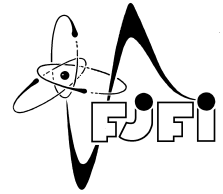




CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering



UNIVERSITY STUDENTS' COMPETITION (SVOČ)

OPTIMAL DISCRETE HARDY INEQUALITIES OF HIGHER ORDER

JAKUB WACLAWEK

ABSTRACT. We prove sufficient conditions on a parameter sequence to determine optimal weights in inequalities for an integer power ℓ of the discrete Laplacian Δ on the half-line. By a concrete choice of the parameter sequence, we obtain explicit optimal discrete Hardy weights of order ℓ . For $\ell = 1$, we rediscover the optimal Hardy weight of Keller–Pinchover–Pogorzelski. For $\ell = 2$, we improve upon the best known Rellich weights due to Gerhat–Krejčířík–Štampach and Huang–Ye. For $\ell \geq 3$, our main result proves a conjecture by Gerhat–Krejčířík–Štampach and improves the classical discrete weights due to Huang–Ye to optimal weights. In an expanded form, this work will be used as the author's bachelor's thesis.

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1. INTRODUCTION

During 1915, in pursuit of finding an elementary proof of the *Hilbert inequality*, Hardy discovered a theorem of great beauty and simplicity which states that whenever the series $\sum_{n=1}^{\infty} a_n^2$ converges, the series of cumulative averages $\sum_{n=1}^{\infty} (A_n/n)^2$, where $A_n := \sum_{k=1}^n a_k$, also converges. This initiated a century of further research. Other great mathematicians, such as E. Landau, G. Pólya, I. Schur, and M. Riesz also significantly contributed to the subsequent development of the Hardy inequality (we refer the reader to [10] for a thorough historical survey). Denoting $u_n := A_n$ or equivalently $a_n := u_n - u_{n-1}$, the *classical discrete Hardy inequality* reads

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}|^2 \geq \frac{1}{4} \sum_{n=1}^{\infty} \frac{|u_n|^2}{n^2} \quad (1.1)$$

for all finitely supported sequences $u \in C_0(\mathbb{N}_0)$ with $u_0 = 0$. Note that if (1.1) holds true for all $u \in C_0(\mathbb{N})$ with $u_0 = 0$, it is also true for all complex sequences $u \in C(\mathbb{N}_0)$ with $u_0 = 0$. In the continuous setting the Hardy inequality asserts that

$$\int_0^{\infty} |\varphi'(x)|^2 dx \geq \frac{1}{4} \int_0^{\infty} \frac{|\varphi(x)|^2}{x^2} dx \quad (1.2)$$

for any φ from the Sobolev space $H^1(0, \infty)$ with $\varphi(0) = 0$.

One can easily observe the analogies between the above inequalities. In fact, it can be shown that the inequalities are equivalent, meaning that the continuous inequality implies the discrete one and vice versa. Moreover, it is a well known fact that the constant $1/4$ on the right-hand side of both (1.1) and (1.2) is optimal, in the sense that it cannot be replaced by a strictly bigger constant. Additionally, the weight $1/(4x^2)$ in (1.2) is known to be critical, meaning that if (1.2) holds with $1/(4x^2)$ replaced by a measurable function $\rho(x) \geq 1/(4x^2)$ for a.e. $x > 0$, then $\rho(x) = 1/(4x^2)$ for a.e. $x > 0$. In contrast, the opposite was recently discovered concerning the discrete version of the Hardy inequality. In [6], M. Keller, Y. Pinchover, and F. Pogorzelski discovered the improved Hardy inequality which reads

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}|^2 \geq \sum_{n=1}^{\infty} \rho_n^{\text{KPP}} |u_n|^2, \quad (1.3)$$

where the weight $1/(4n^2)$ was replaced by the point-wise bigger sequence

$$\rho_n^{\text{KPP}} := 2 - \sqrt{\frac{n-1}{n}} - \sqrt{\frac{n+1}{n}} > \frac{1}{4n^2}. \quad (1.4)$$

Additionally, the authors proved in [7] that the weight ρ^{KPP} is *optimal*. The notion of optimality is a rather strong property which was introduced in a discrete graph setting in [8] (see Definition 1 below). In particular, the optimality of ρ^{KPP} implies criticality, meaning that ρ^{KPP} cannot be further improved by a point-wise bigger sequence such that the analogue of (1.3) would still hold.

The classical Hardy inequalities (1.1) and (1.2) can be interpreted in the sense of quadratic forms in $\ell^2(\mathbb{N})$ or $L^2(0, \infty)$ respectively as lower bounds

$$-\Delta \geq \rho \quad (1.5)$$

for the discrete and the continuous Dirichlet Laplacian on the half-line, where ρ stands for the operator of multiplication by either the discrete or the continuous Hardy weight. In this paper, we use the following definition of the *discrete Laplacian*,

$$(\Delta u)_n := \begin{cases} u_{n-1} - 2u_n + u_{n+1} & \text{if } n \geq 2, \\ -2u_1 + u_2 & \text{if } n = 1, \end{cases}$$

acting on the space of complex sequences $u \in C(\mathbb{N})$ indexed by $\mathbb{N}$. The definition of Δ differs by a sign from the definition of the combinatorial Laplacian used for example by the authors of [7, 5]. With our sign convention, $-\Delta$ is a non-negative operator on spaces of square summable sequences, hence the Hardy inequality takes the form $-\Delta \geq \rho$ on respective spaces in both the discrete as well as the continuous setting.

Furthermore, lower bounds, analogous to (1.5), were found for higher integer powers of the continuous Dirichlet Laplacian, which we shall call the *continuous Hardy inequalities of higher order*. In 1954, F. Rellich discovered (see [12]) the Hardy inequality of order 2 (also called the *Rellich inequality*) which reads

$$\int_0^{\infty} |\varphi''(x)|^2 dx \geq \frac{9}{16} \int_0^{\infty} \frac{|\varphi(x)|^2}{x^4} dx,$$

where $\varphi \in H^2(0, \infty)$ with $\varphi(0) = \varphi'(0) = 0$. Later, in 1961, Birman generalized the Hardy and Rellich inequalities in [1], deriving the Hardy inequality of an arbitrary order $\ell \in \mathbb{N}$ (also called the *Birman*

inequality)

$$\int_0^\infty \left| \varphi^{(\ell)}(x) \right|^2 dx \geq \frac{((2\ell)!)^2}{16^\ell (\ell!)^2} \int_0^\infty \frac{|\varphi(x)|^2}{x^{2\ell}} dx, \quad (1.6)$$

which holds true for any $\varphi \in H^\ell(0, \infty)$ with $\varphi(0) = \dots = \varphi^{(\ell-1)}(0) = 0$. Moreover, the constant on right-hand side of (1.6) is known to be the best possible. Thereby arose a natural question whether a discrete analogue of this statement also holds true and whether it admits improvement similarly to the classical Hardy inequality. The *classical discrete Hardy inequality of order ℓ* reads

$$\sum_{n=\lceil \ell/2 \rceil}^\infty \left| (-\Delta)^{\ell/2} u_n \right|^2 \geq \frac{((2\ell)!)^2}{16^\ell (\ell!)^2} \sum_{n=\ell}^\infty \frac{|u_n|^2}{n^{2\ell}} \quad (1.7)$$

for any $u \in \ell^2(\mathbb{N})$ satisfying $u_1 = \dots = u_{\ell-1} = 0$, where $\lceil x \rceil$ denotes the lowest integer greater or equal to $x \in \mathbb{R}$ and

$$(-\Delta)^{\ell/2} := \begin{cases} (-\Delta)^m & \text{if } \ell = 2m \in 2\mathbb{N}, \\ \nabla \circ (-\Delta)^m & \text{if } \ell = 2m + 1 \in 2\mathbb{N}_0 + 1. \end{cases} \quad (1.8)$$

Here we use the notation of the *discrete gradient* and *divergence*

$$(\nabla u)_n := \begin{cases} u_n - u_{n-1} & \text{if } n \geq 2, \\ u_1 & \text{if } n = 1, \end{cases} \quad \text{and} \quad (\operatorname{div} u)_n := u_{n+1} - u_n \text{ for } n \in \mathbb{N}.$$

Notice that on the Hilbert space $\ell^2(\mathbb{N})$, we have $\nabla^* = -\operatorname{div}$, thus $\Delta = \operatorname{div} \circ \nabla$ and $-\Delta = \nabla^* \circ \nabla$. Therefore, $-\Delta$ is a self-adjoint operator, which is determined by its quadratic form $\langle \cdot, \Delta \cdot \rangle$, and the discrete Hardy inequality can be indeed interpreted as (1.5). Let us also remark that the weight (1.4) in the improved Hardy inequality (1.3) can be written as

$$\rho_n^{\text{KPP}} = \frac{(-\Delta)\sqrt{n}}{\sqrt{n}}$$

for all $n \in \mathbb{N}$.

First steps toward proving (1.7) were made in [4], where B. Gerhat, D. Krejčířík, and F. Štampach studied the case of $\ell = 2$, i.e. the *discrete Rellich inequality*. Not only that they proved the inequality, but via a matrix factorization method they were also able to show that the inequality can be improved to

$$\sum_{n=2}^\infty |(-\Delta)u_n|^2 \geq \sum_{n=2}^\infty \rho_n^{\text{GKS}} |u_n|^2$$

for all $u \in \ell^2(\mathbb{N})$ with $u_1 = 0$, where

$$\rho_n^{\text{GKS}} := \frac{(-\Delta)^2 n^{3/2}}{n^{3/2}} = 6 - 4 \left(1 - \frac{1}{n}\right)^{3/2} - 4 \left(1 + \frac{1}{n}\right)^{3/2} + \left(1 - \frac{2}{n}\right)^{3/2} + \left(1 + \frac{2}{n}\right)^{3/2} > \frac{9}{16n^4}$$

for all $n \geq 2$. However, the *optimality* of $\rho^{(2)}$ was not asserted. The authors of [4] also conjectured that the discrete Hardy inequalities of higher order (1.7) can be improved to

$$\sum_{n=\lceil \ell/2 \rceil}^\infty \left| (-\Delta)^{\ell/2} u_n \right|^2 \geq \sum_{n=\ell}^\infty \frac{(-\Delta)^\ell n^{\ell-1/2}}{n^{\ell-1/2}} |u_n|^2 \quad (1.9)$$

for all $\ell \in \mathbb{N}$. Furthermore, the authors demonstrated that the weights in (1.9) improve upon the classical Birman weights on the right-hand side of (1.7).

Later, in [5] X. Huang and D. Ye proved the inequality (1.7) using a weighted analogue of the classical Hardy inequality (1.1) and demonstrated that the assumption $u_1 = \dots = u_{\ell-1} = 0$ is fundamentally necessary for the inequality (1.7) to hold, answering another question from [4], see also [3] in this regard. Moreover, they found yet another discrete Rellich weight

$$\rho_n^{\text{HY}} = \frac{9}{16n^4} + \frac{15}{16n^5} + \frac{213}{128n^6} + \mathcal{O}\left(\frac{1}{n^7}\right), \quad n \rightarrow \infty, \quad (1.10)$$

which is asymptotically bigger than the previous best known weight

$$\rho_n^{\text{GKS}} = \frac{9}{16n^4} + \frac{105}{128n^6} + \mathcal{O}\left(\frac{1}{n^7}\right), \quad n \rightarrow \infty,$$

but commented that it can still be improved. A question was raised whether the coefficient 15/16 by the second term on the right-hand side of (1.10) is sharp.

The aim of this paper is to construct optimal weights for the discrete Hardy inequality of order $\ell \in \mathbb{N}$, establishing an improved optimal version of (1.7). Moreover, we will answer all the questions mentioned above by analyzing properties of concrete discrete Hardy weights of higher order in greater detail.

The paper is organized as follows. In Section 2, we formulate our main results by means of Theorems 1–5, all of which shall be proven in Section 3 in Sections 3.1–3.5 respectively. We complement our main results with several remarks on more general families of discrete Hardy weights of higher order addressing their non-uniqueness and optimality in Section 4.

2. MAIN RESULTS

In this section, we explain how to construct optimal Hardy weights of any order $\ell \in \mathbb{N}$. Theorems 1–3 provide sufficient conditions on a *parameter sequence* $\mathbf{g}$ to give rise to an optimal discrete Hardy weight of any order. With an explicit choice of $\mathbf{g} = \mathbf{g}^{(\ell)}$, depending on the order $\ell \in \mathbb{N}$ of the given inequality, we obtain a concrete Hardy weight of order ℓ in Theorem 4. Several properties of these weights will be described in Theorem 5.

It turns out to be advantageous, to consider complex sequences indexed by $\mathbb{Z}$, with zero entries up to a certain index, rather than by $\mathbb{N}$. For this reason, we introduce the following subspaces of the space of complex sequences $C(\mathbb{Z})$. Namely, we denote

$$H^\ell := \{u \in C(\mathbb{Z}) \mid u_n = 0 \text{ for all } n < \ell\}$$

and

$$\mathcal{H}^\ell := H^\ell \cap \ell^2(\mathbb{Z})$$

endowed with the Euclidean inner product. Accordingly to our previous notation,

$$\mathcal{H}_0^\ell := \mathcal{H}^\ell \cap C_0(\mathbb{Z})$$

denotes finitely supported sequences in $\mathcal{H}^\ell$. With a slight abuse of notation, we further define the *discrete gradient* and *divergence* acting on complex sequences in $C(\mathbb{Z})$ as

$$(\nabla u)_n := u_n - u_{n-1} \quad \text{and} \quad (\operatorname{div} u)_n := u_{n+1} - u_n \quad (2.1)$$

for all $n \in \mathbb{Z}$. Naturally, we define the *discrete Laplacian* acting on $C(\mathbb{Z})$ as $\Delta := \operatorname{div} \circ \nabla$, i.e.

$$(\Delta u)_n := u_{n-1} - 2u_n + u_{n+1}$$

for all $n \in \mathbb{N}$. The half-integer powers of the Laplacian are defined similarly as in (1.8). The main reason for this notation is that ∇ , div , and Δ commute on $C(\mathbb{Z})$ (we omit $\circ$ when composing difference operators to simplify the notation below). Nevertheless, it is important to realize that the operators $(-\Delta)^\ell$ coincide with those *canonically* defined in Section 1 on the space

$$\mathcal{H}^\ell(\mathbb{N}) := \{u \in \ell^2(\mathbb{N}) \mid u_1 = \dots = u_{\ell-1} = 0\}.$$

Let us recall the definition of *optimality* adopted from [8] and [7], formulated for higher integer powers of the Laplacian acting on the space H^ℓ .

Definition 1. Let $\ell \in \mathbb{N}$. A positive sequence $\{\rho_n\}_{n=\ell}^\infty$ is called a *discrete Hardy weight of order ℓ* if and only if the *discrete Hardy inequality of order ℓ*

$$\sum_{n=\lceil \ell/2 \rceil}^\infty \left| (-\Delta)^{\ell/2} u_n \right|^2 \geq \sum_{n=\ell}^\infty \rho_n |u_n|^2 \quad (2.2)$$

holds for any $u \in \mathcal{H}_0^\ell$. In addition, the weight ρ is said to be *optimal* if it exhibits the following three properties:

- (i) *Criticality*: The weight ρ is called *critical* if for any Hardy weight $\tilde{\rho}$ of order ℓ such that $\tilde{\rho}_n \geq \rho_n$ for all $n \geq \ell$ it follows that $\tilde{\rho} = \rho$.
- (ii) *Non-attainability*: The weight ρ is called *non-attainable* if whenever the equality in (2.2) is attained for $u \in H^\ell$ such that the right-hand side is finite, i.e. $\sqrt{\rho}u \in \mathcal{H}^\ell$, necessarily $u \equiv 0$.
- (iii) *Optimality near infinity*: The weight ρ is called *optimal near infinity* if for any $M \geq \ell$ and $\varepsilon > 0$ there exists $u \in \mathcal{H}_0^M$ such that

$$\sum_{n=\lceil \ell/2 \rceil}^\infty \left| (-\Delta)^{\ell/2} u_n \right|^2 < (1 + \varepsilon) \sum_{n=\ell}^\infty \rho_n |u_n|^2. \quad (2.3)$$

Remark 2.1. We complement the definition with several explanatory remarks.

