The spectral pair for Schrödinger operators with complex integrable potentials

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Mathematical aspects of the physics with non-self-adjoint operators
CIRM, Marseille
April 20–24, 2026

Based on: A. Pushnitski, F. Štampach: `arXiv:2601.05112` (2026).

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$$\boxed{H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q}$$

on $\mathrm{Dom}\, H = \{f \in L^2(\mathbb{R}_+) \mid f, f' \text{ are a.c. on } \mathbb{R}_+, -f'' + qf \in L^2(\mathbb{R}_+), \text{ and } f'(0) + \alpha f(0) = 0\}$,

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- Boundary functionals:

$$\ell_\alpha(f) := \frac{\overline{\alpha} f'(0) - f(0)}{\sqrt{1+|\alpha|^2}}, \qquad \ell_\alpha^\perp(f) := \frac{f'(0) + \alpha f(0)}{\sqrt{1+|\alpha|^2}}, \qquad \begin{pmatrix} \ell_\infty(f) = f'(0), \\ \ell_\infty^\perp(f) = f(0) \end{pmatrix}$$

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where φ and θ are solutions (fundamental system) of the eigenvalue equation

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satisfying the boundary conditions: $\begin{cases} \ell_\alpha(\varphi) = -1, \\ \ell_\alpha^\perp(\varphi) = 0, \end{cases}$ $\begin{cases} \ell_\alpha(\theta) = 0, \\ \ell_\alpha^\perp(\theta) = 1. \end{cases}$

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- The spectral measure: The m-function is Herglotz–Nevanlinna, therefore

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- Borg–Marchenko uniqueness theorem (1949–51): $(q, \alpha) \mapsto \sigma$ is injective.

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- σ on $\mathbb{R}_-$: $\lambda = -\kappa^2 < 0$, with $\kappa > 0$, is an eigenvalue of $H \iff \ell_\alpha^\perp(f_{+\mathrm{i}}(\cdot, \mathrm{i}\kappa)) = 0$ and

$$\sigma(\{\lambda\}) = \frac{|\ell_\alpha(f_{+\mathrm{i}}(\cdot, \mathrm{i}\kappa))|^2}{\|f_{+\mathrm{i}}(\cdot, \mathrm{i}\kappa)\|_{L^2}^2}, \qquad \text{where } \kappa = \sqrt{|\lambda|} > 0.$$

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- Hermitisation of H:
$$\begin{pmatrix} 0 & H \\ H^* & 0 \end{pmatrix} \quad \text{in } L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+).$$

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$$\boxed{\mathbf{H} = -\epsilon \frac{\mathrm{d}^2}{\mathrm{d}x^2} + Q,} \quad \text{with } \epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix},$$

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- Let Φ, Θ be the 2×2 matrix-valued solutions (the fundamental system) of

$$-\epsilon F'' + QF = \lambda F, \qquad \text{on } \mathbb{R}_+$$

satisfying the Cauchy data: $L_\alpha(\Phi) = -I$, $L_\alpha^\perp(\Phi) = 0$ and $L_\alpha(\Theta) = 0$, $L_\alpha^\perp(\Theta) = \epsilon$.

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$$\Theta(\cdot, \lambda) - \Phi(\cdot, \lambda)M(\lambda) \in L^2(\mathbb{R}_+; \mathbb{C}^{2,2}).$$
- Function $M : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2,2}$ is Herglotz–Nevanlinna, therefore
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Theorem

There exists a unique **even** positive measure ν on $\mathbb{R}$ and a unique **odd** complex-valued function $\psi \in L^\infty(\nu)$ satisfying $\|\psi\|_\infty \leq 1$ such that
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We call (ν, ψ) the spectral pair of H.

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Theorem (Jost solutions)

For all $k \in \mathbb{C} \setminus \{0\}$, there exist Jost solutions $F_{\pm 1}$, $F_{\pm i}$ to $(*)$ with the asymptotics:

$$F_{\pm 1}(x,k) = e^{\pm kx} e_- + o(e^{\pm kx}), \qquad x \to \infty,$$
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- Let $k > 0$; while F_{-1} is uniquely determined by its asymptotics, but the other Jost solutions are **not**. E.g., $F_{+\mathrm{i}}$ and $F_{+\mathrm{i}} + cF_{-1}$ have the same asymptotics for all $c \in \mathbb{C}$. A more accurate description of the Jost solutions $F_{\pm 1}$, $F_{\pm \mathrm{i}}$ should be as classes.

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- The Jost solutions are constructed as solutions to Fredholm (**not Voltera**) type integral equations, e. g.

$$F_{+\mathrm{i}}(x,k) = \mathrm{e}^{\mathrm{i}kx} e_{+} - \frac{1}{k} \int_{x}^{\infty} \sin(k(x-y)) P_{+} Q(y) F_{+\mathrm{i}}(y,k) \mathrm{d}y + \frac{1}{2k} \int_{0}^{\infty} \mathrm{e}^{-k|x-y|} P_{-} Q(y) F_{+\mathrm{i}}(y,k) \mathrm{d}y$$

for $0 \leq \arg k < \pi/4$ and $x \gg 0$.

