Optimal Hardy inequalities for powers of the discrete Laplacian on the half-line

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and

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Geometric and analytic functional inequalities on discrete spaces

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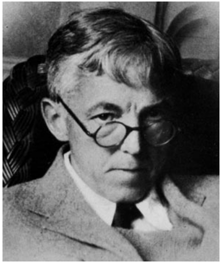
- The classical discrete p-Hardy inequality:

$$\boxed{\sum_{n=1}^{\infty}\left|\frac{1}{n}\sum_{k=1}^{n}v_k\right|^p \le q^p\sum_{n=1}^{\infty}|v_n|^p} \qquad \left(\tfrac{1}{p}+\tfrac{1}{q}=1\right)$$



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- In the difference form ($u_0=0$):

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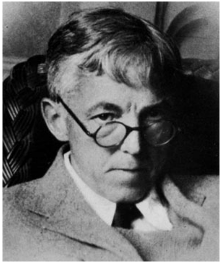

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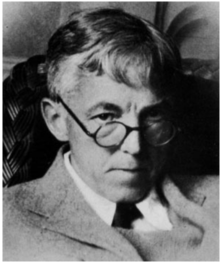

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Theorem [Fischer–Keller–Pogorzelski, 2023]: The discrete p-Hardy weight w^{H} can be improved to

$$\boxed{w_n^{\mathrm{FKP}} = \left[1 - \left(1 - \frac{1}{n}\right)^{1/q}\right]^{p-1} - \left[\left(1 + \frac{1}{n}\right)^{1/q} - 1\right]^{p-1} > \frac{1}{q^p n^p}.}$$

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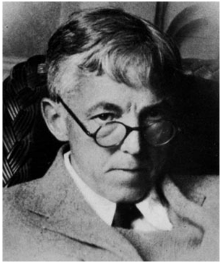

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- For $p = 2$, the optimal weight w^{FKP} found earlier in [Keller–Pinchover–Pogorzelski, 2018]

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holds for all $u \in W^{\ell,p}(\mathbb{R}_+)$ with $u(0_+) = u'(0_+) = \cdots = u^{(\ell-1)}(0_+) = 0$, where

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- For arbitrary $\ell \in \mathbb{N}$: case $p = 2$ [Huang–Ye, 2024]; case $p > 1$ [Š–W, 2026].

DEFINITION: For a (non-specified) general $\ell \in \mathbb{N}$, any non-negative sequence $\rho = \{\rho_n\}_{n=\ell}^{\infty}$ is called a discrete p-Hardy–Rellich–Birman weight, if and only if the inequality

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- Motivated by [Devyver–Fraas–Pinchover, 2014] and [Keller–Pinchover–Pogorzelski, 2018].

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- The definition of $-\Delta_p^{(\ell)}$ stems from a polarization of the ℓ^p-norm of $\nabla^\ell u$:

$$\sum_{n=\ell}^{\infty} \big|\nabla^\ell u_n\big|^p = \sum_{n=\ell}^{\infty} \overline{u_n}\,(-\Delta_p^{(\ell)} u)_n$$

for $u\in C_0(\mathbb{N})$ with $u_1=\cdots=u_{\ell-1}=0$.

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THEOREM [Š–W]: If a parameter sequence $\mathfrak{g}$ satisfies (MON), (SUPHAR), (ASYM), then $\forall u \in C_0(\mathbb{N})$ with $u_1 = \cdots = u_{\ell-1} = 0$, it holds that

$$\boxed{\sum_{k=1}^{\ell} R_k^{(\ell)}(p; \mathfrak{g}, u) \gtrsim \sum_{n=\ell}^{\infty} |\nabla^\ell u_n|^p - \sum_{n=\ell}^{\infty} \frac{-\Delta_p^{(\ell)} \mathfrak{g}_n}{\mathfrak{g}_n^{p-1}} |u_n|^p \gtrsim \sum_{k=1}^{\ell} R_k^{(\ell)}(q; \mathfrak{g}, u) \geq 0}$$

and $\rho(\mathfrak{g}) = -\Delta_p^{(\ell)} \mathfrak{g} / \mathfrak{g}^{p-1}$ is an optimal discrete p-Hardy–Rellich–Birman weight.

