n \rightarrow \infty,$$

where $c_0 \neq 0$, for any $m \in \{0, \dots, 2\ell\}$. Furthermore, the generalized binomial theorem implies

$$(\operatorname{div}^m \mathbf{g}_n)^{p-1} = \sum_{j=m}^{2\ell} \tilde{c}_j n^{(p-1)(\ell-m-1/p)-j} + \mathcal{O}\left(n^{(p-1)(\ell-m-1/p)-2\ell-1}\right) \quad \text{as } n \rightarrow \infty, \quad (1.29)$$

where $\tilde{c}_0 \neq 0$. Using the shift operator (1.21) and the definition of the p -Laplacian (1.7), we may rewrite the numerator as

$$-\Delta_p^{(\ell-1-k)} \operatorname{div}^{k+1} \mathbf{g}_n = (-1)^{\ell+1+k} S^{k+1-\ell} \operatorname{div}^{\ell-1-k} (\operatorname{div}^\ell \mathbf{g}_n)^{p-1},$$

hence applying claims (i) and (iii) of Lemma 1.13 to the sequence $(\operatorname{div}^\ell \mathbf{g})^{p-1}$, we get

$$-\Delta_p^{(\ell-1-k)} \operatorname{div}^{k+1} \mathbf{g}_n \lesssim n^{1/p+k-\ell} \quad \text{and} \quad (\operatorname{div}^{k+1} \mathbf{g}_n)^{p-1} \simeq n^{(p-1)(\ell-1/p-k-1)}.$$

Inequality (1.28) follows immediately.

Next, we examine the asymptotic behavior of the second factor. Namely, we prove that

$$\mathcal{R}(p, \operatorname{div}^k \mathbf{g}; \operatorname{div}^k u^N)_n \lesssim \frac{n^{-p(\frac{1}{p}+k-\ell+1)}}{\ln^{\min\{p,2\}} N} \quad (1.30)$$

for all $p > 1$, where the unspecified constant does not depend on n nor N . First, we establish the inequality

$$\left| \sqrt{\frac{\operatorname{div}^k \mathbf{g}_n}{\operatorname{div}^k \mathbf{g}_{n+1}}} \operatorname{div}^k u_{n+1}^N - \sqrt{\frac{\operatorname{div}^k \mathbf{g}_{n+1}}{\operatorname{div}^k \mathbf{g}_n}} \operatorname{div}^k u_n^N \right| \lesssim \frac{n^{\ell-k-1-1/p}}{\ln N} \quad (1.31)$$

for all $N \geq 2$, $n \geq \ell - k$, and $k \in \{0, \dots, \ell - 1\}$. In fact, this result implicitly appeared in [26, Section 2.4] but for the sake of completeness, we will include the derivation of the inequality nonetheless. By assumption (A4) and claims (i) and (iii) of Lemma 1.13, we find that

$$\sqrt{\frac{\operatorname{div}^k \mathbf{g}_n}{\operatorname{div}^k \mathbf{g}_{n+1}}} \lesssim 1,$$

and therefore the left-hand side of (1.31) can be estimated as

$$\left| \sqrt{\frac{\operatorname{div}^k \mathbf{g}_n}{\operatorname{div}^k \mathbf{g}_{n+1}}} \left| \operatorname{div}^k (\xi^N \mathbf{g})_{n+1} - \frac{\operatorname{div}^k \mathbf{g}_{n+1}}{\operatorname{div}^k \mathbf{g}_n} \operatorname{div}^k (\xi^N \mathbf{g})_n \right| \lesssim \left| \operatorname{div}^k (\xi^N \mathbf{g})_{n+1} - \frac{\operatorname{div}^k \mathbf{g}_{n+1}}{\operatorname{div}^k \mathbf{g}_n} \operatorname{div}^k (\xi^N \mathbf{g})_n \right|.$$

Invoking the Leibnitz formula from Lemma 1.15 the last expression rewrites as

$$\begin{aligned} & \left| \sum_{j=0}^k \binom{k}{j} (\operatorname{div}^j M^{k-j} \mathbf{g})_{n+1} (\operatorname{div}^{k-j} M^j \xi^N)_{n+1} - \frac{\operatorname{div}^k \mathbf{g}_{n+1}}{\operatorname{div}^k \mathbf{g}_n} \sum_{j=0}^k \binom{k}{j} (\operatorname{div}^j M^{k-j} \mathbf{g})_n (\operatorname{div}^{k-j} M^j \xi^N)_n \right| \\ & \leq \sum_{j=0}^k \binom{k}{j} (\operatorname{div}^j M^{k-j} \mathbf{g})_{n+1} \left| (\operatorname{div}^{k-j} M^j \xi^N)_{n+1} - \frac{\operatorname{div}^k \mathbf{g}_{n+1}}{\operatorname{div}^k \mathbf{g}_n} \frac{\operatorname{div}^j M^{k-j} \mathbf{g}_n}{\operatorname{div}^j M^{k-j} \mathbf{g}_{n+1}} (\operatorname{div}^{k-j} M^j \xi^N)_n \right|. \end{aligned}$$

Yet another application of (A4) and formulas from Lemma 1.13 reveals that

$$(\operatorname{div}^j M^{k-j} \mathbf{g})_{n+1} \lesssim n^{\ell-1/p-j} \quad \text{and} \quad \frac{\operatorname{div}^k \mathbf{g}_{n+1}}{\operatorname{div}^k \mathbf{g}_n} \frac{\operatorname{div}^j M^{k-j} \mathbf{g}_n}{\operatorname{div}^j M^{k-j} \mathbf{g}_{n+1}} = \frac{1 + \mathcal{O}(1/n)}{1 + \mathcal{O}(1/n)} \frac{1 + \mathcal{O}(1/n)}{1 + \mathcal{O}(1/n)} = 1 + p_n^{(k,j)},$$

where $p_n^{(k,j)} = \mathcal{O}(1/n)$, as $n \rightarrow \infty$, i.e. $p_n^{(k,j)} \lesssim 1/n$, with $p^{(k,k)} \equiv 0$. Altogether, we deduce the upper bound of the left-hand side of (1.31)

$$\sum_{j=0}^k \binom{k}{j} n^{\ell-1/p-j} \left(\left| (\operatorname{div}^{k+1-j} M^j \xi^N)_n \right| + \left| p_n^{(k,j)} (\operatorname{div}^{k-j} M^j \xi^N)_n \right| \right) \quad (1.32)$$

for all $N \geq 2$, $n \geq \ell$, and $k \in \{0, \dots, \ell-1\}$. In order to obtain the desired estimate (1.31), we have to inspect also the asymptotic behavior of the terms with $\operatorname{div}^m \xi$ in (1.32). To this end, note that for all $x \in [N, N^2]$, we have

$$(\xi^N)'(x) = -\eta' \left(\frac{2 \ln N - \ln x}{\ln N} \right) \frac{1}{x \ln N},$$

thus $|(\xi^N)'(x)| \lesssim 1/(x \ln N)$, where we estimated η' with $\max_{x \in [0,1]} |\eta'(x)|$. It is straightforward to generalize this bound to higher-order derivatives, i.e.

$$|(\xi^N)^{(m)}(x)| \lesssim \frac{1}{x^m \ln N}$$

for all $m \in \mathbb{N}$ and $x > 0$. Consequently, Lemma 1.14 implies that

$$|\operatorname{div}^m \xi_n^N| \lesssim \frac{1}{n^m \ln N},$$

which is true for any $m, n \in \mathbb{N}$ and $N \geq 2$. By monotonicity of the right-hand side in the variable n , we also have

$$|\operatorname{div}^m M^j \xi_n^N| \lesssim \frac{1}{n^m \ln N} \quad (1.33)$$

for any $j \in \mathbb{N}_0$, from which we infer further upper estimate of (1.32)

$$\sum_{j=0}^{k-1} \binom{k}{j} n^{\ell-1/p-j} \left(\frac{1}{n^{k+1-j} \ln N} + \frac{1}{n} \frac{1}{n^{k-j} \ln N} \right) + n^{\ell-1/p-k} \frac{1}{n \ln N} \lesssim \frac{n^{\ell-k-1-1/p}}{\ln N},$$

finally arriving at the (1.31).

If $p \in (1, 2]$, inequality (1.30) follows immediately from (1.31) and definition (1.4) of the corresponding remainders. If $p \in (2, \infty)$, we must estimate also the second factor in the defining formula (1.5). In particular, we show that

$$\left| \operatorname{div}^{k+1} u_n^N \right| + \frac{|\operatorname{div}^k u_n^N|}{\operatorname{div}^k \mathfrak{g}_n} \operatorname{div}^{k+1} \mathfrak{g}_n \lesssim n^{\ell-k-1-1/p}. \quad (1.34)$$

For any $k \in \{0, \dots, \ell\}$, Lemma 1.15 implies

$$\operatorname{div}^k u_n^N = \operatorname{div}^k (\xi^N \mathfrak{g})_n = \sum_{j=0}^k \binom{k}{j} (\operatorname{div}^j M^{k-j} \mathfrak{g})_n (\operatorname{div}^{k-j} M^j \xi^N)_n.$$

Using claims (ii) and (iii) of Lemma 1.13, together with the inequality (1.33), we infer that

$$(\operatorname{div}^j M^{k-j} \mathfrak{g})_n \lesssim n^{\ell-1/p-j} \quad \text{and} \quad (\operatorname{div}^{k-j} M^j \xi^N)_n \lesssim n^{j-k}$$

for all $N \geq 2$, $n \geq \ell - k$, and $j \in \{0, \dots, k\}$. Note that the second estimate for $j = k$ follows from the inequality $\xi^N \leq 1$. Altogether, we arrive at

$$\operatorname{div}^k u_n^N \lesssim n^{\ell-1/p-k}.$$

Moreover, employing claim (iii) of Lemma 1.13 once again, we find that

$$\frac{\operatorname{div}^{k+1} \mathbf{g}_n}{\operatorname{div}^k \mathbf{g}_n} \lesssim \frac{1}{n}.$$

Combining the above inequalities yields the estimate (1.34). Recalling definition (1.5), inequalities (1.31) and (1.34) amount to the desired inequality (1.30) for $p > 2$.