- (i) Criticality means that the Hardy weight ρ of order ℓ cannot be further improved by a point-wise greater weight. However, in [9] it was shown that there exist infinitely many critical Hardy weights of order 1. Analogous property will be observed in Section 4 for Hardy weights of an arbitrary order ℓ .
- (ii) Non-attainability of a Hardy weight ρ of order ℓ means that in the weighted ℓ^2 -space of complex sequences such that $\sqrt{\rho}u \in \mathcal{H}^\ell$, the inequality (2.2) is strict unless $u \equiv 0$.

- (iii) Optimality of ρ near infinity means that the constant 1 on the right-hand side of (2.2) cannot be further improved, even if the space $\mathcal{H}_0^\ell$ is replaced by $\mathcal{H}_0^M$ for an arbitrary $M \geq \ell$. Equivalently, it can be formulated as

$$\inf_{u \in \mathcal{H}_0^M(\mathbb{N}) \setminus \{0\}} \frac{\sum_{n=\lceil \ell/2 \rceil}^{\infty} |(-\Delta)^{\ell/2} u_n|^2}{\sum_{n=\ell}^{\infty} \rho_n |u_n|^2} = 1. \quad (2.4)$$

Remark 2.2. The inequality (2.2) can be straightforwardly extended to all $u \in \mathcal{H}^\ell$ (or even to all $u \in H^\ell$) and therefore to all $u \in \mathcal{H}^\ell(\mathbb{N})$ with $(-\Delta)^{\ell/2}$ also acting on the respective spaces. Since

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} |(-\Delta)^{\ell/2} u_n|^2 = \sum_{n=\ell}^{\infty} \overline{u_n} (-\Delta)^\ell u_n = \langle u, (-\Delta)^\ell u \rangle,$$

the inequality (2.2) can be equivalently written as a lower bound for the integer power of the discrete Laplacian

$$(-\Delta)^\ell \geq \rho$$

on the space $\mathcal{H}^\ell$, where we again identify ρ with the corresponding multiplication operator.

In [9] an abstract identity was derived for generating Hardy weights of order 1. Our starting point shall be a generalized weighted analogue of this identity, which first appeared in [5]. For the readers convenience we will restate the equality here allowing complex sequences together with its prove, which is listed in Section 3.0.

Theorem 0. *Let $V \in C(\mathbb{Z})$ and $\mathbf{g} \in H^1$ such that $\mathbf{g}_n > 0$ for all $n \geq 1$. Then for any $u \in \mathcal{H}_0^1$, we have the identity*

$$\sum_{n=1}^{\infty} V_n |\nabla u_n|^2 + \sum_{n=1}^{\infty} \frac{\operatorname{div}(V \nabla \mathbf{g})_n}{\mathbf{g}_n} |u_n|^2 = \sum_{n=1}^{\infty} V_{n+1} \left| \sqrt{\frac{\mathbf{g}_n}{\mathbf{g}_{n+1}}} u_{n+1} - \sqrt{\frac{\mathbf{g}_{n+1}}{\mathbf{g}_n}} u_n \right|^2.$$

Via iteration of this identity we will obtain our first main result, which is an analogous identity for the quadratic form of $(-\Delta)^\ell$ with $\ell \in \mathbb{N}$. The key idea for deriving such identities lies in a suitable choice of the parametric sequences $\mathbf{g}$ in the individual steps of the iterative process, which generate convenient weights V in terms of $\mathbf{g}$. For this reason, it is necessary to impose some positivity assumptions on the parameter sequence $\mathbf{g}$. In fact, in Theorems 1–3 the assumptions on $\mathbf{g}$ will be gradually strengthened, because of which they will be numbered separately.

Theorem 1. *Let $\ell \in \mathbb{N}$. Suppose that*

$$\mathbf{g} \in H^\ell \text{ such that } \operatorname{div}^k \mathbf{g}_n > 0 \text{ for all } n \geq \ell - k \text{ and } k \in \{0, \dots, \ell - 1\}. \quad (\mathbf{A1})$$

Then for any $u \in \mathcal{H}_0^\ell$, we have the identity

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} |(-\Delta)^{\ell/2} u_n|^2 = \sum_{n=\ell}^{\infty} \frac{(-\Delta)^\ell \mathbf{g}_n}{\mathbf{g}_n} |u_n|^2 + \sum_{k=0}^{\ell-1} \mathcal{R}_k^{(\ell)}(\mathbf{g}; u), \quad (2.5)$$

where

$$\mathcal{R}_k^{(\ell)}(\mathbf{g}; u) := \sum_{n=\ell-k}^{\infty} \frac{(-\Delta)^{\ell-1-k} \operatorname{div}^{k+1} \mathbf{g}_n}{\operatorname{div}^{k+1} \mathbf{g}_n} \left| \sqrt{\frac{\operatorname{div}^k \mathbf{g}_n}{\operatorname{div}^k \mathbf{g}_{n+1}}} \operatorname{div}^k u_{n+1} - \sqrt{\frac{\operatorname{div}^k \mathbf{g}_{n+1}}{\operatorname{div}^k \mathbf{g}_n}} \operatorname{div}^k u_n \right|^2. \quad (2.6)$$

(For $k = \ell - 1$, the coefficient in front of the absolute value in (2.6) is to be understood as 1.)

Remark 2.3. In fact, Theorem 1 can be seen as a corollary of a weighted analogue of the identity (2.5), which may be of independent interest. Namely, if $\ell \in \mathbb{N}$ and $\mathbf{g}$ satisfies the assumptions $(\mathbf{A1})$, then for any weight $V \in C(\mathbb{Z})$, we have the identity

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} V_{n-\lceil \ell/2 \rceil+1} |(-\Delta)^{\ell/2} u_n|^2 = (-1)^\ell \sum_{n=\ell}^{\infty} \frac{\operatorname{div} \nabla^{\ell-1} (V(\operatorname{div}^{\ell-1} \nabla \mathbf{g}))_n}{\mathbf{g}_n} |u_n|^2 + \sum_{k=0}^{\ell-1} \mathcal{R}_k^{(\ell)}(V; \mathbf{g}; u), \quad (2.7)$$

where $\mathcal{R}_k^{(\ell)}(V; \mathbf{g}; u)$ is defined as

$$(-1)^{\ell+k+1} \sum_{n=\ell-k}^{\infty} \frac{\operatorname{div} \nabla^{\ell-2-k} (V(\operatorname{div}^{\ell-2-k} \nabla \operatorname{div}^{k+1} \mathbf{g}))}{\operatorname{div}^{k+1} \mathbf{g}_n} \left| \sqrt{\frac{\operatorname{div}^k \mathbf{g}_n}{\operatorname{div}^k \mathbf{g}_{n+1}}} \operatorname{div}^k u_{n+1} - \sqrt{\frac{\operatorname{div}^k \mathbf{g}_{n+1}}{\operatorname{div}^k \mathbf{g}_n}} \operatorname{div}^k u_n \right|^2; \quad (2.8)$$

for $k = \ell - 1$, the coefficient in front of the absolute value in (2.8) is to be understood as V_{n+1} . Notice that for $V \equiv 1$ this statement becomes Theorem 1. The differences between the first term on the right-hand side of (2.5) and (2.7) and the coefficients in front of the absolute value in (2.6) and (2.8) is due to the fact, that the multiplicative operator V does not commute with ∇ nor div . Similarly, the reason for shifting the indices of V on the left-hand side of (2.7) emerged from the non-commutativity of V and the *shift operator* (3.1), in order to obtain more concise formulas. The proof of this generalization is analogous to the proof of Theorem 1 in Section 3.1 and is therefore omitted.

By imposing additional assumptions on the parameter sequence $\mathbf{g}$, we may ensure non-negativity of the remainders on the right-hand side of (2.5) obtaining an abstract discrete Hardy inequality of order ℓ .

Theorem 2. *Let $\ell \in \mathbb{N}$. Suppose (A1) and, in addition,*

$$(-\Delta)^{\ell-k} \text{div}^k \mathbf{g}_n \geq 0 \text{ for all } n \geq \ell + 1 - k \text{ and } k \in \{1, \dots, \ell - 1\}. \quad (\text{A2})$$

Then for all $u \in \mathcal{H}_0^\ell$, we have the inequality

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left| (-\Delta)^{\ell/2} u_n \right|^2 \geq \sum_{n=\ell}^{\infty} \rho_n(\mathbf{g}) |u_n|^2, \quad (2.9)$$

where $\rho(\mathbf{g}) := (-\Delta)^\ell \mathbf{g}/\mathbf{g}$. If moreover,

$$(-\Delta)^\ell \mathbf{g}_n > 0 \text{ for all } n \geq \ell, \quad (\text{A2}')$$

then $\rho(\mathbf{g}) > 0$, i.e. $\rho(\mathbf{g})$ is a discrete Hardy weight of order ℓ .

Remark 2.4. If the parameter sequence $\mathbf{g}$ satisfies the assumptions (A1) and (A2), the remainder terms on the right-hand side of (2.5) can be interpreted as a norm

$$\mathcal{R}_k^{(\ell)}(\mathbf{g}; u) = \left\| R_k^{(\ell)}(\mathbf{g})u \right\|^2,$$

where

$$R_k^{(\ell)}(\mathbf{g})u_n := \sqrt{\frac{(-\Delta)^{\ell-1-k} \text{div}^{k+1} \mathbf{g}_n}{\text{div}^{k+1} \mathbf{g}_n}} \left(\sqrt{\frac{\text{div}^k \mathbf{g}_n}{\text{div}^k \mathbf{g}_{n+1}}} \text{div}^k u_{n+1} - \sqrt{\frac{\text{div}^k \mathbf{g}_{n+1}}{\text{div}^k \mathbf{g}_n}} \text{div}^k u_n \right) \quad (2.10)$$

for all $k \in \{0, \dots, \ell - 1\}$ and $n \geq \ell - k$ (for $k = \ell - 1$ the coefficient in front of the parentheses is to be understood as 1). Since $(-\Delta)^\ell$ is a bounded operator, the identity (2.5) can be extended to all $u \in \mathcal{H}^\ell$ (in fact, it can be extended even further, see Proposition 9). Recalling Remark 2.2, (after shifting the index n in the remainders) the identity can be interpreted as

$$(-\Delta)^\ell - \rho(\mathbf{g}) = \sum_{k=0}^{\ell-1} R_k^{(\ell)*}(\mathbf{g}) R_k^{(\ell)}(\mathbf{g}) \quad (2.11)$$

on the space $\mathcal{H}^\ell$, where $\rho(\mathbf{g}) = (-\Delta)^\ell \mathbf{g}/\mathbf{g}$ is identified with the corresponding multiplicative operator. For $\ell = 1$, such factorization has been used to provide an alternative proof of the optimal discrete Hardy inequality in [4]. For $\ell = 2$, the authors of [4] factorized the matrix $(-\Delta)^2 - \rho(\mathbf{g})$, with $\mathbf{g}_n = n^{3/2}$, into a single remainder matrix of a form R^*R , where R is a tridiagonal matrix. As the remainder were sought in terms of a single matrix rather than two matrices, its entries could not be found explicitly. The idea was to decompose the pentadiagonal matrix $(-\Delta)^2 - \rho(\mathbf{g})$ into a product of a tridiagonal matrix and its adjoint reducing the order of the corresponding difference operators. The idea behind the factorization (2.11) is similar, however, its main novelty is that the order is reduced successively giving rise to more remainder terms on the right expressed explicitly in terms of the parameter sequence $\mathbf{g}$. In addition, the diagonal term $\rho(\mathbf{g})$, i.e. the actual Hardy weight of order ℓ , is also identified in terms of $\mathbf{g}$. It turns out, that such a concrete description is crucial for the proof of optimality of the weight $\rho(\mathbf{g})$ in the following theorem.

Under additional requirements on the asymptotic behaviour of $\mathbf{g}_n$, as $n \rightarrow \infty$, and strict positivity in assumption (A2) for $k = 1$, we can assert optimality of the discrete Hardy weight $\rho(\mathbf{g})$.

Theorem 3. *Let $\ell \in \mathbb{N}$. Suppose (A1), (A2), (A2'), and, in addition, suppose that $\mathbf{g}$ admits the asymptotic expansion*

$$\mathbf{g}_n = \sum_{j=0}^{2\ell} \alpha_j n^{\ell-1/2-j} + \mathcal{O}\left(n^{-\ell-3/2}\right) \text{ for some } \alpha_j \in \mathbb{R} \text{ with } \alpha_0 \neq 0, \quad (\text{A3})$$

as $n \rightarrow \infty$. Then the discrete Hardy weight $\rho(\mathbf{g}) = (-\Delta)^\ell \mathbf{g}/\mathbf{g}$ of order ℓ is critical and optimal near infinity. If moreover,

$$(-\Delta)^{\ell-1} \text{div} \mathbf{g}_n > 0 \text{ for all } n \geq \ell, \quad (\text{A3}')$$

then $\rho(\mathbf{g})$ is also non-attainable and therefore optimal.

It remains to find a concrete parameter sequence $\mathbf{g}$, which satisfies all the assumptions of the above theorems. To this end, for given $\ell \in \mathbb{N}$, we define

$$\mathbf{g}_n^{(\ell)} := \sqrt{n} \prod_{j=1}^{\ell-1} (n-j) \quad (2.12)$$

for all $n \in \mathbb{N}_0$ and $\mathbf{g}_n := 0$ for all $n < 0$.

Theorem 4. *Let $\ell \in \mathbb{N}$. The sequence $\rho^{(\ell)}$, given by*

$$\rho_n^{(\ell)} := \frac{(-\Delta)^\ell \mathbf{g}_n^{(\ell)}}{\mathbf{g}_n^{(\ell)}} \quad (2.13)$$

for all $n \geq \ell$, where $\mathbf{g}^{(\ell)}$ is given by (2.12), is an optimal discrete Hardy weight of order ℓ .

Our final theorem summarizes properties of the optimal weight $\rho^{(\ell)}$ in greater detail. A remarkable property is that $\rho_n^{(\ell)}$ has a convergent series representation in negative powers of n with all coefficients being non-negative (in fact, for $\ell \geq 2$ the coefficients are positive) for all $n \geq \ell$. Consequently, using more terms of the truncated series representation always produces a tighter inequality.

In order to formulate this theorem, we introduce several combinatorial numbers. First, we will make use of the *binomial number* and the *Pochhammer symbol* defined by standard formulas

$$\binom{\nu}{n} := \frac{\nu(\nu-1)\dots(\nu-n+1)}{n!} \quad \text{and} \quad (\nu)_n := \nu(\nu+1)\dots(\nu+n-1)$$

for all $\nu \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Next, we denote the *Stirling numbers of the first kind* as

$$s(n, k) := (-1)^{n+k} \sum_{1 \leq i_1 < \dots < i_{n-k} < n} i_1 i_2 \dots i_{n-k} \quad (2.14)$$

and the *Stirling numbers of the second kind* as

$$S(n, k) := \sum_{\substack{j_1, \dots, j_k \geq 0 \\ j_1 + \dots + j_k = n-k}} 1^{j_1} 2^{j_2} \dots k^{j_k} \quad (2.15)$$

for all $n \in \mathbb{N}$ and $k \in \{0, \dots, n-1\}$, see [2, § 26.8]. By convention, we also set $s(n, n) = S(n, n) := 1$ for all $n \in \mathbb{N}_0$ and $s(n, k) := 0$ whenever $k < 0$. Moreover, we employ the numbers

$$X_m^{(\ell)} := \sum_{j=-\ell}^{\ell} \binom{2\ell}{\ell+j} (-1)^j j^m \quad (2.16)$$

for all $m, \ell \in \mathbb{N}$ with $X_0^{(\ell)} := 0$ for all $\ell \in \mathbb{N}$.