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- Let $k > 0$; while F_{-1} is uniquely determined by its asymptotics, but the other Jost solutions are **not**. E.g., $F_{+\mathrm{i}}$ and $F_{+\mathrm{i}} + cF_{-1}$ have the same asymptotics for all $c \in \mathbb{C}$. A more accurate description of the Jost solutions $F_{\pm 1}$, $F_{\pm \mathrm{i}}$ should be as classes.
- The Jost solutions are constructed as solutions to Fredholm (**not Voltera**) type integral equations, e. g.

$$F_{+\mathrm{i}}(x,k) = \mathrm{e}^{\mathrm{i}kx} e_{+} - \frac{1}{k}\int_x^\infty \sin(k(x-y)) P_{+} Q(y) F_{+\mathrm{i}}(y,k) \mathrm{d}y + \frac{1}{2k}\int_0^\infty \mathrm{e}^{-k|x-y|} P_{-} Q(y) F_{+\mathrm{i}}(y,k) \mathrm{d}y$$

  for $0 \leq \arg k < \pi/4$ and $x \gg 0$.
- Analytic and asymptotic properties of the Jost solutions follows...

The spectrum of **H**

Theorem

The spectrum of $\mathbf{H}$

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Goal: To describe

- the density $\mathrm{d}\nu(\lambda)/\mathrm{d}\lambda$ and $\psi(\lambda)$, if $\lambda \neq 0$ is not a point mass of ν,
- $\nu(\{\lambda\})$ and $\psi(\lambda)$, if $\lambda \neq 0$ is a point mass of ν,

in terms of the Jost solutions F_{-1}, $F_{+\mathrm{i}}$, with $k = \sqrt{|\lambda|} > 0$.

The distinguished solution $\mathbf{e}$

Proposition

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1. There exists a unique solution $\mathbf{e} = \mathbf{e}(x, k)$ to the anti-linear eigenvalue equation

$$-\mathbf{e}'' + q\mathbf{e} = -k^2\overline{\mathbf{e}}$$

satisfying the asymptotics

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$$\mathbf{e}(x,k) = \mathrm{e}^{-kx} + o(\mathrm{e}^{-kx}), \quad x \to \infty.$$
2. The Jost solutions F_{-1} and $F_{+\mathrm{i}}$ can be expressed as
$$F_{-1}(x,k) = \begin{pmatrix} \overline{\mathbf{e}(x,k)} \\ -\mathbf{e}(x,k) \end{pmatrix}, \qquad F_{+\mathrm{i}}(x,\mathrm{i}k) = \begin{pmatrix} \overline{\mathbf{e}(x,k)} \\ \mathbf{e}(x,k) \end{pmatrix}.$$

Formulas for the spectral pair

Theorem (F. Š., A. Pushnitski)

Suppose $k > 0$.

Formulas for the spectral pair

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Suppose $k > 0$.

1. If $\lambda = k^2$ is not an eigenvalue of $\mathbf{H}$, then

$$\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}(\lambda) = \frac{2k}{\pi}\frac{|\ell_\alpha^\perp(\mathbf{e})|^2}{|\det\{L_\alpha^\perp(F_{+\mathrm{i}}), L_\alpha^\perp(F_{-1})\}|^2}, \quad \psi(\lambda) = \frac{\ell_\alpha^\perp(\mathbf{e})}{\overline{\ell_\alpha^\perp(\mathbf{e})}}.$$

   The determinant in the denominator is unambiguously defined (i.e. it does not depend on the choice of the Jost solution $F_{+\mathrm{i}}$) and non-vanishing.

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   The determinant in the denominator is unambiguously defined (i.e. it does not depend on the choice of the Jost solution $F_{+\mathrm{i}}$) and non-vanishing.

2. If $\lambda = k^2$ is an eigenvalue of $\mathbf{H}$, then $\ell_\alpha^\perp(\mathbf{e}) = 0$, $\ell_\alpha(\mathbf{e}) \neq 0$ and we have

$$\nu(\{\lambda\}) = \frac{|\ell_\alpha(\mathbf{e})|^2}{2\|\mathbf{e}\|^2}, \quad \psi(\lambda) = -\frac{\overline{\ell_\alpha(\mathbf{e})}}{\ell_\alpha(\mathbf{e})}.$$

Related literature

Hankel operators:

- P. Gérard, S. Grellier: *The cubic Szegő equation and Hankel operators*, Astérisque **389** (2017).
- P. Gérard, A. Pushnitski, S. Treil: *An inverse spectral problem for non-compact Hankel operators with simple spectrum*, J. Anal. Math. **154** (2024).

Jacobi operators:

- A. Pushnitski, F. Š.: *An inverse spectral problem for non-self-adjoint Jacobi matrices*, Int. Math. Res. Not. **2024** (2024).
- A. Pushnitski, F. Š.: *A functional model and tridiagonalisation for symmetric anti-linear operators*, J. Operator Theory **95** (2026).

Schrödinger operators:

- A. Pushnitski, F. Š.: *The Borg–Marchenko uniqueness theorem for complex potentials*, accepted in J. Math. Pures Appl. (2026).
- A. Pushnitski, F. Š.: *Schrödinger operators on the half-line with integrable complex potentials* (2026), `arXiv:2601.05112`.

THANK YOU!