Finally, bearing inequalities (1.28) and (1.30) in mind, for all N sufficiently large, we obtain

$$\begin{aligned} \mathcal{R}_k^{(\ell)}(p, \mathbf{g}; u^N) &= \sum_{n=N-k}^{N^2} \frac{-\Delta_p^{(\ell-1-k)} \operatorname{div}^{k+1} \mathbf{g}_n}{(\operatorname{div}^{k+1} \mathbf{g}_n)^{p-1}} \mathcal{R}(p, \operatorname{div}^k \mathbf{g}; \operatorname{div}^k u^N)_n \lesssim \frac{1}{\ln^{\min\{p,2\}} N} \sum_{n=N-k}^{N^2} \frac{1}{n} \\ &\lesssim \frac{1}{\ln^{\min\{p,2\}} N} \int_N^{N^2} \frac{1}{n} \, dn = \frac{1}{\ln^{\min\{p,2\}-1} N} \end{aligned}$$

for all $k \in \{0, \dots, \ell-1\}$, which coincides with the aforementioned inequalities (1.27). The proof of criticality is complete.

b) *Proof of optimality near infinity*: Fix an arbitrary $M \geq \ell$. Recalling the definition of optimality near infinity (iii) and inequality (1.9), we prove that the weight $\rho(\mathbf{g})$ enjoys this property via finding a sequence $\{u^N\}_{N \geq M}$ of real-valued sequences in $\mathcal{H}_0^M$ such that

$$\lim_{N \rightarrow \infty} \frac{\sum_{k=0}^{\ell-1} \mathcal{R}_k^{(\ell)}(p, \mathbf{g}; u^N)}{\sum_{n=\ell}^{\infty} \rho_n(\mathbf{g}) |u_n^N|^p} = 0. \quad (1.35)$$

Once again, the idea is to find a suitable regularization of the parameter sequence $\mathbf{g}$ so that it becomes an element of the space $\mathcal{H}_0^M$. In particular, we set $u^N := \xi^N \mathbf{g}$, where ξ is a smooth bump function determined by

$$\xi^N(x) := \begin{cases} 0 & \text{if } x \leq N, \\ \eta\left(\frac{\ln x - \ln N}{\ln N}\right) & \text{if } N < x \leq N^2, \\ 1 & \text{if } N^2 < x \leq 2N^2, \\ \eta\left(\frac{\ln(2N^3) - \ln x}{\ln N}\right) & \text{if } 2N^2 < x \leq 2N^3, \\ 0 & \text{if } x > 2N^3, \end{cases}$$

where the function η exhibits the same properties as in (1.26). Hence, indeed, $u^N \in \mathcal{H}_0^M$ for all $N \geq M$.

It is straightforward to show that inequality (1.33) still holds, i.e.

$$|M^j \xi_n^N| \lesssim 1 \quad \text{and} \quad |\operatorname{div}^m M^j \xi_n^N| \lesssim \frac{1}{n^m \ln N}$$

for all $m, n \in \mathbb{N}$, $j \in \mathbb{N}_0$, and $N \geq M$. Therefore, the same arguments as in the proof of criticality a) apply, and we obtain the estimates (1.27) even with u^N defined above. Subsequently

$$\mathcal{R}_k^{(\ell)}(p, \mathbf{g}; u^N) \lesssim \frac{1}{\ln^{\min\{p,2\}} N} \left(\sum_{n=N-k}^{N^2} \frac{1}{n} + \sum_{n=2N^2-k}^{2N^3} \frac{1}{n} \right) \lesssim \frac{1}{\ln^{\min\{p,2\}-1} N}$$

for all $k \in \{0, \dots, \ell-1\}$ and N sufficiently large, where the unspecified constant does not depend on N nor n , but may depend on p and k .

In order to finish the proof, we show that with our choice of u^N , the denominator in (1.35) is bounded from below by a positive constant. The above inequality then yields the desired limit. First,

we inspect the asymptotic behavior of the weight $\rho_n(\mathfrak{g}) = -\Delta_p^{(\ell)} \mathfrak{g}_n / \mathfrak{g}_n^{p-1}$ as $n \rightarrow \infty$. Recalling definition (1.7), the numerator can be equivalently written as

$$-\Delta_p^{(\ell)} \mathfrak{g}_n = (-1)^\ell S^{-\ell} \operatorname{div}^\ell (\operatorname{div}^\ell \mathfrak{g}_n)^{p-1}.$$

Therefore, claims (i) and (iii) of Lemma 1.13, together with the expansion (1.29), imply

$$\rho_n(\mathfrak{g}) \gtrsim \frac{n^{1/p-1-\ell}}{n^{(p-1)(\ell-1/p)}} = \frac{1}{n^{\ell p}}$$

for all n sufficiently large. Bearing (A4) in mind, we conclude that

$$\sum_{n=\ell}^{\infty} \rho_n(\mathfrak{g}) |u_n^N|^p \geq \sum_{n=N^2}^{2N^2} \rho_n(\mathfrak{g}) \mathfrak{g}_n^p \gtrsim \sum_{n=N^2}^{2N^2} \frac{n^{\ell p-1}}{n^{\ell p}} \geq \int_{N^2}^{2N^2} \frac{1}{n} \, dn = \ln 2$$

completing the proof of optimality near infinity and consequently the proof of Theorem 1.11. $\square$

Chapter 2

Explicit p -Hardy weights of any order

In this chapter, we attempt to analyze the properties of the specific parameter sequences $\mathbf{g}^{(\ell,p)}$ given by (1.12). Namely, we concentrate on verifying assumptions (A1)–(A4) in order to deduce that the weights $\rho^{(\ell,p)}$ defined by (1.13) are critical p -Hardy weights optimal near infinity by Theorem 1.11. Assumptions (A2) and (A3) will be proven only for some $\ell \in \mathbb{N}$ and $p > 1$; see Theorem 2.4. Finally, a hypothesis will be formulated regarding the general setting in Conjecture 2.5. In order not to distract the reader from the main findings of this chapter, the proofs will be worked out separately in Section 2.1 once again.

Firstly, by the defining formula (1.12) and [6, Eq. (26.8.7)], we obtain

$$\mathbf{g}_n^{(\ell,p)} = n^{1-1/p} \prod_{j=1}^{\ell-1} (n-j) = \sum_{j=0}^{\ell} s(\ell, j) n^{j-1/p} \quad (2.1)$$

for all $n \geq 0$, where $s(\ell, j)$ are the Stirling numbers of the first kind defined by the combinatorial formula

$$s(n, k) := (-1)^{n+k} \sum_{1 \leq i_1 < \dots < i_{n-k} < n} i_1 i_2 \dots i_{n-k}$$

for all $n \in \mathbb{N}$ and $k \in \{0, \dots, n-1\}$; see [6, § 26.8]. By convention, we also set $s(n, n) := 1$ for all $n \in \mathbb{N}_0$ and $s(n, k) := 0$ whenever $k < 0$. One immediately observes that $\mathbf{g}^{(\ell,p)} \in H^\ell$ satisfies the assumption (A4).

Next, we verify assumption (A1) together with a strict inequality in assumption (A2) for $k = \ell$ in the following proposition.

Proposition 2.1. *Suppose $\ell \in \mathbb{N}$ and $p > 1$. Then, we have*

$$\operatorname{div}^k \mathbf{g}_n^{(\ell,p)} > 0 \quad (2.2)$$

for all $n \geq \ell - k$ and $k \in \{0, \dots, \ell\}$.

Henceforth, we focus on the assumptions (A2) and (A3). In particular, we investigate the inequalities

$$-\Delta_p^{(\ell-k)} \operatorname{div}^k \mathbf{g}_n^{(\ell,p)} \geq 0 \quad \text{for all } n \geq \ell - k \text{ and } k \in \{0, \dots, \ell\}.$$

Notice that for $k = \ell$ a strict inequality is already a part of Proposition 2.1. Using the definition (1.7) of the discrete p -Laplacian and shifting the index n , we rewrite the latter inequality as

$$(-1)^k \operatorname{div}^k (\operatorname{div}^\ell \mathbf{g}_n^{(\ell,p)})^{p-1} \geq 0 \quad \text{for all } n \geq 0 \text{ and } k \in \{0, \dots, \ell\}, \quad (2.3)$$

which naturally brings us to the concept of *completely monotonic* functions.

Intermezzo: Completely monotonic functions

A function $f : (0, \infty) \rightarrow \mathbb{R}$ is called *completely monotonic* if and only if it is of the class C^∞ and it satisfies the *Bernstein–Hausdorff–Widder condition*

$$(-1)^k f^{(k)}(x) \geq 0 \quad \text{for all } k \geq 0 \text{ and } x > 0. \quad (2.4)$$

The set of completely monotonic functions shall be denoted as

$$\mathcal{L} := \{f : (0, \infty) \rightarrow \mathbb{R} \mid f \text{ is completely monotonic}\}. \quad (2.5)$$

The notion of completely monotonic function dates back to 1914 and S. Bernstein, who studied functions on $(-\infty, 0)$ with non-negative derivatives of all orders, i.e. functions satisfying (2.4) after omitting the alternating sign $(-1)^k$. Such functions are often called *absolutely monotonic* in literature. This, however, coincides with complete monotonicity up to a change of sign in the variable. Since then, the properties of completely monotonic function have been thoroughly studied as they arise frequently in a wide range of mathematical disciplines, such as the moment problem, probabilistic theory and mathematical analysis. Here we discuss their surprising relevance also in the context of Hardy inequalities.