Remark 2.5. For $m \in \mathbb{N}$ odd and $\ell \in \mathbb{N}$, we have

$$X_m^{(\ell)} = \sum_{j=-\ell}^{\ell} \binom{2\ell}{\ell+j} (-1)^j j^m = \sum_{j=-\ell}^{\ell} \binom{2\ell}{\ell-j} (-1)^{-j} (-j)^m = - \sum_{j=-\ell}^{\ell} \binom{2\ell}{\ell+j} (-1)^j j^m = -X_m^{(\ell)},$$

hence $X_m^{(\ell)} = 0$. Furthermore, in [4] the authors found that

$$X_m^{(\ell)} = 0 \text{ for all } m < 2\ell \quad (2.17)$$

and that for the remaining values, we have

$$X_{2\ell+2m}^{(\ell)} = (-1)^\ell (2\ell)! \sum_{1 \leq k_1 \leq \dots \leq k_m \leq \ell} (k_1 k_2 \dots k_m)^2, \quad (2.18)$$

where $m \in \mathbb{N}$ and $X_{2\ell}^{(\ell)} = (-1)^\ell (2\ell)!$. Expression (2.18) reveals the nontrivial fact that $(-1)^\ell X_{2\ell+2m}^{(\ell)}$ is a positive integer for all $m \in \mathbb{N}_0$.

Lastly, we define the numbers

$$r_m^{(\ell)} := \sum_{j=2\ell}^m \binom{\ell+j-m-1/2}{j} s(\ell, \ell+j-m) X_j^{(\ell)} \quad (2.19)$$

for all $\ell \in \mathbb{N}$ and $m \geq 2\ell$.

Remark 2.6. Since $s(\ell, j) = 0$ whenever $j \leq 0$ and $X_j^{(\ell)} = 0$ whenever j is odd, we may restrict the range of the summation index j in formula (2.19) even further. It turns out, that after this restriction, each summand is positive and the formula reads

$$r_m^{(\ell)} = \sum_{\substack{j=\max(2\ell, m-\ell+1) \\ j \equiv 0 \pmod{2}}}^m \binom{\ell+j-m-1/2}{j} s(\ell, \ell+j-m) X_j^{(\ell)}.$$

Consequently, it holds true that for all $\ell \geq 2$ and $m \geq 2\ell$, we have $r_m^{(\ell)} > 0$ and if $\ell = 1$, then $r_{2m+1}^{(1)} = 0$ and $r_{2m}^{(1)} > 0$ for all $m \geq 1$, which will be shown in Section 3.5.2.

Theorem 5. *Let $\ell \in \mathbb{N}$ and $\rho^{(\ell)}$ defined by the formulas (2.13) and (2.12), then the following properties hold true.*

(i) *The weight sequence $\rho^{(\ell)}$ admits the convergent series expansion*

$$\rho_n^{(\ell)} = \sum_{k=2\ell}^{\infty} \frac{A_k^{(\ell)}}{n^k} \quad (2.20)$$

for all $n \geq \ell$, where the coefficients are defined as

$$A_k^{(\ell)} := \sum_{m=2\ell}^k S(k-m+\ell-1, \ell-1) r_m^{(\ell)}. \quad (2.21)$$

(ii) *For all $\ell \geq 2$ and $k \geq 2\ell$, we have $A_k^{(\ell)} > 0$.*

For $\ell = 1$ and $k \geq 1$, we have $A_{2k+1}^{(1)} = 0$ and $A_{2k}^{(1)} > 0$.

(iii) *For all $n \geq \ell \geq 2$, it is true that*

$$\rho_n^{(\ell)} > \frac{(-\Delta)^\ell n^{\ell-1/2}}{n^{\ell-1/2}} > \left(\frac{1}{2}\right)_\ell^2 \frac{1}{n^{2\ell}}. \quad (2.22)$$

(For $\ell = 1$, the first inequality in (2.22) holds as equality.)

Remark 2.7. We complement Theorem 5 with several remarks.

(i) For the first two coefficients in the expansion (2.20), we have

$$A_{2\ell}^{(\ell)} = \left(\frac{1}{2}\right)_\ell^2 \quad \text{and} \quad A_{2\ell+1}^{(\ell)} = \frac{2\ell^2(\ell-1)}{2\ell-1} \left(\frac{1}{2}\right)_\ell^2$$

for all $\ell \in \mathbb{N}$, which can be computed directly from the formulas (2.21) and (2.19), realizing that $X_{2\ell+1}^{(\ell)} = 0$ and $X_{2\ell}^{(\ell)} = (-1)^\ell (2\ell)!$.

(ii) The first inequality in (2.22) shows that the weight $\rho^{(\ell)}$ improves upon the weights suggested by Gerhart–Krejčířik–Štampach, see (1.9), and proves the conjecture in [4]. Noticing that

$$\left(\frac{1}{2}\right)_\ell^2 = \frac{((2\ell)!)^2}{16^\ell (\ell!)^2},$$

the second inequality in (2.22) shows that $\rho^{(\ell)}$ improves upon the classical discrete Hardy weights (1.7).

(iii) For $\ell = 1$, we rediscovered the optimal Hardy weight (1.4), whose expansion is explicitly known, see [6]. The expansions in the first few terms for $\ell = 2, 3, 4, 5$, as $n \rightarrow \infty$, read

$$\begin{aligned} \rho_n^{(2)} &= \frac{9}{16n^4} + \frac{3}{2n^5} + \frac{297}{128n^6} + \mathcal{O}\left(\frac{1}{n^7}\right), \\ \rho_n^{(3)} &= \frac{225}{64n^6} + \frac{405}{16n^7} + \frac{114975}{1024n^8} + \mathcal{O}\left(\frac{1}{n^9}\right), \\ \rho_n^{(4)} &= \frac{11025}{256n^8} + \frac{4725}{8n^9} + \frac{4879665}{1024n^{10}} + \mathcal{O}\left(\frac{1}{n^{11}}\right), \\ \rho_n^{(5)} &= \frac{893025}{1024n^{10}} + \frac{2480625}{128n^{11}} + \frac{4023077625}{16384n^{12}} + \mathcal{O}\left(\frac{1}{n^{13}}\right). \end{aligned}$$

As was mentioned earlier, in [5], the authors ask whether the constant $15/16$ by the second term in (1.10) is sharp. The above expansion of $\rho_n^{(2)}$ shows that it is not the case.

3. PROOFS

In the course of the proofs worked out below, we will frequently use (besides the operators ∇ , div , and Δ) the *forward shift* operator S acting on $C(\mathbb{Z})$ defined as

$$Su_n := u_{n+1} \quad (3.1)$$

for all $n \in \mathbb{Z}$. Obviously, for any $k \in \mathbb{Z}$ it holds that $S^k u_n = u_{n+k}$ for all $n \in \mathbb{Z}$. Recalling definition (2.1), we see that $\text{div} = S - I$ and $\nabla = I - S^{-1}$, where I stands for the identity operator. In particular, we have identities $\text{div} = S \circ \nabla = \nabla \circ S$, that will be used several times below. Note also that subspaces H^ℓ are not preserved under the action of S and hence neither under the action of div (S is a bijection of H^ℓ onto $H^{\ell-1}$).

3.0. Proof of Theorem 0. Suppose $\mathbf{g} \in H^1$, $\mathbf{g}_n > 0$ for all $n \geq 1$, and $u \in \mathcal{H}_0^1$. For any $n \geq 2$, we have

$$\begin{aligned} \left| \sqrt{\frac{\mathbf{g}_{n-1}}{\mathbf{g}_n}} u_n - \sqrt{\frac{\mathbf{g}_n}{\mathbf{g}_{n-1}}} u_{n-1} \right|^2 &= \frac{\mathbf{g}_{n-1}}{\mathbf{g}_n} |u_n|^2 + \frac{\mathbf{g}_n}{\mathbf{g}_{n-1}} |u_{n-1}|^2 - 2 \text{Re}(u_n u_{n-1}) \\ &= |\nabla u_n|^2 - \frac{\nabla \mathbf{g}_n}{\mathbf{g}_n} |u_n|^2 + \frac{\nabla \mathbf{g}_n}{\mathbf{g}_{n-1}} |u_{n-1}|^2. \end{aligned}$$

Multiplying both sides by V_n and summing over n from 2 to ∞ , we obtain

$$\begin{aligned} \sum_{n=2}^{\infty} V_n \left| \sqrt{\frac{\mathbf{g}_{n-1}}{\mathbf{g}_n}} u_n - \sqrt{\frac{\mathbf{g}_n}{\mathbf{g}_{n-1}}} u_{n-1} \right|^2 &= \sum_{n=2}^{\infty} V_n |\nabla u_n|^2 - \sum_{n=2}^{\infty} V_n \frac{\nabla \mathbf{g}_n}{\mathbf{g}_n} |u_n|^2 + \sum_{n=1}^{\infty} V_{n+1} \frac{\nabla \mathbf{g}_{n+1}}{\mathbf{g}_n} |u_n|^2 \\ &= \sum_{n=2}^{\infty} V_n |\nabla u_n|^2 + \sum_{n=1}^{\infty} \frac{\text{div}(V \nabla \mathbf{g})_n}{\mathbf{g}_n} |u_n|^2 + V_1 \frac{\nabla \mathbf{g}_1}{\mathbf{g}_1} |u_1|^2 \end{aligned}$$

By assumptions, $u_0 = \mathbf{g}_0 = 0$, thus

$$V_1 \frac{\nabla \mathbf{g}_1}{\mathbf{g}_1} |u_1|^2 = V_1 |\nabla u_1|^2.$$

A shift of the summation index n by one on the left-hand side now yields the claim of Theorem 0. $\square$

3.1. Proof of Theorem 1. The proof proceeds by a two-step induction in $\ell \in \mathbb{N}$:

- a) We verify Theorem 1 for $\ell = 1, 2$.
- b) Assuming Theorem 1 to hold for all $\ell \leq 2m$, where $m \in \mathbb{N}$, we prove it for $\ell = 2m + 1$.
- c) Assuming Theorem 1 to hold for all $\ell \leq 2m + 1$, we prove it also for $\ell = 2m + 2$.

The reason to treat even and odd indices ℓ separately stems from the fact that, when lowering a half-integer power of the discrete Laplacian, ∇ or div emerge depending on the parity of ℓ because we have

$$(-\Delta)^{\ell/2} = \begin{cases} \nabla(-\Delta)^{(\ell-1)/2} & \text{if } \ell \in 2\mathbb{N}_0 + 1, \\ -\text{div}(-\Delta)^{(\ell-1)/2} & \text{if } \ell \in 2\mathbb{N}. \end{cases}$$

Since the resulting differences are subtle, parts b) and c) of the proof are analogical and therefore we only briefly indicate the proof of c). Note also that the case $\ell = 2$ is shown only for clarity and illustrative purposes, since it could be omitted after interchanging the order of the steps a) and b).

a) For $\ell = 1$, Theorem 1 coincides with the special case of Theorem 0 with $V \equiv 1$. Next, we suppose $\ell = 2$ and $\mathbf{g}$ to satisfy the assumption **(A1)**, i.e. $\mathbf{g} \in H^2$, $\mathbf{g}_n > 0$ for all $n \geq 2$, and $\text{div} \mathbf{g}_n > 0$ for all $n \geq 1$. Clearly, $\text{div} \mathbf{g} \in H^1$ hence we may apply Theorem 0 with $\mathbf{g}$ replaced by $\text{div} \mathbf{g}$, u replaced by $\text{div} u$, and $V \equiv 1$, from which it follows that

$$\sum_{n=1}^{\infty} |(-\Delta) u_n|^2 = \sum_{n=1}^{\infty} |\text{div} \nabla u_n|^2 = \sum_{n=1}^{\infty} |\nabla(\text{div} u)_n|^2 = \sum_{n=1}^{\infty} \frac{(-\Delta) \text{div} \mathbf{g}_n}{\text{div} \mathbf{g}_n} |\text{div} u_n|^2 + \mathcal{R}_1^{(2)}(\mathbf{g}; u)$$

for all $u \in \mathcal{H}_0^2$, where $\mathcal{R}_1^{(2)}(\mathbf{g}; u)$ is defined by (2.6). Bearing in mind that $\text{div} u = \nabla S u$, we apply Theorem 0 once more, this time with $S \mathbf{g} \in H^1$, $S u \in \mathcal{H}_0^1$, and $V_n := (-\Delta) \text{div} \mathbf{g}_n / \text{div} \mathbf{g}_n$ to the first term on the right-hand side, obtaining the identity

$$\sum_{n=1}^{\infty} |(-\Delta) u_n|^2 = - \sum_{n=1}^{\infty} \frac{\text{div}(V \nabla S \mathbf{g})_n}{S \mathbf{g}_n} |S u_n|^2 + \sum_{n=1}^{\infty} V_{n+1} \left| \sqrt{\frac{S \mathbf{g}_n}{S \mathbf{g}_{n+1}}} S u_{n+1} - \sqrt{\frac{S \mathbf{g}_{n+1}}{S \mathbf{g}_n}} S u_n \right|^2 + \mathcal{R}_1^{(2)}(\mathbf{g}; u)$$

for all $u \in \mathcal{H}_0^2$. Taking also into account that

$$-\text{div}(V \nabla S \mathbf{g}) = \text{div} \left(\frac{\Delta \text{div} \mathbf{g}}{\text{div} \mathbf{g}} \text{div} \mathbf{g} \right) = S(-\Delta)^2 \mathbf{g},$$

we arrive at the desired result

$$\sum_{n=1}^{\infty} |(-\Delta)u_n|^2 = \sum_{n=2}^{\infty} \frac{(-\Delta)^2 \mathbf{g}_n}{\mathbf{g}_n} |u_n|^2 + \mathcal{R}_0^{(2)}(\mathbf{g}; u) + \mathcal{R}_1^{(2)}(\mathbf{g}; u)$$

for all $u \in \mathcal{H}^2$, with $\mathcal{R}_0^{(2)}(\mathbf{g}; u)$ given again by the general definition from (2.6).

b) Suppose $m \in \mathbb{N}$ and assume that Theorem 1 holds true for all $\ell \leq 2m$. Let us consider sequences $u \in \mathcal{H}_0^{2m+1}$ and $\mathbf{g} \in H^{2m+1}$ satisfying the assumption (A1) for $\ell = 2m + 1$. We shall verify (2.5) for $\ell = 2m + 1$. We have

$$\sum_{n=\lceil \frac{2m+1}{2} \rceil}^{\infty} \left| (-\Delta)^{(2m+1)/2} u_n \right|^2 = \sum_{n=m+1}^{\infty} |\nabla(-\Delta)^m u_n|^2 = \sum_{n=m+1}^{\infty} |(-\Delta)^m \nabla u_n|^2 = \sum_{n=m}^{\infty} |(-\Delta)^m \operatorname{div} u_n|.$$

Since $\operatorname{div} u \in \mathcal{H}_0^{2m}$ and $\operatorname{div} \mathbf{g}$ satisfies assumption (A1) for $\ell = 2m$, we may apply the induction hypothesis and obtain

$$\sum_{n=m}^{\infty} |(-\Delta)^m \operatorname{div} u_n|^2 = \sum_{n=2m}^{\infty} \frac{(-\Delta)^{2m} \operatorname{div} \mathbf{g}_n}{\operatorname{div} \mathbf{g}_n} |\operatorname{div} u_n|^2 + \sum_{k=0}^{2m-1} \mathcal{R}_k^{(2m)}(\operatorname{div} \mathbf{g}; \operatorname{div} u).$$