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- Proof of criticality: Let $\widetilde{\rho}$ be a discrete p-H–R–B weight satisfying $\widetilde{\rho} \geq \rho(\mathfrak{g})$. Then

$$0 \leq \sum_{n=\ell}^{\infty} \left(\widetilde{\rho}_n - \rho_n(\mathfrak{g})\right)|u_n|^p \lesssim \sum_{k=1}^{\ell} R_k^{(\ell)}(p,\mathfrak{g},u)$$

  for all sequences $u \in C_0(\mathbb{N}_0)$, $u_0 = \cdots = u_{\ell-1} = 0$. Substituting $u = \mathfrak{g}$, we infer $\boxed{\widetilde{\rho} = \rho(\mathfrak{g}).}$

PROOF OF CRITICALITY: THE MAIN IDEA

- Two sided control: Set $\rho(\mathfrak{g}) := -\Delta_p^{(\ell)}\mathfrak{g}/\mathfrak{g}^{p-1}$. Then

$$\sum_{k=1}^{\ell} R_k^{(\ell)}(q;\mathfrak{g},u) \lesssim \sum_{n=\ell}^{\infty}|\nabla^\ell u_n|^p - \sum_{n=\ell}^{\infty}\rho(\mathfrak{g})|u_n|^p \lesssim \sum_{k=1}^{\ell} R_k^{(\ell)}(p;\mathfrak{g},u),$$

  where $R_k^{(\ell)}(q;\mathfrak{g},u)$ are given by expressions that are complicated but sufficiently descriptive.
- (MON) & (SUPHAR) guarantee $R_k^{(\ell)}(q;\mathfrak{g},u) \geq 0$. So $\rho(\mathfrak{g})$ is a discrete p-H–R–B weight.
- Annihilation by $\mathfrak{g}$: $R_k^{(\ell)}(q;\mathfrak{g},\mathfrak{g}) = 0$ for all $k = 1,\ldots,\ell$.
- Proof of criticality: Let $\widetilde{\rho}$ be a discrete p-H–R–B weight satisfying $\widetilde{\rho} \geq \rho(\mathfrak{g})$. Then

$$0 \leq \sum_{n=\ell}^{\infty}\left(\widetilde{\rho}_n - \rho_n(\mathfrak{g})\right)|u_n|^p \lesssim \sum_{k=1}^{\ell} R_k^{(\ell)}(p,\mathfrak{g},u)$$

  for all sequences $u \in C_0(\mathbb{N}_0)$, $u_0 = \cdots = u_{\ell-1} = 0$. Substituting $u = \mathfrak{g}$, we infer $\boxed{\widetilde{\rho} = \rho(\mathfrak{g}).}$
- Here we are cheating because $\mathfrak{g} \notin C_0(\mathbb{N}_0)$, but one can make it rigorous by an approximation argument.

NEW OPTIMAL DISCRETE p-HARDY–RELLICH–BIRMAN WEIGHTS

$$\boxed{\mathfrak{g}_n^{(\ell,p)} := \frac{\Gamma(n+1/q)}{\Gamma(n-\ell+1)}}$$

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THEOREM [Š–W]: The sequence

$$\boxed{\rho_n^{(\ell,p)} := \frac{-\Delta_p^{(\ell)}\mathfrak{g}_n^{(\ell,p)}}{\left(\mathfrak{g}_n^{(\ell,p)}\right)^{p-1}} = \left(\frac{1}{q}\right)_\ell^{p-1} \frac{1}{(n-\ell+1-1/p)_\ell^{p-1}} \sum_{j=0}^{\ell} \frac{(-\ell)_j (n-\ell+1-1/p)_j^{p-1}}{(n-\ell+1)_j^{p-1}\, j!}}$$

is an optimal discrete p-Hardy–Rellich–Birman weight satisfying

$$\boxed{\rho_n^{(\ell,p)} > \left(\frac{1}{q}\right)_\ell^p \frac{1}{n^{\ell p}}, \quad \forall n \geq \ell.}$$