The often-cited D. V. Widder’s book [27], contains a detailed account on properties of completely monotonic functions. We point out the Bernstein’s Theorem [27, Theorem 12a], which characterizes completely monotonic functions as the Laplace transform of a positive Borel measure. A function $f : (0, \infty) \rightarrow \mathbb{R}$ is completely monotonic if and only if there exists a measure μ on $[0, \infty)$, such that

$$f(x) = \int_{[0, \infty)} e^{-tx} d\mu(t) \quad \text{for all } x > 0.$$

This motivates our notation (2.5).

Moreover, the aforementioned *Hausdorff moment problem* was studied therein. Analogously to the continuous case (2.4), a sequence $\{u_n\}_{n=0}^\infty \subset \mathbb{R}$ is called *completely monotonic* if and only if

$$(-1)^k \operatorname{div}^k u_n \geq 0 \quad \text{for all } k \geq 0 \text{ and } n \geq 0.$$

We denote the set of completely monotonic sequences on $\mathbb{N}_0$ as

$$\mathcal{L}(\mathbb{N}_0) := \{u \in C(\mathbb{N}_0) \mid u \text{ is completely monotonic}\}.$$

The Hausdorff’s Theorem [27, Theorem 4a] provides a necessary and sufficient condition for a sequence u to be completely monotonic, that it admits the integral representation

$$u_n = \int_0^1 t^n d\mu(t) \quad \text{for all } n \geq 0, \quad (2.6)$$

for some finite Borel measure μ on the interval $[0, 1]$.

This formula also explains the connection between completely monotonic functions and sequences. If $f \in \mathcal{L}$ is continuous at 0, i.e. the limit of $f(x)$ as $x \rightarrow 0_+$ is finite, then $f_n := f(n)$ for all $n \geq 0$, where $f(0)$ is defined by the corresponding one-sided limit, defines a completely monotonic sequence $f \in \mathcal{L}(\mathbb{N}_0)$ by Lemma 1.14. On the other hand, if $u \in \mathcal{L}(\mathbb{N}_0)$, one easily observes that

$$u(x) := \int_0^1 t^x d\mu(t),$$

where the measure μ is given by formula (2.6), defines a completely monotonic function $u \in \mathcal{L}$ continuous at 0. Consequently, this reduces many problems concerning completely monotonic sequences to the more trackable continuous case.

Seeing (2.3), we are particularly interested in the complete monotonicity properties of the function $\operatorname{div}^\ell \mathbf{g}^{(\ell,p)}$, which will be investigated in the following proposition. Let us define the natural extension of the sequence $\mathbf{g}^{(\ell,p)}$ to all positive real numbers as

$$\mathbf{g}^{(\ell,p)}(x) := x^{1-1/p} \prod_{j=1}^{\ell-1} (x-j) \quad (2.7)$$

for all $x \geq 0$. Moreover, bearing $\operatorname{div} = \mathbf{S} - \mathbf{I}$ in mind, with a slight abuse of notation, we define the action of the operator div also on the space of real functions on $[0, \infty)$ as

$$\operatorname{div} f(x) := f(x+1) - f(x) \quad \text{for all } x \geq 0.$$

Proposition 2.2. *Suppose $\ell \in \mathbb{N}$ and $p > 1$. Then the function $\operatorname{div}^\ell \mathbf{g}^{(\ell,p)}$, where $\mathbf{g}^{(\ell,p)}$ is given by (2.7), is right-continuous at 0, with $\operatorname{div}^\ell \mathbf{g}^{(\ell,p)}(0) = 0$, and satisfies*

$$(-1)^k (\operatorname{div}^\ell \mathbf{g}^{(\ell,p)})^{(k)}(x) > 0 \quad \text{for all } k \geq 0 \text{ and } x > 0,$$

therefore it is a completely monotonic function.

The generalized Mean Value Theorem 1.14 ensures that $\operatorname{div}^\ell \mathbf{g}^{(\ell,p)} \in \mathcal{L}(\mathbb{N}_0)$ as well, thus inequalities (2.3) are fulfilled if the exponent $p-1$ is omitted, in fact

$$(-1)^k \operatorname{div}^k \operatorname{div}^\ell \mathbf{g}_n^{(\ell,p)} > 0 \quad \text{for all } n \geq 0 \text{ and } k \geq 0.$$

Unfortunately, the complete monotonicity of a function f , does not imply such property for the function f^α , where $\alpha > 0$, hence the previous proposition does not claim inequalities (2.3) for general $p > 1$. However, if we restrict the range of possible $\ell \in \mathbb{N}$ and $p > 1$, we may deduce the following result.

Proposition 2.3. *Assume that one of the following conditions is satisfied:*

- (i) $\ell \in \mathbb{N}$ and $p \geq 2$ is an integer,
- (ii) $\ell \in \{3, 4, 5, 6\}$ and $p > 2$,
- (iii) $\ell = 1$ or $\ell = 2$ and $p > 1$.

Then the sequence $\mathbf{g}^{(\ell,p)}$, given by (1.12), satisfies inequalities (2.3), i.e.

$$(-1)^k \operatorname{div}^k (\operatorname{div}^\ell \mathbf{g}_n^{(\ell,p)})^{p-1} \geq 0$$

for all $n \geq 0$ and $k \in \{0, \dots, \ell\}$.

Recalling that inequalities (2.3) are equivalent to assumptions (A2) and (A3), Propositions 2.1 and 2.3, together with Theorem 1.11, yield the following corollary.

Theorem 2.4. *The sequence $\rho^{(\ell,p)}$, given by (1.13), is a critical and optimal near infinity discrete p -Hardy weight of order ℓ if*

- (i) $\ell \in \mathbb{N}$ and $p \geq 2$ is an integer,
- (ii) $\ell \in \{3, 4, 5, 6\}$ and $p > 2$,
- (iii) $\ell = 1$ or $\ell = 2$ and $p > 1$.

But how can one validate inequalities (2.3), and therefore Theorem 2.4, for any $\ell \in \mathbb{N}$ and $p > 1$? This question appears to have no trivial answer. It might be easier to verify a stronger property of the function $\operatorname{div}^\ell \mathbf{g}^{(\ell,p)}$, instead of dealing with the inequalities in (2.3) separately. In fact, a multitude of numerical tests, with a suitable CAS such as Wolfram Mathematica, suggest that not only $\operatorname{div}^\ell \mathbf{g}^{(\ell,p)} \in \mathcal{L}$ (see Proposition 2.2), but that $\operatorname{div}^\ell \mathbf{g}^{(\ell,p)}$ is *logarithmically completely monotonic*. A function $f : (0, \infty) \rightarrow (0, \infty)$ is said to be *logarithmically completely monotonic* if and only if

$$-(\ln f)' \in \mathcal{L}.$$

Note that by Proposition 2.2, $\operatorname{div}^\ell \mathbf{g}^{(\ell,p)}$ is indeed strictly positive on $(0, \infty)$. This property would imply inequalities (2.3), since it is equivalent to $\operatorname{div}^\ell \mathbf{g}^{(\ell,p)}$ being *infinitely divisible*, i.e.

$$(\operatorname{div}^\ell \mathbf{g}^{(\ell,p)})^\alpha \in \mathcal{L} \quad \text{for any } \alpha > 0,$$

by [21, Theorem 5.9]. Consequently, we expect the following conjecture to hold true.

Conjecture 2.5. *Let $\ell \in \mathbb{N}$ and $p > 1$. The sequence $\rho^{(\ell,p)}$, given by (1.13), i.e.*

$$\rho_n^{(\ell,p)} = \frac{-\Delta_p^{(\ell)} \mathbf{g}_n^{(\ell,p)}}{\left(\mathbf{g}_n^{(\ell,p)}\right)^{p-1}}$$

for all $n \geq \ell$, where $\mathbf{g}^{(\ell,p)}$ is defined by (1.12), is an optimal discrete p -Hardy weight of order ℓ .

Remark 2.6. It is worth noting that, unlike in Theorem 2.4, we additionally hypothesized the non-attainability of $\rho^{(\ell,p)}$ here. Since $\rho^{(1,p)} = \rho^{\text{FKP}}$ defined by (6), for $\ell = 1$, Conjecture 2.5 mimics the results of [9]. Furthermore, for $p = 2$, we have $\rho^{(\ell,2)} = \rho^{(\ell)}$ given by (16), hence, in the linear setting, Conjecture 2.5 was already proven in [26, Theorem 4]. Apart from non-attainability, which we did not discuss in this project, Conjecture 2.5 was proven for the combinations of $\ell \in \mathbb{N}$ and $p > 1$ listed in Theorem 2.4.

2.1 Proofs

In this section, we provide the proofs of Propositions 2.1, 2.2 and 2.3.

2.1.1 Proof of Proposition 2.1

Notice that since $\mathbf{g}^{(\ell,p)} \in H^\ell$ by (2.1), it suffices to show inequality (2.2) for $k = \ell$. Indeed, by definition of discrete divergence (11), $\operatorname{div}^\ell \mathbf{g}^{(\ell,p)} > 0$ on $\mathbb{N}_0$ means that the sequence $\operatorname{div}^{\ell-1} \mathbf{g}^{(\ell)}$ is strictly increasing on $\mathbb{N}_0$. Since $\operatorname{div}^{\ell-1} \mathbf{g}^{(\ell)} \in H^1$, hence $\operatorname{div}^{\ell-1} \mathbf{g}_0^{(\ell)} = 0$, we conclude that $\operatorname{div}^{\ell-1} \mathbf{g}_n^{(\ell)} > 0$ for all $n \geq 1$. Iteration of this argument yields (2.2) for all $k \in \{0, \dots, \ell\}$.