Recalling the general definition of remainders (2.6), we observe that

$$\mathcal{R}_k^{(2m)}(\operatorname{div} \mathbf{g}; \operatorname{div} u) = \mathcal{R}_{k+1}^{(2m+1)}(\mathbf{g}; u) \implies \sum_{k=0}^{2m-1} \mathcal{R}_k^{(2m)}(\operatorname{div} \mathbf{g}; \operatorname{div} u) = \sum_{k=1}^{2m} \mathcal{R}_k^{(2m+1)}(\mathbf{g}; u).$$

Next, we shift the index in the first sum on the right-hand side finding that

$$\sum_{n=\lceil \frac{2m+1}{2} \rceil}^{\infty} \left| (-\Delta)^{(2m+1)/2} u_n \right|^2 = \sum_{n=1}^{\infty} \frac{S^{2m-1}(-\Delta)^{2m} \operatorname{div} \mathbf{g}_n}{S^{2m-1} \operatorname{div} \mathbf{g}_n} |\nabla S^{2m} u_n|^2 + \sum_{k=1}^{2m} \mathcal{R}_k^{(2m+1)}(\mathbf{g}; u). \quad (3.2)$$

Furthermore, we apply Theorem 0 with $S^{2m} u \in \mathcal{H}_0^1$, together with $S^{2m} \mathbf{g} \in H^1$, and the weight

$$V_n := \frac{S^{2m-1}(-\Delta)^{2m} \operatorname{div} \mathbf{g}_n}{S^{2m-1} \operatorname{div} \mathbf{g}_n},$$

to the first term on the right-hand side of (3.2) obtaining

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{S^{2m-1}(-\Delta)^{2m} \operatorname{div} \mathbf{g}_n}{S^{2m-1} \operatorname{div} \mathbf{g}_n} |\nabla S^{2m} u_n|^2 &= - \sum_{n=1}^{\infty} \frac{\operatorname{div}(V \nabla S^{2m} \mathbf{g})_n}{S^{2m} \mathbf{g}_n} |S^{2m} u_n|^2 \\ &\quad + \sum_{n=1}^{\infty} V_{n+1} \left| \sqrt{\frac{S^{2m} \mathbf{g}_n}{S^{2m} \mathbf{g}_{n+1}}} S^{2m} u_{n+1} - \sqrt{\frac{S^{2m} \mathbf{g}_{n+1}}{S^{2m} \mathbf{g}_n}} S^{2m} u_n \right|^2. \end{aligned}$$

Taking also into account that

$$-\operatorname{div}(V \nabla S^{2m} \mathbf{g}) = -\operatorname{div} \left(\frac{S^{2m-1}(-\Delta)^{2m} \operatorname{div} \mathbf{g}}{S^{2m-1} \operatorname{div} \mathbf{g}} S^{2m-1} \operatorname{div} \mathbf{g} \right) = -S^{2m-1} \operatorname{div}(-\Delta)^{2m} \operatorname{div} \mathbf{g} = S^{2m}(-\Delta)^{2m+1} \mathbf{g},$$

we arrive, after shifting the indices, at the formula

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{S^{2m-1}(-\Delta)^{2m} \operatorname{div} \mathbf{g}_n}{S^{2m-1} \operatorname{div} \mathbf{g}_n} |\nabla S^{2m} u_n|^2 &= \sum_{n=2m+1}^{\infty} \frac{(-\Delta)^{2m+1} \mathbf{g}_n}{\mathbf{g}_n} |u_n|^2 \\ &\quad + \sum_{n=2m+1}^{\infty} \frac{(-\Delta)^{2m} \operatorname{div} \mathbf{g}_n}{\operatorname{div} \mathbf{g}_n} \left| \sqrt{\frac{\mathbf{g}_n}{\mathbf{g}_{n+1}}} u_{n+1} - \sqrt{\frac{\mathbf{g}_{n+1}}{\mathbf{g}_n}} u_n \right|^2. \end{aligned}$$

By (2.6), the last term coincides with $\mathcal{R}_0^{(2m+1)}(\mathbf{g}; u)$. Hence, when combined with (3.2), we obtain the identity (2.5) for $\ell = 2m + 1$.

c) Lastly, we assume that Theorem 1 holds for all $\ell \leq 2m + 1$ and verify it for $\ell = 2m + 2$. If we have $u \in \mathcal{H}_0^{2m+2}$ and $\mathbf{g} \in H^{2m+2}$ satisfying the assumption (A1) for $\ell = 2m + 2$, we can write

$$\sum_{n=\lceil \frac{2m+2}{2} \rceil}^{\infty} \left| (-\Delta)^{(2m+2)/2} u_n \right|^2 = \sum_{n=m+1}^{\infty} |(-\Delta)^{m+1} u_n|^2 = \sum_{n=m+1}^{\infty} |\nabla(-\Delta)^m(\operatorname{div} u)_n|^2$$

and apply Theorem 1 for $\ell = 2m + 1$, i.e. the induction hypothesis, with $\operatorname{div} u \in \mathcal{H}_0^{2m+1}$ and $\operatorname{div} \mathbf{g} \in H^{2m+1}$ satisfying the assumption (A1) for $\ell = 2m + 1$. Using the same methods as in b) we would obtain that the identity (2.5) holds for $\ell = 2m + 2$, thus the proof of Theorem 1 is complete. $\square$

3.2. Proof of Theorem 2. The claim is an immediate consequence of Theorem 1. It suffices to notice that the assumptions **(A2)** together with **(A1)** guarantee non-negativity of all the remainders on the right-hand side of (2.5). Assumption **(A2')** then implies positivity of the weight $\rho(\mathbf{g})$. The proof of Theorem 2 is complete. $\square$

3.3. Proof of Theorem 3. We check that, under the respective assumptions, the discrete Hardy weight $\rho(\mathbf{g}) := (-\Delta)^\ell \mathbf{g}/\mathbf{g}$ possesses all the three properties from Definition 1: criticality, non-attainability, and optimality at infinity. The properties will be gradually worked out in three separate steps.

The core idea of this proof is that all the remainders on the right-hand side of (2.5) vanish if we consider $u = \mathbf{g}$. However, $g \notin \mathcal{H}_0^\ell$ so we must find a suitable regularization of $\mathbf{g}$. Furthermore, the asymptotic properties of $\operatorname{div}^k \mathbf{g}$ turn out to be important to assert optimality of the weight $\rho(\mathbf{g})$. For this purpose we will state the following lemmas, whose proofs shall be postponed to Appendix A in order not to distract the reader from our main purpose. Additionally, for more concise formulas, we introduce the *middle operator* acting on $C(\mathbb{Z})$ defined as $M := (S + I)/2$, i.e.

$$Mu_n := \frac{u_n + u_{n+1}}{2}.$$

Lemma 6. *Let $k \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and g be a sequence admitting the asymptotic expansion*

$$g_n = \sum_{j=0}^k a_j n^{\alpha-j} + \mathcal{O}(n^{\alpha-k-1}), \quad n \rightarrow \infty, \quad (3.3)$$

for some $a_j \in \mathbb{R}$ and $a_0 \neq 0$. Then the following claims hold true.

(i) *For all $m \in \mathbb{Z}$, there exist coefficients $a_j^{(m)} \in \mathbb{R}$ such that*

$$S^m g_n = a_0 n^\alpha + \sum_{j=1}^k a_j^{(m)} n^{\alpha-j} + \mathcal{O}(n^{\alpha-k-1}), \quad n \rightarrow \infty.$$

(ii) *For all $m \in \mathbb{N}_0$, there exist coefficients $b_j^{(m)} \in \mathbb{R}$ with $b_0^{(m)} \neq 0$ such that*

$$M^m g_n = \sum_{j=0}^k b_j^{(m)} n^{\alpha-j} + \mathcal{O}(n^{\alpha-k-1}), \quad n \rightarrow \infty.$$

(iii) *For all $m \in \{0, \dots, k\}$, there exist coefficients $c_j^{(m)} \in \mathbb{R}$ with $c_m^{(m)} \neq 0$ such that*

$$\operatorname{div}^m g_n = \sum_{j=m}^k c_j^{(m)} n^{\alpha-j} + \mathcal{O}(n^{\alpha-k-1}), \quad n \rightarrow \infty.$$

The third statement of Lemma 6 asserts that, to some extent, the sequence $\operatorname{div}^k g$, for $\mathbf{g} \in C(\mathbb{Z})$ of the form (3.3), can be treated as a *derivative* $\mathbf{g}^{(k)}$. We will need another auxiliary claim of this sort – a generalization of the Mean Value Theorem, which is most likely known, but we will prove it the Section A.2 of Appendix A for completeness nonetheless.

Lemma 7. *Let $n \in \mathbb{Z}$, $N \in \mathbb{N}$ and $g \in C([n, n + N])$ such that $g \in C^N((n, n + N))$. Then there exists $\xi \in (n, n + N)$ such that*

$$\operatorname{div}^N g_n = g^{(N)}(\xi),$$

where we denoted $g_n := g(n)$.

The last auxiliary identity is the *Leibnitz formula* for the discrete divergence. The multiplication of sequences is to be understood point-wise.

Lemma 8. *For all $m \in \mathbb{N}_0$ and sequences $f, g \in C(\mathbb{Z})$, we have*

$$\operatorname{div}^m(fg) = \sum_{j=0}^m \binom{m}{j} (\operatorname{div}^j M^{m-j} f) (\operatorname{div}^{m-j} M^j g).$$

Now, we are ready to prove Theorem 3.

3.3.1. *Proof of criticality.* Suppose that the sequence $\tilde{\rho} = \{\tilde{\rho}_n\}_{n=\ell}^\infty$ is a discrete Hardy weight of order ℓ satisfying $\tilde{\rho}_n \geq \rho_n(\mathbf{g})$ for all $n \geq \ell$. Using identity (2.5) for the weight $\rho(\mathbf{g})$ together with the discrete Hardy inequality of order ℓ for the weight $\tilde{\rho}$, we find that

$$0 \leq \sum_{n=\ell}^{\infty} (\tilde{\rho}_n - \rho_n(\mathbf{g})) |u_n|^2 \leq \sum_{k=0}^{\ell-1} \mathcal{R}_k^{(\ell)}(\mathbf{g}; u) \quad (3.4)$$

for all $u \in \mathcal{H}_0^\ell$.

As was mentioned earlier, an important fact is that all the remainders are simultaneously annihilated if we set $u = \mathbf{g}$, i.e. $R_k^{(\ell)}(\mathbf{g}; \mathbf{g}) = 0$ for all $k \in \{0, \dots, \ell-1\}$. However, we cannot directly substitute $u = \mathbf{g}$ into (3.4) and conclude from here that $\tilde{\rho} = \rho(\mathbf{g})$ since $\mathbf{g} \notin \mathcal{H}_0^\ell$. This is an issue which is to be overcome by a suitable regularization of $\mathbf{g}$.

Fix an arbitrary $\varepsilon \in (0, 1/2)$ and a smooth function η such that $\eta \equiv 0$ on $(-\infty, \varepsilon)$ and $\eta \equiv 1$ on $(1-\varepsilon, \infty)$. Then for any $N \geq 2$, we introduce $u^N := \xi^N \mathbf{g}$, where $\xi^N := \xi^N(n)$ is a cut-off sequence define as

$$\xi^N(x) := \begin{cases} 1 & \text{if } x \leq N, \\ \eta\left(\frac{2 \ln N - \ln x}{\ln N}\right) & \text{if } N < x \leq N^2, \\ 0 & \text{if } x > N^2. \end{cases} \quad (3.5)$$

Notice that $\xi^N \rightarrow 1$ and hence $u^N \rightarrow \mathbf{g}$ point-wise as $N \rightarrow \infty$. With this choice of $u^N \in \mathcal{H}_0^\ell$, we will show that for all $k \in \{0, \dots, \ell-1\}$ we may estimate

$$0 \leq \mathcal{R}_k^{(\ell)}(\mathbf{g}; u^N) \lesssim \frac{1}{\ln N}, \quad (3.6)$$

where $\lesssim$ means the inequality $\leq$ up to a N -independent multiplicative constant. Invoking Fatou's lemma, we infer from (3.4) and (3.6) that

$$\sum_{n=\ell}^{\infty} (\tilde{\rho}_n - \rho_n(\mathbf{g})) \mathbf{g}_n^2 = \sum_{n=\ell}^{\infty} (\tilde{\rho}_n - \rho_n(\mathbf{g})) \lim_{N \rightarrow \infty} |u_n^N|^2 \leq \liminf_{N \rightarrow \infty} \sum_{n=\ell}^{\infty} (\tilde{\rho}_n - \rho_n(\mathbf{g})) |u_n^N|^2 = 0.$$

Bearing in mind, that all the terms in the sum on the left are non-negative and $\mathbf{g}_n > 0$ for all $n \geq \ell$ by **(A1)**, we may conclude that $\tilde{\rho}_n = \rho_n(\mathbf{g})$ for all $n \geq \ell$, and the proof of criticality of $\rho(\mathbf{g})$ will be complete.

It remains to verify the inequality (3.6). Recalling Remark 2.4, we use the notation

$$\mathcal{R}_k^{(\ell)}(\mathbf{g}; u) = \left\| R_k^{(\ell)}(\mathbf{g}) u \right\|^2 = \sum_{n=\ell-k}^{\infty} \left| R_k^{(\ell)}(\mathbf{g}) u_n \right|^2$$

for all $k \in \{0, \dots, \ell-1\}$, where the operators $R_k^{(\ell)}(\mathbf{g})$ are defined by (2.10). Next, we substitute $u = u^N = \xi^N \mathbf{g}$ into (2.10) and inspect the two factors of

$$\left| R_k^{(\ell)}(\mathbf{g}) u_n^N \right| = \sqrt{\frac{(-\Delta)^{\ell-1-k} \operatorname{div}^{k+1} \mathbf{g}_n}{\operatorname{div}^{k+1} \mathbf{g}_n}} \left| \sqrt{\frac{\operatorname{div}^k \mathbf{g}_n}{\operatorname{div}^k \mathbf{g}_{n+1}}} \operatorname{div}^k (\xi^N \mathbf{g})_{n+1} - \sqrt{\frac{\operatorname{div}^k \mathbf{g}_{n+1}}{\operatorname{div}^k \mathbf{g}_n}} \operatorname{div}^k (\xi^N \mathbf{g})_n \right|$$

separately.

For the first factor, by assumption **(A3)** and claims (i) and (iii) of Lemma 6, we observe that

$$\sqrt{\frac{(-\Delta)^{\ell-1-k} \operatorname{div}^{k+1} \mathbf{g}_n}{\operatorname{div}^{k+1} \mathbf{g}_n}} = \sqrt{\frac{(-1)^{\ell-1-k} \operatorname{div}^{2\ell-k-1} S^{k+1-\ell} \mathbf{g}_n}{\operatorname{div}^{k+1} \mathbf{g}_n}} \lesssim \sqrt{\frac{n^{\ell-1/2-(2\ell-k-1)}}{n^{\ell-1/2-(k+1)}}} = n^{k+1-\ell}$$

for any $n \geq \ell$ and $k \in \{0, \dots, \ell-1\}$, where the unspecified constant is n -independent but may depend on k or ℓ .