THE CASE $p = 2$

- If $p = 2$, we have the new discrete Hardy–Rellich–Birman inequality

$$\boxed{\sum_{n=\ell}^{\infty} |\nabla^{\ell} u_n|^2 \geq \left(\frac{1}{2}\right)_{\ell}^{2} \sum_{n=\ell}^{\infty} \frac{|u_n|^2}{n(n-1/2)(n-1)\cdots(n-\ell+1/2)},}$$

for $u \in C_0(\mathbb{N})$, $u_1 = \cdots = u_{\ell-1} = 0$, with optimal weight.

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- In particular, if $\ell = 1$, we have the new optimal discrete Hardy inequality

$$\boxed{\sum_{n=1}^{\infty} \left|u_n - u_{n-1}\right|^2 \geq \frac{1}{4} \sum_{n=1}^{\infty} \frac{|u_n|^2}{n(n-1/2)},}$$

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- Compare the weight $\rho_n^{(1,2)} = \dfrac{1}{4n(n-1/2)}$ to $\begin{cases} w_n^{\mathrm{H}} = \frac{1}{4n^2}, \\ w_n^{\mathrm{KPP}} = 2 - \sqrt{1 - \frac{1}{n}} - \sqrt{1 + \frac{1}{n}}. \end{cases}$

- For $p = 2$, the Hardy inequality can be understood in the sense of quadratic forms

$$\langle u, -\Delta_{\mathbb{N}} u \rangle_{\ell^2(\mathbb{N})} = \sum_{n=1}^{\infty} \overline{u_n} (-\mathrm{div} \nabla u)_n = \sum_{n=1}^{\infty} |\nabla u_n|^2 \geq \sum_{n=1}^{\infty} \rho_n |u_n|^2 = \langle u, \rho\, u \rangle_{\ell^2(\mathbb{N})} .$$

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- Similarly, we may consider fractional (higher-order) Hardy inequalities $(-\Delta_{\mathbb{N}})^{\alpha} \geq \rho$ for any positive power $\alpha > 0$ of the Laplace operator on $\ell^2(\mathbb{N})$.
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THEOREM [Gerhat–Krejčiřík–Š, 2025]: There exists a non-trivial Hardy weight ρ, satisfying $(-\Delta_{\mathbb{N}})^{\alpha} \geq \rho$ on $\ell^2(\mathbb{N})$, if and only if $\alpha \in (0, 3/2)$.

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THEOREM [Das–Fuente-Fernández, 2026]: For $\alpha \in (0, 1]$ the optimal weights read

$$\rho_\alpha^{\mathrm{DF}}(n) = 4^\alpha \left[\frac{\Gamma\left(\frac{3+2\alpha}{4}\right)}{\Gamma\left(\frac{3-2\alpha}{4}\right)}\right]^2 \frac{\Gamma\left(n + \frac{5-2\alpha}{4}\right)\Gamma\left(n - \frac{1+2\alpha}{4}\right)}{\Gamma\left(n + \frac{5+2\alpha}{4}\right)\Gamma\left(n - \frac{1-2\alpha}{4}\right)} \sim 4^\alpha \left[\frac{\Gamma\left(\frac{3+2\alpha}{4}\right)}{\Gamma\left(\frac{3-2\alpha}{4}\right)}\right]^2 \frac{1}{n^{2\alpha}}.$$

- PROOF: Relies on the criticality theory for Schrödinger operators on graphs developed in [Keller-Pinchover–Pogorzelski, 2018].