In order to finish the proof, we establish the inequality

$$\operatorname{div}^\ell \mathbf{g}_n^{(\ell,p)} > 0$$

for all $n \geq 0$. The key idea is to examine the natural extension of the sequence $\mathbf{g}^{(\ell,p)}$ to all positive real numbers, defined by (2.7), and make use of Lemma 1.14, which implies that for any $n \geq 0$ there exists $\xi \in (n, n + \ell)$, such that

$$\operatorname{div}^\ell \mathbf{g}_n^{(\ell,p)} = \frac{d^\ell \mathbf{g}^{(\ell,p)}}{dx^\ell}(\xi).$$

Furthermore, bearing (2.1) in mind, we have

$$\frac{d^\ell \mathbf{g}^{(\ell,p)}}{dx^\ell}(x) = \sum_{j=1}^{\ell} s(\ell, j) \frac{d^\ell}{dx^\ell} x^{j-1/p} = \sum_{j=1}^{\ell} b_j^{(\ell,p)} x^{j-1/p-\ell}$$

for any $x > 0$, where

$$b_j^{(\ell,p)} := (-1)^{\ell+j} s(\ell, j) \prod_{k=0}^{\ell-1} \left| j - \frac{1}{p} - k \right| > 0$$

for all $j \in \{1, \dots, \ell\}$. The desired inequality follows immediately, hence the proof of Proposition 2.1 is completed. $\square$

2.1.2 Proof of Proposition 2.2

Clearly, $\operatorname{div}^\ell \mathbf{g}^{(\ell,p)}(0) = 0$, and the function is right-continuous at 0, by definition (2.7). Next, consider an arbitrary $k \in \mathbb{N}_0$. Evidently, for any $x > 0$, we have

$$(-1)^k \frac{d^k}{dx^k} \operatorname{div}^\ell \mathbf{g}^{(\ell,p)}(x) = (-1)^k \operatorname{div}^\ell \frac{d^k}{dx^k} \mathbf{g}^{(\ell,p)}(x).$$

Therefore, by Lemma 1.14, we readily infer that there exists $\xi \in (x, x + \ell)$ such that

$$(-1)^k \frac{d^k}{dx^k} \operatorname{div}^\ell \mathbf{g}^{(\ell,p)}(x) = (-1)^k \frac{d^{\ell+k} \mathbf{g}^{(\ell,p)}}{dx^{\ell+k}}(\xi).$$

Furthermore, recalling formula (2.1), we find that

$$(-1)^k \frac{d^{\ell+k} \mathbf{g}^{(\ell,p)}}{dx^{\ell+k}}(x) = (-1)^k \sum_{j=1}^{\ell} s(\ell, j) \frac{d^{\ell+k}}{dx^{\ell+k}} x^{j-1/p} = \sum_{j=1}^{\ell} c_j^{(\ell,p)} x^{j-1/p-\ell-k}$$

for any $x > 0$, where

$$c_j^{(\ell,p)} = (-1)^k (-1)^{\ell+j+k} s(\ell, j) \prod_{m=0}^{\ell+k-1} \left| j - \frac{1}{p} - m \right| > 0$$

for all $j \in \{1, \dots, \ell\}$. Altogether, we conclude that

$$(-1)^k \frac{d^k}{dx^k} \operatorname{div}^\ell \mathbf{g}^{(\ell,p)}(x) > 0 \quad \text{for all } k \geq 0 \text{ and } x > 0,$$

hence the complete monotonicity of the function $\operatorname{div}^\ell \mathbf{g}^{(\ell,p)}$. The proof of Proposition 2.2 is complete. $\square$

2.1.3 Proof of Proposition 2.3

The proof is carried out in three separate steps, each corresponding to the fulfillment of assumptions (i), (ii), and (iii), respectively.

a) *Let $\ell \in \mathbb{N}$ and $p \geq 2$ integer:* As previously noted, a function f being completely monotonic does not necessarily mean that its positive power f^α will also be completely monotonic. However, if $\alpha \in \mathbb{N}$ is an integer, the following observation may be employed.

Lemma 2.7. *For any $f, g \in \mathcal{L}$, we have $fg \in \mathcal{L}$, i.e. $\mathcal{L} \cdot \mathcal{L} \subset \mathcal{L}$*

Proof. The proof follows immediately from the Leibnitz formula

$$(-1)^k (fg)^{(k)} = \sum_{j=0}^k \binom{k}{j} (-1)^j f^{(j)} (-1)^{k-j} g^{(k-j)}$$

for any $k \geq 0$ and definition (2.4). $\square$

Finally, from Proposition 2.2 and Lemma 2.7, we infer that $(\operatorname{div}^\ell \mathbf{g}^{(\ell,p)})^{p-1} \in \mathcal{L}$. Consequently, with the aid of Lemma 1.14, we conclude

$$(-1)^k \operatorname{div}^k (\operatorname{div}^\ell \mathbf{g}_n^{(\ell,p)})^{p-1} \geq 0 \quad \text{for all } n \geq 0 \text{ and } k \geq 0.$$

In particular the inequalities hold for all $k \in \{0, \dots, \ell\}$, which completes the proof of part a).

If $p > 1$ is non-integer, the validation of inequalities (2.3) is far more complicated. Conversely to part a) of this proof, the inequalities in (2.3) will only be established for $k \in \{0, \dots, \ell\}$, for which reason we introduce the following notation; confer with [25].

Let $N \in \mathbb{N}^* := \mathbb{N} \cup \{\infty\}$, the function $f : (0, \infty) \rightarrow \mathbb{R}$ is called *completely monotonic of order N* if and only if it is of the class C^N and

$$(-1)^k f^{(k)}(x) \geq 0 \quad \text{for all } k \in \{0, \dots, N\} \text{ and } x > 0.$$

The set of completely monotonic functions of order N is denoted as $\mathcal{L}^N$, in particular, we have $\mathcal{L}^\infty = \mathcal{L}$. Furthermore, we define the set of functions, whose positive power is completely monotonic, as

$$\mathcal{L}_\alpha^N := \{f : (0, \infty) \rightarrow \mathbb{R} \mid f^\alpha \in \mathcal{L}^N\}$$

for any $\alpha > 0$ and $N \in \mathbb{N}^*$. In particular, we have $\mathcal{L}_1^N = \mathcal{L}^N$ and we abbreviate the notation if $N = \infty$ as $\mathcal{L}_\alpha := \mathcal{L}_\alpha^\infty$.

Invoking Lemma 1.14 once more, we observe that in order to show (2.3), it suffices to establish $\operatorname{div}^\ell \mathbf{g}^{(\ell,p)} \in \mathcal{L}_{p-1}^\ell$, where $\mathbf{g}^{(\ell,p)}$ is to be understood as a function on $(0, \infty)$ defined by (2.7). To this end, we will make use of the following result from [25, Theorem 4].

Lemma 2.8. *Let $0 < \alpha \leq \beta$. Then the following inclusions hold true.*

- (i) *For any $N \in \{0, \dots, 5\}$, we have $\mathcal{L}_\alpha^N \subset \mathcal{L}_\beta^N$.*
- (ii) *For any $N \in \{0, \dots, 6\}$, we have $\mathcal{L}_\alpha \subset \mathcal{L}_\beta^N$.*

Remark 2.9. In [25], the authors also show that both claims of Theorem 2.8 fail to hold for sufficiently large N , thus the following strategy cannot be extended for all $\ell \in \mathbb{N}$.

b) Let $\ell \in \{3, 4, 5, 6\}$ and $p > 2$: Let us recall that $\operatorname{div} \mathbf{g}^{(\ell,p)} \in \mathcal{L}$ by Proposition 2.2. Invoking claim (ii) of Lemma 2.8 with $\alpha = 1$ and $\beta = p - 1 \geq 1$, we observe that $\operatorname{div} \mathbf{g}^{(\ell,p)} \in \mathcal{L}_{p-1}^\ell$. The statement then stems from the generalized Mean Value Theorem 1.14.

c) Let $\ell = 1$ or $\ell = 2$ and $p > 1$: For $\ell = 1$, the inequality is obvious. In fact, one straightforwardly observes that $\mathbf{g}^{(\ell,p)}$ always complies (2.3) with $k = 1$, i.e.

$$(\operatorname{div} \mathbf{g}_n^{(\ell,p)})^{p-1} \geq (\operatorname{div} \mathbf{g}_{n+1}^{(\ell,p)})^{p-1},$$

by Proposition 2.2 and Lemma 1.14. For $\ell = 2$, we shall make use of the following auxiliary statement. See [17, Theorem 7] for the proof.

Lemma 2.10. *Let $N \geq 1$. For any $f \in \mathcal{L}^{N+1}$, the function $f^{1/N}$ is convex.*

Let us denote $f := \operatorname{div}^2 \mathbf{g}^{(2,p)}$. By Proposition 2.2 and Lemma 2.10, we have

$$\left(f^{1/N}(x)\right)' = \frac{1}{N} f^{1/N-1}(x) f'(x) < 0 \quad \text{and} \quad \left(f^{1/N}(x)\right)'' \geq 0$$

for all $N \geq 1$ and $x > 0$, therefore $f \in \mathcal{L}_{1/N}^2$. Employing also claim (i) of Lemma 2.8, we conclude that $f \in \mathcal{L}_\alpha^2$ for any $\alpha > 0$, hence, in particular, $f \in \mathcal{L}_{p-1}^2$. Quick application of Lemma 1.14 reveals the desired result once again. The proof of Proposition 2.3 is complete. $\square$

Chapter 3

Properties of p -Hardy weights

This chapter, which closely follows the results of [28], concentrates on the properties of the optimal p -Hardy weight $\rho^{(p)} := \rho^{(1,p)}$, given by the general definition (16). Let us recall the formula

$$\rho_n^{(p)} = \left(1 - \left(1 - \frac{1}{n}\right)^{1/q}\right)^{p-1} - \left(\left(1 + \frac{1}{n}\right)^{1/q} - 1\right)^{p-1} \quad (3.1)$$

for any $n \geq 1$, where $q > 1$ is the Hölder conjugate index to p , i.e. $1/p + 1/q = 1$.

As previously stated, the optimality of $\rho^{(p)}$ was established by Fisher, Keller, and Pogorzelski in [9]; see also Chapters 1 and 2 for alternative proof excluding non-attainability. The proof of the p -Hardy inequality (5) presented in [9] relies on methods developed in greater generality for graphs in [8]. A substantial part of [9] is devoted to demonstrating that the weight (3.1) improves upon the classical p -Hardy weight, i.e. to the non-trivial inequality

$$\rho_n^{(p)} > \left(\frac{p-1}{p}\right)^p \frac{1}{n^p} \quad (3.2)$$

for any $n \geq 1$ and $p > 1$.