Similarly, we find that

$$\sqrt{\frac{\operatorname{div}^k \mathbf{g}_n}{\operatorname{div}^k \mathbf{g}_{n+1}}} \lesssim 1,$$

and therefore the second factor can be estimated as

$$\left| \sqrt{\frac{\operatorname{div}^k \mathbf{g}_n}{\operatorname{div}^k \mathbf{g}_{n+1}}} \operatorname{div}^k (\xi^N \mathbf{g})_{n+1} - \frac{\operatorname{div}^k \mathbf{g}_{n+1}}{\operatorname{div}^k \mathbf{g}_n} \operatorname{div}^k (\xi^N \mathbf{g})_n \right| \lesssim \left| \operatorname{div}^k (\xi^N \mathbf{g})_{n+1} - \frac{\operatorname{div}^k \mathbf{g}_{n+1}}{\operatorname{div}^k \mathbf{g}_n} \operatorname{div}^k (\xi^N \mathbf{g})_n \right|.$$

Applying Lemma 8 in the last expression, we can rewrite it as

$$\begin{aligned} & \left| \sum_{j=0}^k \binom{k}{j} (\operatorname{div}^j M^{k-j} \mathbf{g})_{n+1} (\operatorname{div}^{k-j} M^j \xi^N)_{n+1} - \frac{\operatorname{div}^k \mathbf{g}_{n+1}}{\operatorname{div}^k \mathbf{g}_n} \sum_{j=0}^k \binom{k}{j} (\operatorname{div}^j M^{k-j} \mathbf{g})_n (\operatorname{div}^{k-j} M^j \xi^N)_n \right| \\ & \leq \sum_{j=0}^k \binom{k}{j} (\operatorname{div}^j M^{k-j} \mathbf{g})_{n+1} \left| (\operatorname{div}^{k-j} M^j \xi^N)_{n+1} - \frac{\operatorname{div}^k \mathbf{g}_{n+1}}{\operatorname{div}^k \mathbf{g}_n} \frac{\operatorname{div}^j M^{k-j} \mathbf{g}_n}{\operatorname{div}^j M^{k-j} \mathbf{g}_{n+1}} (\operatorname{div}^{k-j} M^j \xi^N)_n \right|. \end{aligned}$$

Yet another application of **(A3)** and formulas from Lemma 6 reveals that

$$(\operatorname{div}^j M^{k-j} \mathbf{g})_{n+1} \lesssim n^{\ell-1/2-j} \quad \text{and} \quad \frac{\operatorname{div}^k \mathbf{g}_{n+1}}{\operatorname{div}^k \mathbf{g}_n} \frac{\operatorname{div}^j M^{k-j} \mathbf{g}_n}{\operatorname{div}^j M^{k-j} \mathbf{g}_{n+1}} = \frac{1 + \mathcal{O}(1/n)}{1 + \mathcal{O}(1/n)} \frac{1 + \mathcal{O}(1/n)}{1 + \mathcal{O}(1/n)} = 1 + p_n^{(k,j)},$$

where $p_n^{(k,j)} = \mathcal{O}(1/n)$, as $n \rightarrow \infty$, i.e. $p_n^{(k,j)} \lesssim 1/n$. Altogether, we deduce the upper bound

$$\left| R_k^{(\ell)}(\mathbf{g}) u_n^N \right| \lesssim n^{k+1/2} \sum_{j=0}^k \binom{k}{j} n^{-j} \left(\left| (\operatorname{div}^{k+1-j} M^j \xi^N)_n \right| + \left| p_n^{(k,j)} (\operatorname{div}^{k-j} M^j \xi^N)_n \right| \right) \quad (3.7)$$

for all $N \geq 2$, $n \geq \ell$, and $k \in \{0, \dots, \ell-1\}$, where the unspecified multiplicative constant does not depend on n nor N . In order to complete the proof, we have to estimate also the terms with $\operatorname{div}^m \xi$ in (3.7). To this end, note that for all $x \in [N, N^2]$, we have

$$(\xi^N)'(x) = -\eta' \left(\frac{2 \ln N - \ln x}{\ln N} \right) \frac{1}{x \ln N},$$

thus $|(\xi^N)'(x)| \lesssim 1/(x \ln N)$, where we estimated η' with $\max_{x \in [0,1]} |\eta'(x)|$. By induction, it is straightforward to generalize this bound to higher order derivatives, i.e.

$$\left| (\xi^N)^{(m)}(x) \right| \lesssim \frac{1}{x^m \ln N} \quad (3.8)$$

for all $m \in \mathbb{N}$ and $x > 0$. Consequently, Lemma 7 implies that

$$|\operatorname{div}^m \xi_n^N| \lesssim \frac{1}{n^m \ln N},$$

which is true for any $m, n \in \mathbb{N}$ and $N \geq 2$. As the right-hand side is a decreasing function of n , we also have

$$|\operatorname{div}^m M^j \xi^N| \lesssim \frac{1}{n^m \ln N}$$

for any $j \in \mathbb{N}_0$, from which we infer the needed estimates in (3.7) getting

$$\left| R_k^{(\ell)}(\mathbf{g}) u_n^N \right| \lesssim n^{k+1/2} \sum_{j=0}^k \binom{k}{j} n^{-j} \left(\frac{1}{n^{k+1-j} \ln N} + \frac{1}{n} \frac{1}{n^{k-j} \ln N} \right) \lesssim \frac{1}{\sqrt{n} \ln N} \quad (3.9)$$

for all $N \geq 2$, $n \geq \ell$, and $k \in \{0, \dots, \ell-1\}$. Finally, for sufficiently large N we may estimate

$$\mathcal{R}_k^{(\ell)}(\mathbf{g}; u) = \sum_{n=N-k}^{N^2} \left| R_k^{(\ell)}(\mathbf{g}) u_n^N \right|^2 \lesssim \frac{1}{\ln^2 N} \sum_{n=N-k}^{N^2} \frac{1}{n} \lesssim \frac{1}{\ln^2 N} \int_N^{N^2} \frac{1}{n} \, dn = \frac{1}{\ln N}$$

for all $k \in \{0, \dots, \ell-1\}$ arriving at the desired result (3.6). The proof of criticality is complete.

3.3.2. Proof of non-attainability. For the proof of non-attainability, we first state the following auxiliary claim which asserts that under certain assumptions, the identity from Theorem 1 extends from $\mathcal{H}_0^\ell$ to all sequences in H^ℓ for which the left-hand side of (2.5) is finite.

Proposition 9. *Let $\ell \in \mathbb{N}$. Suppose that assumptions **(A1)**, **(A2)**, and **(A2')** hold. Then the identity (2.5) extends from $\mathcal{H}_0^\ell$ to all sequences from the space*

$$\mathcal{D}^\ell := \left\{ u \in H^\ell \mid \|(-\Delta)^{\ell/2} u\| < \infty \right\}.$$

Proof. We will prove Proposition 9 in three steps.

a) First, we show that the range of $(-\Delta)^{\ell/2} \upharpoonright_{\mathcal{H}_0^\ell}$ is dense in $\mathcal{H}^{\lceil \ell/2 \rceil}$. It follows from the definition of $(-\Delta)^{\ell/2}$ that for any $u \in \mathcal{H}_0^\ell$, we have $(-\Delta)^{\ell/2} u_n = 0$ for all $n < \lceil \ell/2 \rceil$, i.e. $(-\Delta)^{\ell/2} u \in H^{\lceil \ell/2 \rceil}$. Moreover, from the boundedness of the operator $(-\Delta)^{\ell/2}$, we infer that $(-\Delta)^{\ell/2} u \in \ell^2(\mathbb{Z})$, hence $(-\Delta)^{\ell/2} u \in \mathcal{H}^{\lceil \ell/2 \rceil}$. In order to prove the density, we check that if $v \in \mathcal{H}^{\lceil \ell/2 \rceil}$ satisfies

$$\langle v, (-\Delta)^{\ell/2} u \rangle = 0$$

for all $u \in \mathcal{H}_0^\ell$, then $v \equiv 0$. According to the parity of ℓ , we distinguish two cases. Let $\ell = 2m \in 2\mathbb{N}$, then

$$\langle v, (-\Delta)^{\ell/2} u \rangle = \langle v, (-\Delta)^m u \rangle = \langle (-\Delta)^m v, u \rangle$$

for all $u \in \mathcal{H}_0^{2m}$. In particular, by taking $u = \delta_n$, i.e. $u_k = \delta_{nk}$, where δ is the Kronecker delta, for $n \geq 2m$, we observe that v solves the linear difference equation of order $2m$ with constant coefficients

$$(-\Delta)^m v_n = \sum_{j=-m}^m \binom{2m}{m+j} (-1)^j v_{n+j} = 0$$

for all $n \geq 2m$. It is easy to show that the fundamental system of the above linear difference equation consists of sequences $1, n, \dots, n^{2m-1}$. Therefore, we find that

$$v_n = \sum_{j=0}^{2m-1} c_j n^j$$

for all $n \geq m$ and some $c_j \in \mathbb{C}$. Taking also into account that $v \in \ell^2(\mathbb{Z})$, necessarily, we have $c_j = 0$ for all $j \in \{0, \dots, 2m-1\}$ and since $v \in H^m$, we conclude that $v \equiv 0$. The argument for $\ell = 2m+1 \in 2\mathbb{N}_0 + 1$ is analogous. In this case, we have

$$\langle v, (-\Delta)^{\ell/2} u \rangle = \langle v, \nabla(-\Delta)^m u \rangle = -\langle (-\Delta)^m \operatorname{div} v, u \rangle = 0$$

for all $u \in \mathcal{H}_0^{2m+1}$. Again, we consider $u = \delta_n$ for $n \geq 2m+1$, from which we infer that v solves the linear difference equation

$$-(-\Delta)^m \operatorname{div} v_n = \sum_{j=-m}^{m+1} \binom{2m+1}{m+j} (-1)^j v_{n+j} = 0$$

for all $n \geq 2m+1$, whose general solution is given by the linear combination

$$v_n = \sum_{j=0}^{2m} c_j n^j$$

for all $n \geq m+1$ and some $c_j \in \mathbb{C}$. The same argumentation as in the case where ℓ is even then implies $v \equiv 0$.

b) Pick arbitrary $v \in \mathcal{H}^{[\ell/2]}$. By a), there exists a sequence $u^N \in \mathcal{H}_0^\ell$ such that $(-\Delta)^{\ell/2} u^N \rightarrow v$, as $N \rightarrow \infty$, in the norm of $\ell^2(\mathbb{Z})$. In particular, the sequence $\{(-\Delta)^{\ell/2} u^N\}_{N=1}^\infty$ is Cauchy in $\ell^2(\mathbb{Z})$. Recalling Remark 2.4, assumptions **(A1)** and **(A2)** imply that the remainders in (2.5) can be interpreted as a (non-negative) norm of $R_k^{(\ell)}(\mathbf{g})u^N$ and assumption **(A2')** guarantees positivity of the weight $\rho_n(\mathbf{g})$. Thus, identity (2.5) asserts inequalities

$$\left\| \sqrt{\rho(\mathbf{g})}(u^N - u^M) \right\| \leq \left\| (-\Delta)^{\ell/2}(u^N - u^M) \right\| \quad \text{and} \quad \left\| R_k^{(\ell)}(\mathbf{g})(u^N - u^M) \right\| \leq \left\| (-\Delta)^{\ell/2}(u^N - u^M) \right\|$$

for all $k \in \{0, \dots, \ell-1\}$ and $N, M \in \mathbb{N}$. Consequently, the sequences $\{\sqrt{\rho(\mathbf{g})}u^N\}_{N=1}^\infty$ and $\{R_k^{(\ell)}(\mathbf{g})u^N\}_{N=1}^\infty$ are also Cauchy and therefore convergent in $\ell^2(\mathbb{Z})$. Let us denote w the ℓ^2 -limit of $\sqrt{\rho(\mathbf{g})}u^N$, as $N \rightarrow \infty$. Clearly, $w \in \mathcal{H}^\ell$ and $\sqrt{\rho_n(\mathbf{g})}u^N \rightarrow w_n$, as $N \rightarrow \infty$, for every $n \in \mathbb{Z}$. Since $\rho_n(\mathbf{g}) > 0$ for all $n \geq \ell$, there exists $u \in H^\ell$ such that $w = \sqrt{\rho(\mathbf{g})}u$ and $u_n^N \rightarrow u_n$, as $N \rightarrow \infty$, for all $n \in \mathbb{Z}$. Moreover, the limits of all the above mentioned ℓ^2 -convergent sequences must coincide with their point-wise limits, i.e.

$$(-\Delta)^{\ell/2} u^N \rightarrow (-\Delta)^{\ell/2} u, \quad \sqrt{\rho(\mathbf{g})}u^N \rightarrow \sqrt{\rho(\mathbf{g})}u, \quad R_k^{(\ell)}(\mathbf{g})u^N \rightarrow R_k^{(\ell)}(\mathbf{g})u$$

in $\ell^2(\mathbb{Z})$ as $N \rightarrow \infty$. In particular, we have $v = (-\Delta)^{\ell/2} u$, hence $u \in \mathcal{D}^\ell$. Tending $N \rightarrow \infty$ in the identity (2.5) for u_N , i.e. in

$$\left\| (-\Delta)^{\ell/2} u^N \right\|^2 = \left\| \sqrt{\rho(\mathbf{g})}u^N \right\|^2 + \sum_{k=0}^{\ell-1} \left\| R_k^{(\ell)}(\mathbf{g})u^N \right\|^2,$$

yields

$$\left\| (-\Delta)^{\ell/2} u \right\|^2 = \left\| \sqrt{\rho(\mathbf{g})}u \right\|^2 + \sum_{k=0}^{\ell-1} \left\| R_k^{(\ell)}(\mathbf{g})u \right\|^2.$$

c) Now, pick arbitrary $\tilde{u} \in \mathcal{D}^\ell$. By definition, $(-\Delta)^{\ell/2} \tilde{u} \in \mathcal{H}^{[\ell/2]}$. Part b) applied to $(-\Delta)^{\ell/2} \tilde{u}$ implies the existence of a sequence $u \in H^\ell$, such that the above identity holds and $(-\Delta)^{\ell/2} \tilde{u} = (-\Delta)^{\ell/2} u$. In order to finish the prove, it remains to show that for two sequences $u, \tilde{u} \in H^\ell$, we have the implication

$$(-\Delta)^{\ell/2} \tilde{u} = (-\Delta)^{\ell/2} u \implies \tilde{u} = u.$$

By linearity, it suffices to prove that for any $w \in H^\ell$ such that $(-\Delta)^{\ell/2}w = 0$, we have $w \equiv 0$. This can be easily seen from the fact that w solves the linear difference equation

$$(-\Delta)^{\ell/2}w_n = 0$$

for all $n \in \mathbb{Z}$ with the boundary condition $w_n = 0$ for all $n < \ell$. Recursively, we obtain $w_n = 0$ also for all $n \geq \ell$, thus proving Proposition 9. $\square$

Now, suppose that $u \in H^\ell$ fulfills (2.9) as equality whose (both) sides are finite. In view of Proposition 9, assumptions **(A1)**, **(A2)**, and **(A2')** imply that $\mathcal{R}_k^{(\ell)}(\mathbf{g}; u) = 0$ for all $k \in \{0, \dots, \ell - 1\}$. In particular, for $k = 0$ we derive a necessary condition (which is in fact sufficient)

$$\frac{(-\Delta)^{\ell-1} \operatorname{div} \mathbf{g}_n}{\operatorname{div} \mathbf{g}_n} \left| \sqrt{\frac{\mathbf{g}_n}{\mathbf{g}_{n+1}}} u_{n+1} - \sqrt{\frac{\mathbf{g}_{n+1}}{\mathbf{g}_n}} u_n \right|^2 = 0$$

for all $n \geq \ell$, see (2.6). By the additional assumption **(A3')** and also **(A1)**, the prefactor is strictly positive and therefore the sequence u is a solution of the difference equation of the first order

$$\sqrt{\frac{\mathbf{g}_n}{\mathbf{g}_{n+1}}} u_{n+1} - \sqrt{\frac{\mathbf{g}_{n+1}}{\mathbf{g}_n}} u_n = 0$$

for all $n \geq \ell$. However, this equation has the only solution $\mathbf{g}$ up to a multiplicative constant, i.e. $u_n = c \mathbf{g}_n$ for some constant $c \in \mathbb{C}$ and $n \geq \ell$. Obviously, for such u it holds that $\mathcal{R}_k^{(\ell)}(\mathbf{g}; u) = 0$ for all $k \in \{0, \dots, \ell - 1\}$. By assumption **(A3)** and Lemma 6, we have

$$\rho_n(\mathbf{g}) \gtrsim \frac{n^{\ell-1/2-2\ell}}{n^{\ell-1/2}} = n^{-2\ell}$$

and due to the asymptotic behaviour of $\mathbf{g}_n$ we may conclude that

$$\rho_n(\mathbf{g}) \mathbf{g}_n^2 \gtrsim \frac{1}{n^{2\ell}} n^{2\ell-1} = \frac{1}{n} \quad (3.10)$$

for all $n \geq \ell$. Consequently, the right-hand side of (2.9) can be estimated as

$$\sum_{n=\ell}^{\infty} \rho_n(\mathbf{g}) |u_n|^2 \gtrsim \sum_{n=\ell}^{\infty} \frac{1}{n} = \infty,$$

whenever $c \neq 0$, hence the proof of non-attainability is complete.