- Recall that for any integer $\ell \in \mathbb{N}$, the Hardy–Rellich–Birman inequality

$$\sum_{n=\ell}^{\infty} \overline{u_n}(-\Delta_{\mathbb{N}})^{\ell} u_n = \sum_{n=\ell}^{\infty} \left|\nabla^{\ell} u_n\right|^2 \geq \left(\frac{1}{2}\right)_{\ell}^{2} \sum_{n=\ell}^{\infty} \frac{|u_n|^2}{n^{2\ell}}$$

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ON $\mathbb{N}$

$$(-\Delta_{\mathbb{N}}) = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & \ddots & \ddots & \ddots & \end{pmatrix}$$

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On $\mathbb{N}$

$$(-\Delta_{\mathbb{N}})^3 = \begin{pmatrix} 14 & -14 & 6 & -1 & & & & \\ -14 & 20 & -15 & 6 & -1 & & & \\ 6 & -15 & 20 & -15 & 6 & -1 & & \\ -1 & 6 & -15 & 20 & -15 & 6 & -1 & \\ & -1 & 6 & -15 & 20 & -15 & 6 & -1 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

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TWO DIFFERENT LAPLACIANS ON THE DISCRETE HALF-LINE

- Recall that for any integer $\ell \in \mathbb{N}$, the Hardy–Rellich–Birman inequality
$$\sum_{n=\ell}^{\infty} \overline{u_n} (-\Delta_{\mathbb{N}})^{\ell} u_n = \sum_{n=\ell}^{\infty} \left|\nabla^{\ell} u_n\right|^2 \geq \left(\frac{1}{2}\right)_{\ell}^{2} \sum_{n=\ell}^{\infty} \frac{|u_n|^2}{n^{2\ell}}$$
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Two different Laplacians on the discrete half-line

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- Set $(-\Delta)^{\alpha} := P_+^*(-\Delta_{\mathbb{Z}})^{\alpha} P_+$, where $P_+ : \ell^2(\mathbb{N}) \hookrightarrow \ell^2(\mathbb{Z})$, i.e. $\boxed{(-\Delta)^{\alpha} u_n = \sum_{m=1}^{\infty} (-1)^{m+n}\binom{2\alpha}{\alpha+m-n} u_m}$

CONTINUOUS CASE

- Due to the non-locality of the fractional derivatives, the definition depends on whether the functions are formally extended by zero to $\mathbb{R}_-$.

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THEOREM [Bogdan–Dyda, 2011]

(i) Let $\alpha \in (0,1]$. Then $\forall \varphi \in C_0^\infty(\mathbb{R}_+)$ it holds that

$$\boxed{\int_0^\infty \overline{\varphi(x)}(-\Delta_{\mathbb{R}_+})^\alpha \varphi(x)\,\mathrm{d}x \geq \frac{\Gamma^2(\alpha+1/2)-\Gamma(2\alpha)\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{|\varphi(x)|^2}{x^{2\alpha}}\,\mathrm{d}x}$$

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- Due to the non-locality of the fractional derivatives, the definition depends on whether the functions are formally extended by zero to $\mathbb{R}_-$.

Theorem [Bogdan–Dyda, 2011]

(i) Let $\alpha \in (0,1]$. Then $\forall \varphi \in C_0^\infty(\mathbb{R}_+)$ it holds that

$$\boxed{\int_0^\infty \overline{\varphi(x)}(-\Delta_{\mathbb{R}_+})^\alpha \varphi(x)\,\mathrm{d}x \geq \frac{\Gamma^2(\alpha+1/2)-\Gamma(2\alpha)\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{|\varphi(x)|^2}{x^{2\alpha}}\,\mathrm{d}x}$$

(ii) Let $\alpha > 0$. Then $\forall \varphi \in C_0^\infty(\mathbb{R}_+)$ it holds that

$$\boxed{\int_0^\infty \overline{\varphi(x)}(-\Delta_{\mathbb{R}})^\alpha \varphi(x)\,\mathrm{d}x \geq \frac{\Gamma^2(\alpha+1/2)}{\pi}\int_0^\infty \frac{|\varphi(x)|^2}{x^{2\alpha}}\,\mathrm{d}x}$$

- The constants are optimal in both cases.