With the additional assumption, that $p \geq 2$ is an integer, it was further asserted that $\rho^{(p)}$ admits the asymptotic expansion

$$\rho_n^{(p)} = \sum_{k=0}^{\infty} \frac{A_{2k}^{(p)}}{n^{2k+p}} \quad (3.3)$$

for all $n \geq 1$, where all the coefficients $A_{2k}^{(p)}$ are strictly positive. Since $A_0^{(p)} = ((p-1)/p)^p$, the first term coincides with the classical p -Hardy weight, hence inequality (3.2) readily follows. However, it remained an open problem whether (3.3) holds true even for non-integer $p > 1$. In fact, the authors of [9] conjectured that the function $x \mapsto \rho^{(p)}(1/x)$, where

$$\rho^{(p)}(x) := \left(1 - \left(1 - \frac{1}{x}\right)^{1/q}\right)^{p-1} - \left(\left(1 + \frac{1}{x}\right)^{1/q} - 1\right)^{p-1} \quad (3.4)$$

for all $x \in (0, 1)$, is *absolutely monotonic*, i.e. that the derivatives of all orders are positive on $(0, 1)$. Note that this property clearly implies non-negativity of the coefficients in (3.3). Even though this is only true if $p \in \mathbb{N}$, which can be seen from the expansion (3.11), we will prove that the conjecture holds for every $p > 1$ if the function $x \mapsto x^{-p} \rho^{(p)}(1/x)$ is considered instead; see Theorem 3.3 below.

It seems that the properties of complete, resp. absolute, monotonicity emerge, somewhat unexpectedly, as a leitmotif in the context of Hardy inequalities; see also Chapter 2 and Proposition 2.2. Here, we establish the absolute monotonicity of $x \mapsto x^{-p} \rho^{(p)}(1/x)$ exploiting the tools of complex

analysis. Namely, in Proposition 3.2, we show that the function f_p , given by formula (3.6) below, is a *Herglotz–Nevanlinna* function. In analogy to Chapter 2, we include a brief intermezzo to introduce the concept of Herglotz–Nevanlinna functions.

Intermezzo: Herglotz–Nevanlinna functions

Let us denote the upper and lower half-plane by

$$\mathbb{C}_{\pm} := \{z \in \mathbb{C} \mid \pm \operatorname{Im} z > 0\}.$$

A function $f : \mathbb{C}_+ \rightarrow \mathbb{C}$ is called *Herglotz–Nevanlinna* if and only if f is analytic on $\mathbb{C}_+$ and it preserves the upper half-plane, i.e. $\operatorname{Im} f(z) \geq 0$ for all $z \in \mathbb{C}_+$. One of the earliest appearances of Herglotz–Nevanlinna functions traces back to the beginning of the twentieth century, when Pick and Nevanlinna, independently, studied this well-established class of functions in the context of interpolation problem in the complex plane and also the *Stieltjes moment problem*. For this reason these functions are frequently called *Nevanlinna* or *Nevanlinna–Pick* functions in literature.

Ever since, Herglotz–Nevanlinna functions have been used effectively across a wide range of mathematical disciplines both in pure mathematics and in applications. Among these are the spectral theory of Jacobi and Schrödinger operators [23, 24], moment problem and orthogonal polynomials [1], theory of operator monotonic functions [22], canonical systems [20], de Branges’ functional model spaces [5], or theory of self-adjoint extensions [2]. See also [21, Chapter 6], where Herglotz–Nevanlinna functions, continuous up to $(0, \infty)$, known as the *complete Bernstein functions*, were studied in the context of complete monotonicity.

At first sight, the definition may not seem very restrictive. Nevertheless, it has strong implications, which make Herglotz–Nevanlinna functions such a potent tool in mathematics. In particular, Nevanlinna observed that these functions can be described via an integral representation involving only three parameters: two numbers and a positive Borel measure; see Theorem 3.1. In fact, Nevanlinna extended an analogous result due to Herglotz, who studied representation of analytic functions in the unit disk with a positive real part, now commonly referred to as *Herglotz functions*. This historical connection is one of the reasons why this class of functions bears the names of both mathematicians.

In the following theorem, we summarize selected properties of Herglotz–Nevanlinna functions that will be important for our purposes.

Theorem 3.1. *Let $f : \mathbb{C}_+ \rightarrow \mathbb{C}$ be a Herglotz–Nevanlinna function. Then the following claims hold true:*

- (i) *There exist unique constants $a \in \mathbb{R}$, $b \geq 0$, and a unique Borel measure μ on $\mathbb{R}$ satisfying*

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$$

such that

$$f(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t) \quad \text{for all } z \in \mathbb{C}_+. \quad (3.5)$$

- (ii) *If the limit of $\operatorname{Im} f(x + i\varepsilon)$ for $\varepsilon \rightarrow 0+$ exists and is finite for all $x \in \mathbb{R}$, then the measure μ is purely absolutely continuous and its density reads*

$$\frac{d\mu}{dx}(x) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0+} \operatorname{Im} f(x + i\varepsilon) \quad \text{for all } x \in \mathbb{R}.$$

(iii) If μ is a compactly supported finite measure, then

$$f(z) = bz + a - \int_{\mathbb{R}} \frac{t}{1+t^2} d\mu(t) - \frac{\mu(\mathbb{R})}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \quad \text{as } z \rightarrow \infty.$$

These results are standard and the proofs of claims (i) and (ii) can be found for instance in [24, §3.4]. Note that, in general, the integral in the formula (3.5) cannot be split into the sum of two integrals

$$\int_{\mathbb{R}} \frac{d\mu(t)}{t-z} - \int_{\mathbb{R}} \frac{t}{1+t^2} d\mu(t),$$

since both integrals may diverge. If, however, further restrictions are imposed on the measure μ as in claim (iii), the convergence of both integrals can be ensured. Claim (iii) then straightforwardly follows from the expansion of the first integral in negative powers of z as $z \rightarrow \infty$.

Representation (3.5) allows us to extend the function to the lower half-plane $\mathbb{C}_-$, because the integral is well defined for all $z \in \mathbb{C} \setminus \mathbb{R}$. This extension evidently enjoys a symmetry with respect to the Schwarz reflection $f(\bar{z}) = \overline{f(z)}$, hence it is sometimes referred to as *symmetric* or *integral extension* of the Herglotz–Nevanlinna function f . Hereafter, a Herglotz–Nevanlinna functions f is to be understood as $f : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ analytic on $\mathbb{C} \setminus \mathbb{R}$ such that

$$f(\bar{z}) = \overline{f(z)} \quad \text{and} \quad \text{Im } f(z) \geq 0 \quad \text{for all } z \in \mathbb{C}_+.$$

Considering $\rho^{(p)}$, determined by the expression (3.4), as a function of complex variable, we define function $f_p(z) := -z^{p-1}\rho^{(p)}(z)$, i.e.

$$f_p(z) = z^{p-1} \left[\left(\left(1 + \frac{1}{z}\right)^{1/q} - 1 \right)^{p-1} - \left(1 - \left(1 - \frac{1}{z}\right)^{1/q} \right)^{p-1} \right] \quad (3.6)$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$ and $p > 1$, where $1/p + 1/q = 1$. Here and below, the positive power of a complex argument assumes its principle branch, i.e. for $\alpha > 0$, we always make use of the definition

$$z^\alpha := |z|^\alpha e^{i\alpha \text{Arg } z} \quad (3.7)$$

with $\text{Arg } z \in (-\pi, \pi]$. It turns out that f_p extends analytically to $\mathbb{C} \setminus [-1, 1]$, with a branch cut on $[-1, 1]$; see Lemma 3.10.

Furthermore, for $x \in [0, 1]$, we set

$$\omega_p(x) := -\frac{1}{\pi} x^{p-1} \text{Im} \left(1 - e^{i\pi/q} \left(\frac{1}{x} - 1 \right)^{1/q} \right)^{p-1}. \quad (3.8)$$

The value of ω_p at 0 is to be understood as the limit of $\omega_p(x)$ as $x \rightarrow 0_+$. In Lemma 3.11, we show that $\omega_p(0) = 0$ for any $p > 1$. The function ω_p turns out to be the density of a measure whose even moments coincide with the coefficients from the expansion (3.3); see Theorem 3.3 below.

Proposition 3.2. *Let $p > 1$. The function f_p defined by (3.6) is Herglotz–Nevanlinna. Moreover, the density ω_p defined by (3.8) is strictly positive on $(0, 1)$ and the integral representation*

$$f_p(z) = \int_{-1}^1 \frac{\omega_p(|t|)}{t-z} dt \quad (3.9)$$

holds for all $z \in \mathbb{C} \setminus [-1, 1]$.

Recall that a real-valued function g is called *absolutely monotonic* in $(0, 1)$, if it is infinitely-differentiable on $(0, 1)$ and satisfies

$$g^{(k)}(x) \geq 0 \quad \text{for all } k \geq 0 \text{ and } x \in (0, 1).$$

The integral representation (3.9) implies a property of $\rho^{(p)}$ that proves a corrected form of the conjecture from [9]:

Theorem 3.3. *For any $p > 1$, the function*

$$x \mapsto \frac{1}{x^p} \rho^{(p)}\left(\frac{1}{x}\right) \quad (3.10)$$

is absolutely monotonic on $(0, 1)$ and continuous on $[0, 1]$. Moreover, for all $x \in (0, 1]$, we have the convergent expansion

$$\frac{1}{x^p} \rho^{(p)}\left(\frac{1}{x}\right) = \sum_{k=0}^{\infty} A_{2k}^{(p)} x^{2k} \quad (3.11)$$

with the coefficients given by the formula

$$A_{2k}^{(p)} := \int_{-1}^1 t^{2k} \omega_p(|t|) dt > 0$$

for all $k \in \mathbb{N}_0$.