3.3.3. Proof of optimality near infinity. Fix $M \in \mathbb{N}$. As follows from (2.4), this property can be proven by finding a sequence $\{u^N\}_{N \geq M} \subset H_0^M(\mathbb{N}) \setminus \{0\}$ such that

$$\lim_{N \rightarrow \infty} \frac{\sum_{k=0}^{\ell-1} \mathcal{R}_k^{(\ell)}(\mathbf{g}; u^N)}{\sum_{n=\ell}^{\infty} \rho_n(\mathbf{g}) |u_n^N|^2} = 0. \quad (3.11)$$

Such sequence can be obtained by slightly modifying the regularization (3.5) used in the proof of criticality in Section 3.3.1. We define $u^N := \xi^N \mathbf{g}$, where, this time, ξ^N is given by

$$\xi^N(x) := \begin{cases} 0 & \text{if } x \leq N, \\ \eta\left(\frac{\ln x - \ln N}{\ln N}\right) & \text{if } N < x \leq N^2, \\ 1 & \text{if } N^2 < x \leq 2N^2, \\ \eta\left(\frac{\ln(2N^3) - \ln x}{\ln N}\right) & \text{if } 2N^2 < x \leq 2N^3, \\ 0 & \text{if } x > 2N^3, \end{cases}$$

where function η has the same properties as in (3.5). Then, indeed, $u^N \in \mathcal{H}_0^M \setminus \{0\}$ for all $N \geq M$. Note that the inequality (3.8) still holds for any $m \in \mathbb{N}$ and $x > 0$. Therefore, the same estimates as in Section 3.3.1 apply and we get

$$\mathcal{R}_k^{(\ell)}(\mathbf{g}; u^N) = \sum_{n=N-k}^{N^2} \left| R_k^{(\ell)}(\mathbf{g}) u_n^N \right|^2 + \sum_{n=2N^2-k}^{2N^3} \left| R_k^{(\ell)}(\mathbf{g}) u_n^N \right|^2 \lesssim \frac{1}{\ln^2 N} \left(\sum_{n=N-k}^{N^2} \frac{1}{n} + \sum_{n=2N^2-k}^{2N^3} \frac{1}{n} \right) \lesssim \frac{1}{\ln N}$$

for all N sufficiently large and $k \in \{0, \dots, \ell - 1\}$.

On the other hand, seeing (3.10), we may estimate the denominator in (3.11) from below as

$$\sum_{n=\ell}^{\infty} \rho_n(\mathbf{g}) |u_n^N|^2 \geq \sum_{n=N^2}^{2N^2} \rho_n(\mathbf{g}) \mathbf{g}_n^2 \gtrsim \sum_{n=N^2}^{2N^2} \frac{1}{n} \geq \int_{N^2}^{2N^2} \frac{1}{n} dn = \ln 2$$

for all N sufficiently large. Altogether, the above estimates immediately imply (3.11), completing the proof of optimality near infinity and subsequently also the proof of Theorem 3. $\square$

3.4. Proof of Theorem 4. For the parameter sequence $\mathbf{g}^{(\ell)}$ defined by (2.12), we verify below that

- a) $\operatorname{div}^k \mathbf{g}_n^{(\ell)} > 0$ for all $n \geq \ell - k$ and $k \in \{0, \dots, \ell - 1\}$;
- b) $(-\Delta)^{\ell-k} \operatorname{div}^k \mathbf{g}_n^{(\ell)} > 0$ for all $n \geq \ell - k$ and $k \in \{0, \dots, \ell - 1\}$.

Claim a) together with the obvious fact $\mathbf{g}^{(\ell)} \in H^\ell$ means that $\mathbf{g}^{(\ell)}$ fulfills assumption **(A1)**. Claim b) implies that $\mathbf{g}^{(\ell)}$ satisfies assumptions **(A2)**, **(A2')**, and **(A3')**. As it follows trivially from definition (2.12) that $\mathbf{g}^{(\ell)}$ satisfies also **(A3)**, Theorem 3 asserts that the sequence $\rho^{(\ell)}$ given by (2.13) is an optimal discrete Hardy weight of order ℓ .

a) First we show that $\operatorname{div}^\ell \mathbf{g}_n^{(\ell)} > 0$ for all $n \geq 0$. For $x \geq 0$, let us denote

$$\mathbf{g}^{(\ell)}(x) := \sqrt{x} \prod_{j=1}^{\ell-1} (x - j).$$

Then $\mathbf{g}_n^{(\ell)} = \mathbf{g}^{(\ell)}(n)$ for all $n \in \mathbb{N}_0$. By [2, Eq. (26.8.7)], we have

$$\mathbf{g}^{(\ell)}(x) = \sum_{j=1}^{\ell} s(\ell, j) x^{j-1/2}, \quad (3.12)$$

where $s(\ell, j)$ are the Stirling numbers of the first kind defined by formula (2.14). Notice that $(-1)^{\ell+j} s(\ell, j) > 0$ for all $j \in \{1, \dots, \ell\}$. It follows from Lemma 7, that for any $n \in \mathbb{N}_0$ there exists $\xi \in (n, n + \ell)$ such that

$$\operatorname{div}^\ell \mathbf{g}_n^{(\ell)} = \frac{d^\ell \mathbf{g}^{(\ell)}}{dx^\ell}(\xi).$$

Moreover, for any $x > 0$, we have

$$\frac{d^\ell \mathbf{g}^{(\ell)}}{dx^\ell}(x) = \sum_{j=1}^{\ell} s(\ell, j) \frac{d^\ell}{dx^\ell} x^{j-1/2} = \sum_{j=1}^{\ell} b_j^{(\ell)} x^{j-1/2-\ell},$$

where

$$b_j^{(\ell)} := (-1)^{\ell+j} s(\ell, j) \prod_{k=1}^{\ell} \left| j + \frac{1}{2} - k \right| > 0$$

for all $j \in \{1, \dots, \ell\}$. Therefore, we infer that $\operatorname{div}^\ell \mathbf{g}_n^{(\ell)} > 0$ for all $n \in \mathbb{N}_0$, indeed.

Next, notice that, since $\mathbf{g}^{(\ell)} \in H^\ell$, we have

$$\operatorname{div}^k \mathbf{g}_{\ell-k}^{(\ell)} = \mathbf{g}_\ell^{(\ell)} = \sqrt{\ell}(\ell-1)! > 0 \quad (3.13)$$

for every $k \in \{0, \dots, \ell\}$. By definition of the discrete divergence, $\operatorname{div}^\ell \mathbf{g}^{(\ell)} > 0$ on $\mathbb{N}_0$ means that the sequence $\operatorname{div}^{\ell-1} \mathbf{g}^{(\ell)}$ is strictly increasing on $\mathbb{N}_0$. Since $\operatorname{div}^{\ell-1} \mathbf{g}_1^{(\ell)} > 0$ by (3.13), we conclude that $\operatorname{div}^{\ell-1} \mathbf{g}_n^{(\ell)} > 0$ for all $n \geq 1$. Iteration of this argument yields claim a).

b) We make use of Lemma 7 once more. Since

$$(-\Delta)^{\ell-k} \operatorname{div}^k \mathbf{g}_n = (-1)^{\ell-k} \operatorname{div}^{2\ell-k} \mathbf{g}_{n-\ell+k}^{(\ell)},$$

Lemma 7 implies that, for any $n \geq \ell - k$ and $k \in \{0, \dots, \ell\}$, there is $\xi > 0$ such that

$$(-\Delta)^{\ell-k} \operatorname{div}^k \mathbf{g}_n = (-1)^{\ell-k} \frac{d^{2\ell-k} \mathbf{g}^{(\ell)}}{dx^{2\ell-k}}(\xi)$$

Similarly as in part a) of this proof, we find this time that

$$(-1)^{\ell-k} \frac{d^{2\ell-k} \mathbf{g}^{(\ell)}}{dx^{2\ell-k}}(x) = (-1)^{\ell-k} \sum_{j=1}^{\ell} s(\ell, j) \frac{d^{2\ell-k}}{dx^{2\ell-k}} x^{j-1/2} = \sum_{j=1}^{\ell} c_j^{(\ell)} x^{j-1/2-2\ell+k},$$

where

$$c_j^{(\ell)} := (-1)^{\ell-k} s(\ell, j) (-1)^{2\ell-k-j} \prod_{k=1}^{2\ell-k} \left| j + \frac{1}{2} - k \right| = (-1)^{\ell+j} s(\ell, j) \prod_{k=1}^{2\ell-k} \left| j + \frac{1}{2} - k \right| > 0$$

for every $j \in \{1, \dots, \ell\}$. Consequently,

$$(-1)^{\ell-k} \frac{d^{2\ell-k} \mathbf{g}^{(\ell)}}{dx^{2\ell-k}}(x) > 0$$

for all $x > 0$ and $k \in \{0, \dots, \ell\}$, from which claim b) readily follows. The proof of Theorem 4 is complete. $\square$

3.5. Proof of Theorem 5. First, with the aid of the generalized binomial theorem, we find for all $\nu \in \mathbb{R}$ and $n > \ell$ that

$$\begin{aligned} (-\Delta)^\ell n^\nu &= \sum_{j=-\ell}^{\ell} \binom{2\ell}{\ell+j} (-1)^j (n+j)^\nu = n^\nu \sum_{j=-\ell}^{\ell} \binom{2\ell}{\ell+j} (-1)^j \left(1 + \frac{j}{n}\right)^\nu \\ &= n^\nu \sum_{j=-\ell}^{\ell} \binom{2\ell}{\ell+j} (-1)^j \sum_{m=0}^{\infty} \binom{\nu}{m} \frac{j^m}{n^m} = n^\nu \sum_{m=0}^{\infty} \binom{\nu}{m} \left[\sum_{j=-\ell}^{\ell} \binom{2\ell}{\ell+j} (-1)^j j^m \right] \frac{1}{n^m}. \end{aligned}$$

Recalling the definition (2.16) and the property (2.17), we arrive at the identity

$$(-\Delta)^\ell n^\nu = \sum_{m=2\ell}^{\infty} \binom{\nu}{m} \frac{X_m^{(\ell)}}{n^{m-\nu}} \quad (3.14)$$

for all $\nu \in \mathbb{R}$ and $n > \ell$. Moreover, if $\nu > 0$, the convergence of the series in (3.14) can be extended to all $n \geq \ell$ by inspection of the asymptotic behavior of the summand. Indeed, one deduces from (2.16) and the Stirling formula, see [11], that

$$X_{2m}^{(\ell)} \sim 2(-1)^\ell \ell^{2m} \quad \text{and} \quad \binom{\nu}{m} \sim \frac{1}{\Gamma(-\nu)} \frac{(-1)^m}{m^{\nu+1}}$$

as $m \rightarrow \infty$. Therefore, the non-vanishing even summands of (3.14) behave as

$$\binom{\nu}{2m} \frac{X_{2m}^{(\ell)}}{n^{2m-\nu}} \sim \frac{2n^\nu}{\Gamma(-\nu)} \frac{(-1)^\ell}{(2m)^{\nu+1}} \left(\frac{\ell}{n}\right)^{2m},$$

as $m \rightarrow \infty$ and the series in (3.14) converges even for $n = \ell$.

3.5.1. Proof of claim (i). Recalling that

$$\mathfrak{g}_n^{(\ell)} = \sum_{j=1}^{\ell} s(\ell, j) n^{j-1/2}$$

for all $n \geq 0$, see (3.12), and invoking the expansion (3.14), we find that

$$\begin{aligned} (-\Delta)^\ell \mathfrak{g}_n^{(\ell)} &= \sum_{j=1}^{\ell} s(\ell, j) (-\Delta)^\ell n^{j-1/2} = \sum_{j=1}^{\ell} s(\ell, j) \sum_{m=2\ell}^{\infty} \binom{j-1/2}{m} \frac{X_m^{(\ell)}}{n^{m-j+1/2}} \\ &= n^{\ell-1/2} \sum_{m=2\ell}^{\infty} \sum_{j=0}^{\ell-1} \binom{\ell-j-1/2}{m} s(\ell, \ell-j) \frac{X_m^{(\ell)}}{n^{m+j}} \end{aligned}$$

for all $n \geq \ell$. Therefore, for the weight $\rho^{(\ell)}$, we have

$$\begin{aligned} \rho_n^{(\ell)} &= \frac{(-\Delta)^\ell \mathfrak{g}_n^{(\ell)}}{\mathfrak{g}_n^{(\ell)}} = \frac{n^{\ell-1}}{(n-1) \dots (n-\ell+1)} \sum_{m=2\ell}^{\infty} \sum_{j=0}^{\ell-1} \binom{\ell-j-1/2}{m} s(\ell, \ell-j) \frac{X_m^{(\ell)}}{n^{m+j}} \\ &= \frac{n^{\ell-1}}{(n-1) \dots (n-\ell+1)} \sum_{m=2\ell}^{\infty} \sum_{j=m}^{\infty} \binom{\ell+m-j-1/2}{m} s(\ell, \ell+m-j) \frac{X_m^{(\ell)}}{n^j} \\ &= \frac{n^{\ell-1}}{(n-1) \dots (n-\ell+1)} \sum_{j=2\ell}^{\infty} \left[\sum_{m=2\ell}^j \binom{\ell+m-j-1/2}{m} s(\ell, \ell+m-j) X_m^{(\ell)} \right] \frac{1}{n^j} \end{aligned}$$

for all $n \geq \ell$. Seeing also the definition (2.19), we infer that

$$\rho_n^{(\ell)} = \frac{n^{\ell-1}}{(n-1) \dots (n-\ell+1)} \sum_{j=2\ell}^{\infty} \frac{r_j^{(\ell)}}{n^j}$$

for all $n \geq \ell$. The final step is to expand the prefactor in front of the sum in the above expression in terms of negative powers of n , which is to be done using the Stirling numbers of the second kind (2.15). Namely, by [2, Eq. (26.8.11)], we have

$$\frac{n^{\ell-1}}{(n-1) \dots (n-\ell+1)} = \sum_{j=0}^{\infty} \frac{S(j+\ell-1, \ell-1)}{n^j}$$

for all $n \geq \ell \geq 1$. Altogether, we find that

$$\begin{aligned}\rho_n^{(\ell)} &= \sum_{j=0}^{\infty} \frac{S(j+\ell-1, \ell-1)}{n^j} \sum_{j=0}^{\infty} \frac{r_{j+2\ell}^{(\ell)}}{n^{j+2\ell}} = \sum_{k=0}^{\infty} \left[\sum_{m=0}^k S(k-m+\ell-1, \ell-1) r_{m+2\ell}^{(\ell)} \right] \frac{1}{n^{k+2\ell}} \\ &= \sum_{k=2\ell}^{\infty} \left[\sum_{m=2\ell}^k S(k-m+\ell-1, \ell-1) r_m^{(\ell)} \right] \frac{1}{n^k}\end{aligned}$$

for all $n \geq \ell$, from which we extract the formula (2.21) for the coefficients $A_k^{(\ell)}$, thus completing the proof of claim (i).