- We extend the definition of $(-\Delta)^\alpha$ to the formal domain

$$\mathcal{D}_\alpha := \Big\{ u \in C(\mathbb{N}) \;\Big|\; \sum\nolimits_{m=1}^{\infty} \big| (-\Delta)^\alpha_{n,m} u_m \big| < \infty,\ \forall n \in \mathbb{N} \Big\}.$$

SUFFICIENT CONDITIONS FOR A PARAMETER SEQUENCE

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ASSUMPTIONS: Given $\mathfrak{g} = \{\mathfrak{g}_n\}_{n=1}^{\infty} \in \mathcal{D}_\alpha$ [parameter sequence]

THEOREM [Š–W]: If a parameter sequence $\mathfrak{g}$ satisfies $\mathfrak{g}_n > 0$ for all $n \geq 1$, then $\forall u \in C_0(\mathbb{N})$ it holds that

$$\boxed{\sum_{n=1}^{\infty} \overline{u_n}(-\Delta)^\alpha u_n - \sum_{n=1}^{\infty} \frac{(-\Delta)^\alpha \mathfrak{g}_n}{\mathfrak{g}_n} |u_n|^2 = -\sum_{1 \leq n < m} (-\Delta)^\alpha_{m,n}\, \mathfrak{g}_m\, \mathfrak{g}_n \left| \frac{u_m}{\mathfrak{g}_m} - \frac{u_n}{\mathfrak{g}_n} \right|^2 \overset{?}{\geq} 0}$$

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$$\nabla^k \mathfrak{g}_n > 0 \quad \text{for all } n \geq 1 \text{ and } k \in \{0, \dots, \lceil \alpha \rceil - 1\} \tag{MON'}$$

$$\nabla^k \mathfrak{g} \in \mathcal{D}_{\alpha - k} \text{ and } (-\Delta)^{\alpha - k} \nabla^k \mathfrak{g}_n > 0 \quad \text{for all } n \geq 1 \text{ and } k \in \{0, \dots, \lceil \alpha \rceil - 1\} \tag{SUPHAR'}$$

Theorem [Š–W]: If a parameter sequence $\mathfrak{g}$ satisfies (MON') and (SUPHAR'), then $\forall u \in C_0(\mathbb{N})$ it holds that

$$\boxed{\sum_{n=1}^{\infty} \overline{u_n} (-\Delta)^\alpha u_n - \sum_{n=1}^{\infty} \frac{(-\Delta)^\alpha \mathfrak{g}_n}{\mathfrak{g}_n} |u_n|^2 = \sum_{k=1}^{\lceil \alpha \rceil} R_k^{(\alpha)}(\mathfrak{g}, u) \geq 0}$$

and therefore $\rho(\mathfrak{g}) = (-\Delta)^\alpha \mathfrak{g} / \mathfrak{g}$ is a fractional Hardy weight.

- Bearing the definition of $\mathfrak{g}^{(\ell,p)}$ in mind, we set

$$\boxed{\mathfrak{g}_n^{(\alpha)} := \frac{\Gamma(n+\alpha-1/2)}{\Gamma(n)}}$$

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is an optimal discrete fractional Hardy weight.

COROLLARY [Š–W]: For all $u \in \ell^2(\mathbb{N})$, with $u_n = 0$ for all $n < \lceil\alpha\rceil$, we have the inequality

$$\boxed{\sum_{n=\lceil\alpha\rceil}^{\infty} \overline{u_n}(-\Delta)^\alpha u_n \geq \frac{\Gamma^2(\alpha+1/2)}{\pi} \sum_{n=\lceil\alpha\rceil}^{\infty} \frac{|u_n|^2}{n^{2\alpha}}}$$

and the constant $\Gamma^2(\alpha+1/2)/\pi$ is sharp.

LITERATURE

BASED ON:

- F. Š., J. W.: *Optimal discrete Hardy–Rellich–Birman inequalities*, accepted in J. Anal. Math.
- F. Š., J. W.: *Optimal discrete p-Hardy–Rellich–Birman inequalities*, arXiv:2605.25238.
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Thank you!