Remark 3.4. In fact, the derivatives of (3.10) of all orders are strictly positive in $(0, 1)$.

As an immediate consequence of Theorem 3.3, we get the expansion (3.3) for all $p > 1$. Lastly, we derive an explicit but combinatorially complicated formula for the coefficients $A_{2k}^{(p)}$ for all $k \geq 0$ and $p > 1$.

Proposition 3.5. *For any $p > 1$ and $k \in \mathbb{N}_0$, we have*

$$A_{2k}^{(p)} = 2 \left(\frac{p-1}{p}\right)^{p-1} \sum_{n=1}^{2k+1} \binom{p-1}{n} \sum_{\substack{r_1, \dots, r_n \geq 1 \\ r_1 + \dots + r_n = 2k+1}} \frac{(1/p)_{r_1}}{(r_1+1)!} \cdots \frac{(1/p)_{r_n}}{(r_n+1)!}, \quad (3.12)$$

where $(\nu)_n$ is the Pochhammer symbol (9).

Remark 3.6. If $p > 1$ is an integer, formula (3.12) simplifies to

$$A_{2k}^{(p)} = 2 \sum_{j=0}^{p-1} (-1)^{j+p} \binom{p-1}{j} \binom{j(p-1)/p}{2k+p},$$

which is a consequence of the expansion (3.18) below.

Remark 3.7. A straightforward computation based on (3.12) gives the first three even moments:

$$\begin{aligned} A_0^{(p)} &= \left(\frac{p-1}{p}\right)^p, \\ A_2^{(p)} &= \left(\frac{p-1}{p}\right)^p \frac{3p-1}{8p}, \\ A_4^{(p)} &= \left(\frac{p-1}{p}\right)^p \frac{(5p-1)(43p^2+p-6)}{1152p^3}. \end{aligned}$$

To conclude this chapter, we formulate a conjecture about the asymptotic expansion of the p -Hardy weights of an arbitrary order, introduced in Theorem 2.4, resp. in Conjecture 2.5. This hypothesis extends the results of Theorem 3.3 from the case $\ell = 1$ to all $\ell \in \mathbb{N}$.

Conjecture 3.8. *Let $\rho^{(\ell,p)}$ be defined by formulas (1.12) and (1.13), then the following properties hold true.*

(i) *The weight sequence $\rho^{(\ell,p)}$ admits a convergent series expansion*

$$\rho_n^{(\ell,p)} = \sum_{k=0}^{\infty} \frac{A_k^{(\ell,p)}}{n^{\ell p+k}} \quad \text{for all } n \geq \ell. \quad (3.13)$$

(ii) *For all $\ell \geq 2$ and $k \geq 0$, we have $A_k^{(\ell,p)} > 0$.*

For $\ell = 1$ and $k \geq 0$, we have $A_{2k+1}^{(1,p)} = 0$ and $A_{2k}^{(1,p)} > 0$.

(iii) *For all $n \geq \ell$, it is true that*

$$\rho_n^{(\ell,p)} > \left(\frac{p-1}{p} \right)_{\ell}^p \frac{1}{n^{\ell p}}.$$

It is straightforward to show that the leading term in the asymptotic expansion (3.13) is

$$\left(\frac{p-1}{p} \right)_{\ell}^p,$$

therefore claim (iii) is an immediate consequence of claim (i). If the latter inequality is to be confirmed (together with Conjecture 2.5), we could infer an extraordinary result, that the *classical discrete p -Hardy inequality of order ℓ* (or *classical p -Birman inequality*)

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left| (-\Delta)^{\ell/2} u_n \right|^p \geq \left(\frac{p-1}{p} \right)_{\ell}^p \sum_{n=\ell}^{\infty} \frac{|u_n|^p}{n^{\ell p}} \quad (3.14)$$

holds for all $u \in \mathcal{H}_0^{\ell}$.

3.1 Proofs

3.1.1 A warm up: the case of integer p

In the context of the discrete p -Hardy inequalities, key properties of the function f_p are the integral formula (3.9), positivity of ω_p on $(0, 1)$, and finite (actually vanishing) side limits $\omega_p(0) = \omega_p(1) = 0$, since they directly lead to the expansion (3.3) with positive coefficients. Notably, when $p > 1$ is an integer, these properties can be established using elementary methods, bypassing the need for Herglotz–Nevanlinna theory. As an exercise and an alternative demonstration of the results from [9], we offer a concise direct proof.

Proof (of formula (3.9) for integer p). Assuming $p \in \mathbb{N}$, $p > 1$, we first check the positivity of ω_p on $(0, 1)$. Using the binomial formula, we may rewrite the definition (3.8) as

$$\omega_p(x) = -\frac{1}{\pi} \sum_{j=0}^{p-1} \binom{p-1}{j} (-1)^j \sin\left(\frac{j\pi}{q}\right) (1-x)^{j/q} x^{p-1-j/q}. \quad (3.15)$$

From here, it is immediate that $\omega_p(x) > 0$ for all $x \in (0, 1)$ since $(-1)^j \sin(j\pi/q) < 0$ for all $1 \leq j \leq p-1$. Moreover, the side-limit values $\omega_p(0) = \omega_p(1) = 0$ also readily follow.

While it is obvious that the integral in (3.9) is analytic in $\mathbb{C} \setminus [-1, 1]$, one can verify the same for f_p by inspection of the defining expression (3.6); this is shown in Lemma 3.10 below in detail. Therefore it suffices to verify (3.9) for $z \in \mathbb{C}$ in a neighborhood of ∞ . Equation (3.9) rewrites equivalently as

$$-\frac{1}{z}f_p\left(\frac{1}{z}\right) = \int_{-1}^1 \frac{\omega_p(|t|)}{1-tz} dt = 2 \int_0^1 \frac{\omega_p(t)}{1-t^2z^2} dt. \quad (3.16)$$

We prove (3.16) by expanding both sides into a power series in z , assuming $|z| < 1$, and showing coincidence of the corresponding coefficients.

a) *Expansion of the left-hand side of (3.16)*: By applying the (generalized) binomial theorem in (3.6) twice, we find

$$\begin{aligned} -\frac{1}{z}f_p\left(\frac{1}{z}\right) &= \frac{1}{z^p} \sum_{j=0}^{p-1} \binom{p-1}{j} (-1)^j \left((1-z)^{j/q} - (-1)^{p-1} (1+z)^{j/q} \right) \\ &= \frac{1}{z^p} \sum_{j=0}^{p-1} \binom{p-1}{j} (-1)^j \sum_{k=0}^{\infty} \binom{j/q}{k} \left((-z)^k - (-1)^{p-1} z^k \right) \end{aligned}$$

for all $z \in \mathbb{C}$ with $|z| < 1$. Thus, we arrive at the formula

$$-\frac{1}{z}f_p\left(\frac{1}{z}\right) = \begin{cases} 2 \sum_{n=0}^{\infty} z^{2n-p} \sum_{j=0}^{p-1} (-1)^j \binom{p-1}{j} \binom{j/q}{2n} & \text{for } p \in 2\mathbb{N}, \\ 2 \sum_{n=0}^{\infty} z^{2n+1-p} \sum_{j=0}^{p-1} (-1)^{j+1} \binom{p-1}{j} \binom{j/q}{2n+1} & \text{for } p \in 2\mathbb{N}-1. \end{cases} \quad (3.17)$$

Notice that for any $k \in \mathbb{N}$, $\alpha > 0$, and $m \in \{0, 1, \dots, k-1\}$, we have

$$\sum_{j=0}^k \binom{k}{j} \binom{\alpha j}{m} (-1)^j = 0,$$

which is a consequence of the identity

$$\sum_{j=0}^k \binom{k}{j} j^m (-1)^j = 0,$$

which holds true for any $m \in \{0, 1, \dots, k-1\}$ by [11, Eq. 0.154.3]. As a result, the terms in (3.17) corresponding to $n = 0, 1, \dots, \lfloor (p-2)/2 \rfloor$ vanish. Using this observation and shifting the index n , we obtain the expansion

$$-\frac{1}{z}f_p\left(\frac{1}{z}\right) = 2 \sum_{n=0}^{\infty} z^{2n} \sum_{j=0}^{p-1} (-1)^{j+p} \binom{p-1}{j} \binom{j/q}{2n+p} \quad (3.18)$$

for any $z \in \mathbb{C}$ with $|z| < 1$.

b) *Expansion of the right-hand side of (3.16)*: For $z \in \mathbb{C}$ with $|z| < 1$, we have

$$\int_0^1 \frac{\omega_p(t)}{1-t^2z^2} dt = \sum_{n=0}^{\infty} z^{2n} \int_0^1 t^{2n} \omega_p(t) dt.$$

Moreover, bearing (3.15) in mind, we obtain

$$\int_0^1 \frac{\omega_p(t)}{1-t^2 z^2} dt = \sum_{n=0}^{\infty} z^{2n} \left(\frac{1}{\pi} \sum_{j=0}^{p-1} \binom{p-1}{j} (-1)^{j+1} \sin\left(\frac{j\pi}{q}\right) \int_0^1 (1-t)^{j/q} t^{p-1-j/q+2n} dt \right).$$

The latter integral can be computed using the Euler beta and gamma functions as

$$\int_0^1 (1-t)^{j/q} t^{p-1-j/q+2n} dt = B\left(2n+p-\frac{j}{q}, 1+\frac{j}{q}\right) = \frac{\Gamma\left(2n+p-\frac{j}{q}\right) \Gamma\left(1+\frac{j}{q}\right)}{\Gamma(2n+p+1)},$$

which further simplifies to

$$\int_0^1 (1-t)^{j/q} t^{p-1-j/q+2n} dt = (-1)^{p+1} \binom{j/q}{2n+p} \frac{\pi}{\sin(j\pi/q)}$$

by employing the identity $\Gamma(x+1) = x\Gamma(x)$ and Euler's reflection formula. Altogether, we deduce for the right-hand side of (3.16) the same expansion as in (3.18). The proof of (3.9) for integer $p > 1$ follows. $\square$

Remark 3.9. The crucial simplifications inherent in the preceding method are not applicable for non-integer values of $p > 1$; cf. Proposition 3.5. This discrepancy suggests that establishing (3.9) for non-integer $p > 1$ requires a different strategy.