3.5.2. *Proof of claim (ii).* For $\ell = 1$, the claim is an immediate consequence of the explicitly known expansion

$$\rho_n^{(1)} = \rho_n^{\text{KPP}} = \sum_{k=1}^{\infty} \binom{4k}{2k} \frac{1}{(4k-1)2^{4k-1}} \frac{1}{n^{2k}} \quad (3.15)$$

for all $n \geq 1$, from which we deduce that

$$A_{2k+1}^{(1)} = 0 \quad \text{and} \quad A_{2k}^{(1)} = \binom{4k}{2k} \frac{1}{(4k-1)2^{4k-1}} > 0$$

for all $k \in \mathbb{N}$.

Suppose $\ell \geq 2$. Recalling the definition (2.15), we have $S(k-m+\ell-1, \ell-1) > 0$ for all $\ell \geq 2$ and $k \geq m$, hence in order to prove that $A_k^{(\ell)} > 0$ for all $k \geq 2\ell$, it suffices to show, that $r_m^{(\ell)} > 0$ for all $m \geq 2\ell$ which is done in the rest of this part. Recalling formulas (2.14) and (2.18), we find by inspection of the sign of each of the three terms in the sum from (2.19) that

$$(-1)^\ell X_j^{(\ell)} \geq 0, \quad (-1)^{j+m} s(\ell, \ell+j-m) \geq 0, \quad (-1)^{\ell+m} \binom{\ell+j-m-1/2}{j} \geq 0 \quad (3.16)$$

for all $m \geq 2\ell$ and $m \geq j \geq 2\ell$. Taking also into account that $X_j^{(\ell)} = 0$ whenever j is odd, we see that each summand from the sum for coefficients $r_m^{(\ell)}$ in (2.19) is non-negative. Consequently, $r_m^{(\ell)} \geq 0$ for all $m \geq 2\ell$. Moreover, we can estimate $r_m^{(\ell)}$ from below by the last non-vanishing summand which corresponds to index $j = m$ if m is even and $j = m-1$ if m is odd. If $m \geq 2\ell$ is even, then

$$r_m^{(\ell)} \geq \binom{\ell-1/2}{m} s(\ell, \ell) X_m^{(\ell)} = \left| \binom{\ell-1/2}{m} X_m^{(\ell)} \right| > 0,$$

by (2.18). If $m \geq 2\ell$ is odd, then

$$r_m^{(\ell)} \geq \binom{\ell-3/2}{m-1} s(\ell, \ell-1) X_{m-1}^{(\ell)} = \left| \binom{\ell-3/2}{m-1} \binom{\ell}{2} X_{m-1}^{(\ell)} \right| > 0,$$

by (2.18) again and the fact that $s(\ell, \ell-1) = -\ell(\ell-1)/2$. In total, we verify that $r_m^{(\ell)} > 0$ for all $m \geq 2\ell$ and the proof of claim (ii) is complete.

3.5.3. *Proof of claim (iii).* The case for $\ell = 1$ is an immediate consequence of the expansion (3.15). Suppose $n \geq \ell \geq 2$. The inequality

$$\frac{(-\Delta)^\ell n^{\ell-1/2}}{n^{\ell-1/2}} > \left(\frac{1}{2}\right)_\ell \frac{1}{n^{2\ell}}$$

has been already proven in [4]. It can be also deduced by using (3.14) with $\nu = \ell - 1/2$ and noticing that each summand corresponding to m odd is vanishing while each summand corresponding to m even is positive, see (3.16). Then we may estimate

$$\frac{(-\Delta)^\ell n^{\ell-1/2}}{n^{\ell-1/2}} > \binom{\ell-1/2}{2\ell} \frac{X_{2\ell}^{(\ell)}}{n^{2\ell}} = \left(\frac{1}{2}\right)_\ell \frac{1}{n^{2\ell}}$$

by (2.18) and a little algebra.

Next, we verify the inequality

$$\rho_n^{(\ell)} > \frac{(-\Delta)^\ell n^{\ell-1/2}}{n^{\ell-1/2}}.$$

We show that the summands in

$$(-\Delta)^\ell \mathfrak{g}_n^{(\ell)} = \sum_{j=1}^{\ell} s(\ell, j) (-\Delta)^\ell n^{j-1/2}$$

are all positive. First, recall that $(-1)^{\ell+j}s(\ell, j) > 0$ for all $j \in \{1, \dots, \ell\}$, see (2.14). Second, Lemma 7 implies the existence of $\xi \in (n - \ell, n + \ell)$, hence $\xi > 0$, such that

$$(-\Delta)^\ell n^{j-1/2} = (-1)^\ell \frac{d^{2\ell}}{dx^{2\ell}} \Big|_{x=\xi} x^{j-1/2} = (-1)^\ell \prod_{k=1}^{2\ell} \left(j + \frac{1}{2} - k \right) \xi^{j-2\ell-1/2}.$$

It follows that $(-1)^{j+\ell}(-\Delta)^\ell n^{j-1/2} > 0$ for all $j \in \{1, \dots, \ell\}$. Consequently, we may estimate

$$(-\Delta)^\ell \mathbf{g}_n^{(\ell)} > s(\ell, \ell)(-\Delta)^\ell n^{\ell-1/2} = (-\Delta)^\ell n^{\ell-1/2},$$

from which we conclude that

$$\rho_n^{(\ell)}(\mathbf{g}) = \frac{(-\Delta)^\ell \mathbf{g}_n^{(\ell)}}{\mathbf{g}_n^{(\ell)}} > \frac{n^{\ell-1}}{(n-1) \dots (n-\ell+1)} \frac{(-\Delta)^\ell n^{\ell-1/2}}{n^{\ell-1/2}} > \frac{(-\Delta)^\ell n^{\ell-1/2}}{n^{\ell-1/2}}.$$

The proof of Theorem 5 is complete. □

4. MORE GENERAL FAMILIES OF DISCRETE HARDY WEIGHTS OF HIGHER ORDER

The concrete parameter sequence $\mathbf{g}^{(\ell)}$, defined by the formula (2.12), was considered in Theorem 4, because it gives rise to the corresponding optimal weight $\rho^{(\ell)}$ which is of a relatively simple form. However, these are not the only optimal discrete Hardy weights. In this section, we intend to emphasize that the abstract formulation of Theorems 1–3 can be used to derive more general discrete Hardy weights of higher order.

4.1. Countable sets of optimal Hardy weights of higher order. In Theorem 4, we found a concrete optimal Hardy weight $\rho^{(\ell)}$ of order ℓ , but it is not the only optimal weight that we can obtain using our method. The following generalization of Theorem 4 explains how to generate more optimal Hardy weights. In fact, for every $\ell \in \mathbb{N}$, we find countably infinite set of optimal discrete Hardy weights of order ℓ .

Proposition 10. *Let $\ell \in \mathbb{N}$. For any $k \in \mathbb{N}_0$, the sequence $\rho^{(\ell, k)}$, given by*

$$\rho_n^{(\ell, k)} := \frac{(-\Delta)^\ell \operatorname{div}^k \mathbf{g}_n^{(\ell+k)}}{\operatorname{div}^k \mathbf{g}_n^{(\ell+k)}}$$

for all $n \geq \ell$, where $\mathbf{g}^{(m)}$ is defined by (2.12) for any $m \in \mathbb{N}$, is a optimal discrete Hardy weight of order ℓ .

Proof. In the proof of Theorem 4 in Section 3.4, we verified, that the sequence $\mathbf{g}^{(\ell)}$ defined by (2.12) satisfies the assumptions

- a) $\mathbf{g} \in H^\ell$,
- b) $\operatorname{div}^k \mathbf{g}_n > 0$ for all $n \geq \ell - k$ and $k \in \{0, \dots, \ell - 1\}$,
- c) $(-\Delta)^{\ell-k} \operatorname{div}^k \mathbf{g}_n > 0$ for all $n \geq \ell - k$ and $k \in \{0, \dots, \ell - 1\}$,

from which we concluded that $\mathbf{g}^{(\ell)}$ also satisfies the assumptions **(A1)**, **(A2)**, **(A2')**, and **(A3')** for $\ell \in \mathbb{N}$. Moreover, it is obvious, that if an arbitrary sequence $\mathbf{g}$ meets the assumptions a), b), and c), for $\ell = m \geq 2$, then the sequence $\operatorname{div} \mathbf{g}$ satisfies all three assumptions a), b), and c) for $\ell = m - 1$. By induction, it follows that for all $\ell \in \mathbb{N}$ and $k \in \mathbb{N}_0$, the sequences $\operatorname{div}^k \mathbf{g}^{(\ell+k)}$ fulfill the assumptions **(A1)**, **(A2)**, **(A2')**, and **(A3')** for ℓ .

Furthermore, it follows from claim (iii) of Lemma 6 that, for all $k \in \mathbb{N}_0$, the sequences $\operatorname{div}^k \mathbf{g}^{(\ell+k)}$ satisfy also the assumption **(A3)** for $\ell \in \mathbb{N}$. Theorem 3 then implies optimality of the weights $\rho^{(\ell, k)}$. □

Remark 4.1. Evidently, Proposition 10 asserts Theorem 4, for any $\ell \in \mathbb{N}$, since $\rho^{(\ell, 0)} = \rho^{(\ell)}$. Furthermore, for $\ell = 2$ and $k = 1$, we have the optimal discrete Rellich weight

$$\rho_n^{(2, 1)} = \frac{(-\Delta)^2 \operatorname{div} \mathbf{g}_n^{(3)}}{\operatorname{div} \mathbf{g}_n^{(3)}} = \frac{9}{16n^4} + \frac{63}{40n^5} + \frac{9357}{3200n^6} + \mathcal{O}\left(\frac{1}{n^7}\right).$$

Notice that the coefficient 63/40 by the second term improves upon the analogous term 3/2 in the expansion of $\rho^{(2)}$, see claim (iii) of Remark 2.7. On the other hand, for the weight $\rho^{(2, 1)}$ it does not hold, that all the terms in the expansion are non-negative (unlike for $\rho^{(2)}$).

4.2. q -generalized Hardy weights of higher order. In Theorem 3, we restricted ourselves to parametric sequences that satisfy $\mathbf{g}_n \sim n^{\ell-1/2}$, as $n \rightarrow \infty$, see (A3). However, an analogous claim holds even if we let the asymptotic behaviour of $\mathbf{g}_n$, as $n \rightarrow \infty$, depend on a real parameter $s \geq 1/2$.

Proposition 11. *Let $\ell \in \mathbb{N}$. Suppose (A1), (A2), (A2'), and, in addition, suppose that $\mathbf{g}$ admits the asymptotic expansion*

$$\mathbf{g}_n = \sum_{j=0}^{2\ell} \alpha_j n^{\ell-s-j} + \mathcal{O}(n^{-\ell-s-1}) \text{ for some } \alpha_j \in \mathbb{R} \text{ with } \alpha_0 \neq 0 \text{ and } s \geq 1/2 \quad (\text{A3}'')$$

as $n \rightarrow \infty$. Then the discrete Hardy weight $\rho(\mathbf{g}) = (-\Delta)^\ell \mathbf{g}/\mathbf{g}$ of order ℓ is critical.

Proof. The proof of Proposition 11 can be done in a similar manner as the proof of criticality in Section 3.3.1, with only difference being the resulting estimate

$$\left| R_k^{(\ell)}(\mathbf{g}) u_n^N \right| \lesssim \frac{1}{n^s \ln N},$$

that can be further estimated from above with $1/(\sqrt{n} \ln N)$, for all $n \geq \ell$ and $N \geq 2$, as in (3.9) nonetheless. The details are left as an exercise for the reader.

Second way of proving Proposition 11, is to restrict ourselves to the case $s > 1/2$, since the claim for $s = 1/2$ is a part of Theorem 3. Now, similarly as in Section 3.3.1, suppose that the sequence $\tilde{\rho} = \{\tilde{\rho}_n\}_{n=\ell}^\infty$ is a discrete Hardy weight of order ℓ satisfying $\tilde{\rho}_n \geq \rho_n(\mathbf{g})$ for all $n \geq \ell$. Using identity (2.5), with the extension provided by Proposition 9, for the weight $\rho(\mathbf{g})$ together with the discrete Hardy inequality of order ℓ for the weight $\tilde{\rho}$ (which can be straightforwardly extended to all $u \in H^\ell$), we find that

$$0 \leq \sum_{n=\ell}^{\infty} (\tilde{\rho}_n - \rho_n(\mathbf{g})) |u_n|^2 \leq \sum_{k=0}^{\ell-1} \mathcal{R}_k^{(\ell)}(\mathbf{g}; u)$$

for all $u \in \mathcal{D}^\ell$. Invoking Lemma 6, we observe that $(-\Delta)^{\ell/2} \mathbf{g}_n \sim n^{-s}$, hence $\mathbf{g} \in \mathcal{D}^\ell$. Substituting $u = \mathbf{g}$ into the above inequality, we conclude that $\tilde{\rho}_n = \rho_n(\mathbf{g})$ for all $n \geq \ell$, since all the remainders on the right-hand side are annihilated with this particular choice of u . $\square$

Proposition 11 can be used to derive more Hardy weights of any order. For a parameter $q > 0$, we set

$$\mathbf{g}_n^{(\ell)}(q) := n^q \prod_{j=1}^{\ell-1} (n-j) \quad (4.1)$$

for all $n \geq 0$ and $\mathbf{g}_n^{(\ell)}(q) = 0$ if $n < 0$.

Proposition 12. *Let $\ell \in \mathbb{N}$ and $q \in (0, 1)$. Then the sequence $\rho^{(\ell)}(q)$ given by*

$$\rho_n^{(\ell)}(q) := \frac{(-\Delta)^\ell \mathbf{g}_n^{(\ell)}(q)}{\mathbf{g}_n^{(\ell)}(q)} \quad (4.2)$$

for all $n \geq \ell$, where $\mathbf{g}^{(\ell)}(q)$ is defined by (4.1), is a discrete Hardy weight of order ℓ . Furthermore, $\rho^{(\ell)}(q)$ is critical if and only if $q \in (0, 1/2]$, non-attainable if and only if $q \in [1/2, 1)$ and optimal near infinity if and only if $q = 1/2$.

Remark 4.2. If $q = 1$, $\rho^{(\ell)}(1) \equiv 0$ for all $\ell \geq 1$ and therefore it is meaningful to consider only $q \in (0, 1)$. In the case $\ell = 1$, weight $\rho^{(1)}(q)$ appeared already in [9]. Clearly, $\mathbf{g}^{(\ell)}$ defined in (2.12) corresponds to $q = 1/2$, thus $\rho^{(\ell)}(1/2) = \rho^{(\ell)}$ defined by (2.13). Moreover, Proposition 12 asserts that the weight $\rho^{(\ell)}(q)$ is optimal if and only if $q = 1/2$.

Proof. For $q \in (0, 1)$, claims a) and b) from the proof of Theorem 4 in Section 3.4 can be verified in an analogous fashion. Consequently, $\mathbf{g}^{(\ell)}(q)$ meets the assumptions (A1), (A2), and (A2'), hence $\rho^{(\ell)}(q)$ is a discrete Hardy weight of order ℓ for all $q \in (0, 1)$ by Theorem 2. Let us discuss the optimality of $\rho^{(\ell)}(q)$.

a) *Criticality:* Assumption (A3'') is fulfilled if $q \in (0, 1/2]$, and therefore the weight $\rho^{(\ell)}(q)$ is critical by Proposition 11 for $q \in (0, 1/2]$.