3.1.2 Proof of Proposition 3.2

Take an arbitrary $p > 1$. First, we deduce preliminary properties of the function f_p defined by (3.6).

Lemma 3.10. *Function f_p extends analytically to the cut-plane $\mathbb{C} \setminus [-1, 1]$, satisfies the symmetry relations*

$$\overline{f_p(z)} = f_p(\bar{z}) \quad \text{and} \quad f_p(-z) = -f_p(z) \quad (3.19)$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$, and admits the asymptotic expansion

$$f_p(z) = -\left(\frac{p-1}{p}\right)^p \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \quad \text{as } z \rightarrow \infty. \quad (3.20)$$

Proof. First, we verify the symmetry formulas (3.19). Since the principal branch of the power function (3.7) enjoys the symmetry $\bar{z}^\alpha = \overline{z^\alpha}$ for all $z \in \mathbb{C} \setminus (-\infty, 0]$ the first symmetry formula from (3.19) for f_p follows readily from its definition (3.6). To prove the second symmetry from (3.19), observe that, for any $\alpha > 0$, we have

$$(-z)^\alpha = e^{\mp i\alpha\pi} z^\alpha \quad \text{if } z \in \mathbb{C}_\pm.$$

It suffices to take $z \in \mathbb{C}_+$, for which we have

$$\pm 1 \mp \left(1 \mp \frac{1}{z}\right)^{1/q} \in \mathbb{C}_-; \quad (3.21)$$

here it is important that $q > 1$. Now, we find

$$\begin{aligned} f_p(-z) &= (-z)^{p-1} \left[\left(\left(1 - \frac{1}{z}\right)^{1/q} - 1 \right)^{p-1} - \left(1 - \left(1 + \frac{1}{z}\right)^{1/q} \right)^{p-1} \right] \\ &= e^{-i(p-1)\pi} z^{p-1} \left[e^{i(p-1)\pi} \left(1 - \left(1 - \frac{1}{z}\right)^{1/q} \right)^{p-1} - e^{i(p-1)\pi} \left(\left(1 + \frac{1}{z}\right)^{1/q} - 1 \right)^{p-1} \right] \\ &= -f_p(z) \end{aligned}$$

for any $z \in \mathbb{C}_+$, which implies that $f_p(-z) = -f_p(z)$ for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Second, we prove the analyticity of f_p on $\mathbb{C} \setminus [-1, 1]$. Since the function $z \mapsto z^\alpha$ defined by (3.7) is analytic in $\mathbb{C} \setminus (-\infty, 0]$, the analyticity of f_p in $\mathbb{C} \setminus \mathbb{R}$ is immediate from its definition (3.6). To show that f_p is analytic on $\mathbb{C} \setminus [-1, 1]$, it suffices to show that f_p extends continuously to $\mathbb{R} \setminus [-1, 1]$. For any $x > 1$, one computes, by inspection of the expression (3.6), that

$$\lim_{\substack{z \rightarrow x \\ z \in \mathbb{C} \setminus \mathbb{R}}} f_p(z) = x^{p-1} \left[\left(\left(1 + \frac{1}{x} \right)^{1/q} - 1 \right)^{p-1} - \left(1 - \left(1 - \frac{1}{x} \right)^{1/q} \right)^{p-1} \right]. \quad (3.22)$$

The continuity of f_p on $(1, \infty)$ follows. The continuity of f_p on $(-\infty, -1)$ is a consequence of the already proven formulas (3.19).

Lastly, we derive the asymptotic expansion (3.20). Due to symmetries (3.19), we may confine z to $\mathbb{C}_+ \cup (1, \infty)$. Since $(zw)^\alpha = z^\alpha w^\alpha$ for any $z \in \mathbb{C}_+ \cup (1, \infty)$ and $w \in \mathbb{C}_- \cup (1, \infty)$, we may rewrite (3.6), bearing (3.21) in mind, as

$$f_p(z) = \left(z \left(1 + \frac{1}{z} \right)^{1/q} - z \right)^{p-1} - \left(z - z \left(1 - \frac{1}{z} \right)^{1/q} \right)^{p-1} \quad (3.23)$$

for all $z \in \mathbb{C}_+ \cup (1, \infty)$. Now, using the generalized binomial theorem twice, one readily expands

$$\begin{aligned} f_p(z) &= \frac{1}{q^{p-1}} \left[\left(1 + \left(\frac{1}{q} - 1 \right) \frac{1}{2z} + \mathcal{O}\left(\frac{1}{z^2}\right) \right)^{p-1} - \left(1 - \left(\frac{1}{q} - 1 \right) \frac{1}{2z} + \mathcal{O}\left(\frac{1}{z^2}\right) \right)^{p-1} \right] \\ &= \frac{1}{q^{p-1}} \left[\left(\frac{1}{q} - 1 \right) \frac{p-1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \right] \end{aligned}$$

as $z \rightarrow \infty$. Recalling that $1/p + 1/q = 1$, the last formula simplifies to (3.20). $\square$

In the second lemma, we explore the continuation of f_p onto the interval $[-1, 1]$ from the upper half-plane $\mathbb{C}_+$.

Lemma 3.11. *The function f_p extends continuously to the interval $[-1, 1]$ from $\mathbb{C}_+$. Moreover, for $x \in \mathbb{R}$, we have*

$$\lim_{\substack{z \rightarrow x \\ z \in \mathbb{C}_+}} \operatorname{Im} f_p(z) = \begin{cases} \pi \omega_p(|x|), & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases} \quad (3.24)$$

where ω_p is defined by (3.8); here the value for $x = 0$ is given by the respective limit $\omega_p(0) = 0$.

Proof. By (3.19), it suffices to consider $x \geq 0$. For $x \in (0, 1]$, one computes from (3.6) and definition (3.7) that

$$\lim_{\substack{z \rightarrow x \\ z \in \mathbb{C}_+}} f_p(z) = x^{p-1} \left[\left(\left(1 + \frac{1}{x} \right)^{1/q} - 1 \right)^{p-1} - \left(1 - e^{i\pi/q} \left(\frac{1}{x} - 1 \right)^{1/q} \right)^{p-1} \right]. \quad (3.25)$$

For $x = 0$, the limit is 0, as one can see from the expression (3.23), where both summands vanish for $z \rightarrow 0$ since

$$z \left(1 \pm \frac{1}{z} \right)^{1/q} = \mathcal{O}\left(z^{1/p}\right) \quad \text{as } z \rightarrow 0.$$

Consequently, f_p extends continuously to the interval $[-1, 1]$ from $\mathbb{C}_+$.

Since the first term on the right-hand side of (3.25) is real, we have

$$\lim_{\substack{z \rightarrow x \\ z \in \mathbb{C}_+}} \operatorname{Im} f_p(z) = -x^{p-1} \operatorname{Im} \left(1 - e^{i\pi/q} \left(\frac{1}{x} - 1 \right)^{1/q} \right)^{p-1},$$

which coincides with $\pi\omega_p(x)$ for all $x \in (0, 1]$; see definition (3.8). If $x > 1$, we already know from (3.22) that the limit of $f_p(z)$, for $z \rightarrow x$, exists and is real. This proves identity (3.24). $\square$

Lastly, we show that the density ω_p is strictly positive on the open interval $(0, 1)$.

Lemma 3.12. *For all $x \in (0, 1)$, $\omega_p(x) > 0$.*

Proof. By definition (3.8), it suffices to verify that

$$\operatorname{Im} \left(1 - e^{i\pi/q} \left(\frac{1}{x} - 1 \right)^{1/q} \right)^{p-1} < 0$$

for all $x \in (0, 1)$. Denoting $t := (1/x - 1)^{1/q}$ and using (3.7), this inequality is true if

$$\sin \left((p-1) \operatorname{Arg} \left(1 - e^{i\pi/q} t \right) \right) < 0 \quad (3.26)$$

for all $t > 0$. We verify (3.26) separately for two cases when $p \in (1, 2)$ and $p \geq 2$.

Suppose first that $p \geq 2$. Then $\pi/2 \leq \pi/q < \pi$, $\operatorname{Re}(1 - e^{i\pi/q} t) > 0$, and we find

$$0 > \operatorname{Arg} \left(1 - e^{i\pi/q} t \right) = \arctan \left(\frac{-t \sin(\pi/q)}{1 - t \cos(\pi/q)} \right) > \arctan \left(\tan \left(\frac{\pi}{q} \right) \right) = \frac{\pi}{q} - \pi = -\frac{\pi}{p}.$$

Therefore the argument of the sine in (3.26) satisfies

$$0 > (p-1) \operatorname{Arg} \left(1 - e^{i\pi/q} t \right) > \frac{1-p}{p} \pi = -\frac{\pi}{q} > -\pi,$$

hence inequality (3.26) holds true.

Second, suppose $p \in (1, 2)$. Then $0 < \pi/q < \pi/2$ and $\operatorname{Re}(1 - e^{i\pi/q} t) = 1 - t \cos(\pi/q)$ changes sign as function of $t > 0$. Let us denote by t_0 the unique positive point for which $\operatorname{Re}(1 - e^{i\pi/q} t_0) = 0$. If $t < t_0$, then $\operatorname{Re}(1 - e^{i\pi/q} t) > 0$, and we have

$$-\frac{\pi}{2} < \operatorname{Arg} \left(1 - e^{i\pi/q} t \right) = \arctan \left(\frac{-t \sin(\pi/q)}{1 - t \cos(\pi/q)} \right) < 0,$$

because the argument of the inverse tangent is negative. If $t > t_0$, then $\operatorname{Re}(1 - e^{i\pi/q} t) < 0$, and we have

$$-\frac{\pi}{2} > \operatorname{Arg} \left(1 - e^{i\pi/q} t \right) = -\pi + \arctan \left(\frac{-t \sin(\pi/q)}{1 - t \cos(\pi/q)} \right) > -\pi,$$

because the argument of the inverse tangent is positive. If $t = t_0$, we infer that $\operatorname{Arg}(1 - e^{i\pi/q} t) = -\pi/2$ because $\operatorname{Im} \operatorname{Arg}(1 - e^{i\pi/q} t) = -t \sin(\pi/q) < 0$. In total, we see that

$$-\pi < -(p-1)\pi < (p-1) \operatorname{Arg} \left(1 - e^{i\pi/q} t \right) < 0,$$

where the assumption $p \in (1, 2)$ was used. Therefore the inequality (3.26) remains true. $\square$

The final auxiliary claim reduces the verification of the inequality $\operatorname{Im} f_p(z) \geq 0$ for all $z \in \mathbb{C}_+$ to checking it along the real line. It can be straightforwardly deduced from the Phragmén–Lindelöf principle [15, Sec. III.C].

Theorem 3.13. *Let f be a bounded analytic function on $\mathbb{C}_+$ continuous up to the real line $\mathbb{R}$. If $\operatorname{Im} f(x) \geq 0$ for all $x \in \mathbb{R}$, then $\operatorname{Im} f(z) \geq 0$ for all $z \in \mathbb{C}_+$.*

Proof (of Proposition 3.2). First, we prove that f_p is Herglotz–Nevanlinna. By Lemma 3.10, function f_p is analytic on $\mathbb{C} \setminus \mathbb{R}$. Therefore f_p is Herglotz–Nevanlinna if $\operatorname{Im} f_p(z) \geq 0$ for all $z \in \mathbb{C}_+$. According to Lemmas 3.10 and 3.11, f_p extends continuously to $\mathbb{R}$ from $\mathbb{C}_+$ and is bounded on $\mathbb{C}_+$, which stems from the asymptotic expansion (3.20). Thus, utilizing Theorem 3.13, it suffices to show that the limit values of $\operatorname{Im} f_p(z)$, as $z \rightarrow x$ from $\mathbb{C}_+$, are nonnegative for all $x \in \mathbb{R}$. This is justified by Lemmas 3.11 and 3.12.

As the positivity of ω_p in $(0, 1)$ is established by Lemma 3.12, it remains to prove the integral representation (3.9). Since f_p is Herglotz–Nevanlinna Theorem 3.1 applies, providing us with numbers $a_p \in \mathbb{R}$, $b_p \geq 0$, and measure μ_p such that the integral representation of claim (i) holds. By claim (ii) of Theorem 3.1 and Lemma 3.11, the measure μ_p is supported in $[-1, 1]$, purely absolutely continuous, and its density reads

$$\frac{d\mu_p}{dx}(x) = \omega_p(|x|)$$

for all $x \in (-1, 1)$. As the density ω_p is continuous on $[0, 1]$, μ_p is a finite measure. Lastly, the claim (iii) of Theorem 3.1 combined with the expansion (3.20) implies

$$b_p = 0, \quad a_p = \int_{-1}^1 \frac{t}{1+t^2} d\mu_p(t), \quad \text{and moreover} \quad \mu_p(\mathbb{R}) = \left(\frac{p-1}{p}\right)^p.$$

In total, we arrive at the integral representation of the form (3.9), as claimed. The proof of Proposition 3.2 is complete. $\square$

3.1.3 Proof of Theorem 3.3

One infers from (3.9) the integral formula

$$\frac{1}{x^p} \rho^{(p)}\left(\frac{1}{x}\right) = -\frac{1}{x} f_p\left(\frac{1}{x}\right) = \int_{-1}^1 \frac{\omega_p(|t|)}{1-tx} dt \quad (3.27)$$

for all $x \in (-1, 1)$. This is an infinitely-differentiable function of x on $(-1, 1)$ and, by the dominated convergence, its k -th derivative reads

$$\frac{d^k}{dx^k} \left[\frac{1}{x^p} \rho^{(p)}\left(\frac{1}{x}\right) \right] = k! \int_{-1}^1 \frac{t^k \omega_p(|t|)}{(1-tx)^{k+1}} dt = k! \int_0^1 \left(\frac{1}{(1-tx)^{k+1}} + \frac{(-1)^k}{(1+tx)^{k+1}} \right) t^k \omega_p(t) dt$$

for all $x \in (-1, 1)$. Since the expression in the round brackets in the last integral is positive for all $t, x \in (0, 1)$ and $k \geq 0$ the function in (3.27) is absolutely monotonic on $(0, 1)$. The continuity on the closed interval $[0, 1]$ also follows immediately from (3.27).

Expanding the integrand on the right-hand side of (3.27) into the power series in the variable xt and interchanging the sum and the integral yields

$$\frac{1}{x^p} \rho^{(p)}\left(\frac{1}{x}\right) = \sum_{k=0}^{\infty} x^k \int_{-1}^1 t^k \omega_p(|t|) dt,$$

where the series on the right converges for all $x \in (-1, 1)$. Since the last integral vanishes for k odd, we arrive at the claimed expansion (3.11) for all $x \in (0, 1)$. The extension of (3.11) to $x = 1$ follows from the Monotone Convergence Theorem since all the coefficients are non-negative, in fact $A_{2k}^{(p)} > 0$ for all $p > 1$ and $k \in \mathbb{N}_0$. The proof of Theorem 3.3 is complete. $\square$

3.1.4 Proof of Proposition 3.5

It follows from (3.9) and (3.11) that, for any $|z| > 1$, we have

$$f_p(z) = - \sum_{k=0}^{\infty} \frac{A_{2k}^{(p)}}{z^{2k+1}}. \quad (3.28)$$

We prove the formula (3.12) by expanding the analytic function f_p in a neighborhood of ∞ . First, we expand each of the two terms from the defining expression (3.23). For the first term, assuming $|z|$ is sufficiently large, we get

$$\begin{aligned} \left(z \left(1 + \frac{1}{z} \right)^{1/q} - z \right)^{p-1} &= \left(\sum_{m=0}^{\infty} \binom{1/q}{m+1} \frac{1}{z^m} \right)^{p-1} \\ &= \frac{1}{q^{p-1}} \left(1 + \sum_{m=1}^{\infty} \frac{(1-1/q)_m}{(m+1)!} \frac{(-1)^m}{z^m} \right)^{p-1} \\ &= \frac{1}{q^{p-1}} \sum_{n=0}^{\infty} \binom{p-1}{n} \left(\sum_{m=1}^{\infty} \frac{(1/p)_m}{(m+1)!} \frac{(-1)^m}{z^m} \right)^n \\ &= \frac{1}{q^{p-1}} \left(1 + \sum_{n=1}^{\infty} \binom{p-1}{n} \sum_{k=1}^{\infty} \Gamma_k^{(n)} \frac{(-1)^k}{z^k} \right), \end{aligned}$$

where

$$\Gamma_k^{(n)} := \sum_{\substack{r_1, \dots, r_n \geq 1 \\ r_1 + \dots + r_n = k}} \frac{(1/p)_{r_1}}{(r_1+1)!} \cdots \frac{(1/p)_{r_n}}{(r_n+1)!}.$$

Analogously, we deduce

$$\left(z - z \left(1 - \frac{1}{z} \right)^{1/q} \right)^{p-1} = \frac{1}{q^{p-1}} \left(1 + \sum_{n=1}^{\infty} \binom{p-1}{n} \sum_{k=1}^{\infty} \Gamma_k^{(n)} \frac{1}{z^k} \right).$$

Notice that $\Gamma_s^{(n)} = 0$ if $n > s$. Then, by (3.23), we have

$$f_p(z) = \frac{1}{q^{p-1}} \sum_{n=1}^{\infty} \binom{p-1}{n} \sum_{s=n}^{\infty} \Gamma_s^{(n)} \frac{(-1)^s - 1}{z^s} = - \frac{2}{q^{p-1}} \sum_{k=0}^{\infty} \frac{1}{z^{2k+1}} \sum_{n=1}^{2k+1} \binom{p-1}{n} \Gamma_{2k+1}^{(n)}$$

for all $z \in \mathbb{C}$ with $|z|$ sufficiently large. Comparing the coefficients by the same powers of z on both sides in (3.28) and recalling that $1/p + 1/q = 1$, we arrive at the statement. The proof of Proposition 3.5 is complete. $\square$

Conclusion

This research project advanced the understanding of discrete p -Hardy inequalities of higher order $\ell \in \mathbb{N}$ on the half-line, broadening the scope of existing results beyond the cases $\ell = 1$, with general $p > 1$, and an arbitrary $\ell \in \mathbb{N}$ in the linear case $p = 2$, considered in [9] and [26] respectively. In particular, Chapter 1 introduced a systematic approach for deriving abstract p -Hardy weights of arbitrary order, as formalized in Theorems 1.3, 1.8, and 1.11. Building on this framework, we constructed explicit weights via a concrete choice of the parameter sequence $\mathbf{g}^{(\ell,p)}$, given by formula (1.12). In Chapter 2, monotonicity properties of these sequences were studied in order to verify that they meet assumptions (A1)–(A4). This turned out to be a very non-trivial problem which is yet to be solved in general setting. However, numerous partial results were discovered in Theorem 2.4. Finally, in Chapter 3, we analyzed the p -Hardy weight (3.1) of order 1 in greater detail, revealing a connection to a certain Herglotz–Nevanlinna function and confirming the positive coefficient expansion (3.11) conjectured in [9].

Nonetheless, the work in this intriguing field of mathematics is far from over. As mentioned earlier, it remains an unresolved mystery, whether the weights $\rho^{(\ell,p)}$, given by (1.13), satisfy the Hardy inequality (1.1) and whether they improve upon the *classical* weights suggested in (3.14). Our main queries, regarding these problems are formulated in Conjectures 2.5 and 3.8.

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