On the other hand, if $q \in (1/2, 1)$, $\rho^{(\ell)}(q)$ is not critical because, in this case,

$$\rho_n^{(\ell)}(q) < \rho_n^{(\ell)}(1/2) \text{ for all } n \geq \ell. \quad (4.3)$$

Indeed, seeing definition (4.2) and also (3.12), statement (4.3) can be equivalently written as

$$\sum_{j=1}^{\ell} s(\ell, j) (-\Delta)^\ell n^{j-1+q} < \sum_{j=1}^{\ell} s(\ell, j) n^{q-1/2} (-\Delta)^\ell n^{j-1/2}$$

for all $n \geq \ell$. Recalling that $(-1)^{\ell+j}s(\ell, j) > 0$ for all $\ell \in \mathbb{N}$ and $j \in \{1, \dots, \ell\}$, it suffices to show that

$$(-1)^{\ell+j}(-\Delta)^\ell n^{j-1+q} < (-1)^{\ell+j} n^{q-1/2} (-\Delta)^\ell n^{j-1/2}$$

for all $n \geq \ell$ and $j \in \{1, \dots, \ell\}$. With the aid of the expansion (3.14) and the fact that $X_m^{(\ell)} = 0$ whenever m is odd, the above expression can be written as

$$(-1)^{\ell+j} n^{j-1+q} \sum_{m=\ell}^{\infty} \binom{j-1+q}{2m} \frac{X_{2m}^{(\ell)}}{n^{2m}} < (-1)^{\ell+j} n^{q-1/2} n^{j-1/2} \sum_{m=\ell}^{\infty} \binom{j-1/2}{2m} \frac{X_{2m}^{(\ell)}}{n^{2m}}.$$

Moreover, $(-1)^\ell X_{2m}^{(\ell)} > 0$ for all $m \geq \ell$ by (2.18), hence the inequality (4.3) follows from the fact that

$$(-1)^j \binom{j-1+q}{2m} < (-1)^j \binom{j-1/2}{2m}$$

for all $m \geq \ell$ and $j \in \{1, \dots, \ell\}$. This can be verified, realizing that both expressions are positive, see (3.16), and that $(q+k)(1-q+k) < (k+1/2)^2$, which holds true for all $q \in (1/2, 1)$ and $k \in \mathbb{N}_0$.

b) *Non-attainability*: For $q \in [1/2, 1)$, non-attainability of $\rho^{(\ell)}(q)$ can be proven in an analogous fashion as in Section 3.3.2. Since the sequence $\mathbf{g}^{(\ell)}(q)$ satisfies the assumptions **(A1)** and **(A3')**, which can be seen from the claims a) and b) from the proof of Theorem 4 in Section 3.4 once again, we find that if $u \in H^\ell$ fulfills (2.2) as equality with $\rho = \rho^{(\ell)}(q)$, then $u_n = c \mathbf{g}_n^{(\ell)}(q)$ for some constant $c \in \mathbb{C}$ and all $n \geq \ell$. Moreover, by the expansion (3.14), we have

$$\rho_n^{(\ell)}(q) = \binom{\ell-1+q}{2\ell} \frac{X_{2\ell}^{(\ell)}}{n^{2\ell}} + \mathcal{O}\left(\frac{1}{n^{2\ell+1}}\right) = \frac{(q)_\ell(1-q)_\ell}{n^{2\ell}} + \mathcal{O}\left(\frac{1}{n^{2\ell+1}}\right), \quad (4.4)$$

hence

$$\rho_n^{(\ell)}(q) (\mathbf{g}_n^{(\ell)}(q))^2 \gtrsim \frac{1}{n^{2\ell}} n^{2\ell-2+2q} = \frac{1}{n^{2-2q}} \quad (4.5)$$

for all $n \geq \ell$, which is not a summable sequence for $q \in [1/2, 1)$. Therefore, we conclude that $c = 0$, i.e. $u \equiv 0$, proving the non-attainability.

Conversely, for $q \in (0, 1/2)$, the Hardy inequality of order ℓ (2.2), with the weight $\rho^{(\ell)}(q)$, holds as equality for $u = \mathbf{g}^{(\ell)}(q)$, while both sides are finite. This stems from the fact that

$$(-\Delta)^{\ell/2} \mathbf{g}_n^{(\ell)}(q) \sim \frac{1}{n^{1-q}}, \quad n \rightarrow \infty,$$

hence $\mathbf{g}^{(\ell)}(q) \in \mathcal{D}^\ell$. Proposition 9 then implies, that the identity (2.5) holds for $u = \mathbf{g}^{(\ell)}(q)$, while all the remainders on the right-hand side vanish. Moreover, seeing (4.5), we infer that $\sqrt{\rho^{(\ell)}(q)} \mathbf{g}^{(\ell)}(q) \in \mathcal{H}^\ell$ for all $q \in (0, 1/2)$.

c) *Optimality near infinity*: Theorem 4 asserts optimality near infinity of $\rho^{(\ell)}(1/2)$. The non-optimality near infinity of $\rho^{(\ell)}(q)$ for $q \neq 1/2$ is a consequence of the fact that the constant by the leading term in (4.4) is smaller than for $q = 1/2$, i.e.

$$(q)_\ell(1-q)_\ell < \left(\frac{1}{2}\right)_\ell^2$$

for $q \neq 1/2$. This, can be seen from the definition of the Pochhammer symbol and from the inequality $(q+k)(1-q+k) < (k+1/2)^2$ once again. Consequently, for $q \neq 1/2$ fixed, we find $\varepsilon > 0$ sufficiently small, such that

$$(1+\varepsilon)\rho_n^{(\ell)}(q) \leq \rho_n^{(\ell)}$$

for all n sufficiently large. Then, for all $M \in \mathbb{N}$ sufficiently large and any $u \in \mathcal{H}_0^M$, we have

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left| (-\Delta)^{\ell/2} u_n \right|^2 \geq \sum_{n=M}^{\infty} \rho_n^{(\ell)} |u_n|^2 \geq (1+\varepsilon) \sum_{n=M}^{\infty} \rho_n^{(\ell)}(q) |u_n|^2,$$

contradicting (2.3). $\square$

In addition, more Hardy weights of higher order can be found similarly as in Section 4.1. Since the above proof relies mostly on the claims a) and b) from Section 3.4 and the asymptotic behaviour of $\mathbf{g}^{(\ell)}(q)$ and $\rho^{(\ell)}(q)$, defined by (4.1) and (4.2) respectively, the following proposition can be verified in an analogous fashion as Proposition 10, and therefore we shall state it without further examination of its proof.

Proposition 13. *Let $\ell \in \mathbb{N}$. For any $k \in \mathbb{N}_0$ and $q \in (0, 1)$, the sequence $\rho^{(\ell, k)}(q)$, given by*

$$\rho_n^{(\ell, k)}(q) := \frac{(-\Delta)^\ell \operatorname{div}^k \mathbf{g}_n^{(\ell+k)}(q)}{\operatorname{div}^k \mathbf{g}_n^{(\ell+k)}(q)}$$

for all $n \geq \ell$, where $\mathbf{g}^{(m)}(q)$ is defined by (4.1) for any $m \in \mathbb{N}$, is a discrete Hardy weight of order ℓ . Furthermore, $\rho^{(\ell,k)}(q)$ is critical if $q \in (0, 1/2]$, non-attainable if and only if $q \in [1/2, 1)$ and optimal near infinity if and only if $q = 1/2$.

4.3. Multi-parameter families of optimal discrete Hardy weights of higher order. For $\ell \geq 2$, more general candidates for optimal weights can be found. Following the footsteps of [7, Theorem 0.1], the original motivation for deriving a suitable parametric sequence $\mathbf{g}$, that meets all the assumptions of Theorems 1–3, i.e. the sequence $\mathbf{g}^{(\ell)}$ defined by the formula (2.12), was to seek for a positive sequence $\mathbf{h}$ satisfying the linear difference equation of order 2ℓ with constant coefficients

$$(-\Delta)^\ell \mathbf{h}_n = 0$$

for all $n \geq \ell$, with the boundary condition $\mathbf{h}_0 = \dots = \mathbf{h}_{\ell-1} = 0$, and then taking $\mathbf{g} = \sqrt{\mathbf{h}}$.

It is easy to verify, that the fundamental system of the above linear difference equation consists of the sequences $1, n, \dots, n^{2\ell-1}$. Together with the boundary condition $\mathbf{h}_0 = \dots = \mathbf{h}_{\ell-1} = 0$, this yields a $(\ell - 1)$ -parametric family of sequences (up to a multiplicative constant)

$$\mathbf{h}_n = \prod_{j=0}^{\ell-1} (n - j) \prod_{k=1}^{\ell-1} (n - \alpha_k).$$

for all $n \in \mathbb{N}_0$. If we set $\mathbf{g}_n = \sqrt{\mathbf{h}_n}$, we obtain candidates for constructing optimal Hardy weights of order ℓ . However, we find it difficult to formulate further restrictions in terms of parameters $\alpha_1, \dots, \alpha_{\ell-1}$ so that the sequence $\mathbf{g}_n$ would satisfy the assumptions **(A1)**, **(A2)**, and **(A3')** (the assumption **(A3)** is always satisfied by the generalized binomial theorem). Note that for $\mathbf{g}^{(\ell)}$, we have $\alpha_k = k$ for all $k \in \{1, \dots, \ell - 1\}$.

For example, if $\ell = 2$, we have the candidate

$$\mathbf{g}_n(\alpha) = \sqrt{n(n-1)(n-\alpha)}.$$

We have tested numerically, that approximately for $\alpha \in (0.847, 1.306)$ all the above listed assumptions are satisfied, hence the sequence

$$\rho_n^{(2)}(\alpha) = \frac{(-\Delta)^2 \mathbf{g}_n(\alpha)}{\mathbf{g}_n(\alpha)}$$

for all $n \geq 2$, is a optimal Hardy weight of order 2, i.e. optimal Rellich weight, by Theorem 3.

APPENDIX A. PROOF OF LEMMAS 6–8

A.1. Proof of Lemma 6. In order to prove Lemma 6, it suffices to show the first point for $m = \pm 1$, which can be seen from the Taylor series expansion

$$(n \pm 1)^\beta = n^\beta \left(1 \pm \frac{1}{n}\right)^\beta = n^\beta \sum_{j=0}^{\infty} \binom{\beta}{j} \left(\frac{\pm 1}{n}\right)^j = n^\beta + \sum_{j=1}^k c_j n^{\beta-j} + \mathcal{O}(n^{\alpha-k-1}), \quad n \rightarrow \infty.$$

The rest readily follows by induction, but we leave the technical details as an easy exercise for the reader. $\square$

A.2. Proof of Lemma 7. Let p be a polynomial of degree less or equal to N such that $p_{n+j} = g_{n+j}$ for all $j \in \{0, \dots, N\}$. Here and below, we use the notation $p_n := p(n)$ and $g_n := g(n)$. Notice that

$$\operatorname{div}^N g_n = \sum_{j=0}^N \binom{N}{j} (-1)^{N-j} g_{n+j} = \sum_{j=0}^N \binom{N}{j} (-1)^{N-j} p_{n+j} = \operatorname{div}^N p_n.$$

Next, we will show that

$$\operatorname{div}^N p_n = p^{(N)}(n).$$

First, we prove that if the degree of p is less or equal to N , then the polynomial $\operatorname{div} p(x) = p(x+1) - p(x)$ is of degree less or equal to $N - 1$. Indeed, for $p(x) = \sum_{k=0}^n a_k x^k$, where $a_k \in \mathbb{R}$, we have

$$\operatorname{div} p(x) = \sum_{k=0}^n a_k ((x+1)^k - x^k) = \sum_{k=0}^n a_k \left(\sum_{j=0}^k \binom{k}{j} x^j - x^k \right) = \sum_{k=0}^n a_k \sum_{j=0}^{k-1} \binom{k}{j} x^j.$$

Consequently, $\operatorname{div}^N p = c$ for some constant $c \in \mathbb{R}$ and therefore $\operatorname{div}^M p = 0$ for all $M > N$. Because both div and d/dx are linear operators on respective spaces, it suffices to show that $(\operatorname{div}^N x^N) = (x^N)^{(N)} = N!$.

We will prove this by induction once again. For $N = 0$ the statement holds trivially. Assuming the claim holds for $N - 1$ we have

$$\begin{aligned} \operatorname{div}^N x^N &= \operatorname{div}^{N-1}(\operatorname{div} x^N) = \operatorname{div}^{N-1}((x+1)^N - x^N) = \sum_{k=0}^{N-1} \binom{N}{k} (\operatorname{div}^{N-1} x^k) \\ &= N(\operatorname{div}^{N-1} x^{N-1}) = N(N-1)! = N!. \end{aligned}$$

As $p_{n+j} - g_{n+j} = 0$ for all $j \in \{0, \dots, N\}$, we infer from Rolle's theorem, that for all $j \in \{0, \dots, N-1\}$ there exists $c_j \in (n+j, n+j+1)$ such that $p'(c_j) - g'(c_j) = 0$. By iteration of Rolle's theorem, we obtain $\xi \in (n, n+N)$ such that

$$g^{(N)}(\xi) = p^{(N)}(\xi) = p^{(N)}(n) = \operatorname{div}^N p_n = \operatorname{div}^N g_n.$$

□

A.3. Proof of Lemma 8. The proof of Lemma 8 proceeds by induction. The statement is trivially true for $m = 0$. For $m = 1$, we have

$$\operatorname{div}(fg) = (S - I)(fg) = \begin{cases} (Sf)(Sg) - fg \pm (Sf)g = (Sf)(\operatorname{div}g) + (\operatorname{div}f)g \\ (Sf)(Sg) - fg \pm f(Sg) = f(\operatorname{div}g) + (\operatorname{div}f)(Sg) \end{cases}$$

Adding the two lines above and dividing by 2, we verify the statement for $m = 1$. Next, for the case $m + 1$ we infer from the induction hypothesis that

$$\begin{aligned} \operatorname{div}^{m+1}(fg) &= \operatorname{div}(\operatorname{div}^m(fg)) = \operatorname{div} \left(\sum_{j=0}^m \binom{m}{j} (\operatorname{div}^j M^{m-j} f)(\operatorname{div}^{m-j} M^j g) \right) \\ &= \sum_{j=0}^m \binom{m}{j} (\operatorname{div}^{j+1} M^{m-j} f)(\operatorname{div}^{m-j} M^{j+1} g) + \sum_{j=0}^m \binom{m}{j} (\operatorname{div}^j M^{m+1-j} f)(\operatorname{div}^{m+1-j} M^j g) \\ &= (\operatorname{div}^{m+1} f)(M^{m+1} g) + (M^{m+1} f)(\operatorname{div}^{m+1} g) + \sum_{j=1}^m \left[\binom{m}{j-1} + \binom{m}{j} \right] (\operatorname{div}^j M^{m+1-j} f)(\operatorname{div}^{m+1-j} M^j g) \\ &= \sum_{j=0}^{m+1} \binom{m+1}{j} (\operatorname{div}^j M^{m+1-j} f)(\operatorname{div}^{m+1-j} M^j g), \end{aligned}$$

hence the proof is complete. □

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(Jakub Waclawek) DEPARTMENT OF MATHEMATICS, FACULTY OF NUCLEAR SCIENCES AND PHYSICAL ENGINEERING, CZECH TECHNICAL UNIVERSITY IN PRAGUE, TROJANOVÁ 13, 12000 PRAHA 2, CZECH REPUBLIC
Email address: `waclajak@cvut.cz`