~~The spectral pair~~

~~An inverse spectral problem~~ for NSA Jacobi & Schrödinger operators

joint with Alexander Pushnitski

Articles : "Jacobi case", IMRN, 2024.

"Schrödinger case", J. Math. Pures Appl. (to appear).

1) SA Jacobi operators : Assum. : $b_n \in \mathbb{R}$, $a_n > 0$ (non-deg.)
+ both seq. bounded (for simpli.)

$$J = \begin{bmatrix} b_0 & a_0 & & \\ a_0 & b_1 & a_1 & \\ & a_1 & b_2 & a_2 \\ & & \ddots & \ddots \end{bmatrix} \quad \text{acting on } \ell^2(\mathbb{N}_0)$$

↳ $J = J^*$ bounded, δ_0 is a cyclic vector of J.

Spectral measure : $\mu_J := \langle \delta_0, E_J \delta_0 \rangle$.

Spectral map : $\Lambda : J \longmapsto \mu_J$

Thm. : 1) Injectivity of Λ (uniqueness) : The spectral meas. μ_J determines uniquely J.

2) Surjectivity of Λ (charact.) : If μ is a probability meas. on $\mathbb{R}$ with infinite & compact support, then $\exists J$ s.t. $\Lambda(J) = \mu$.

2) NSA Jacobi op. : Assum. : relax $b_n \in \mathbb{R}$ to $b_n \in \mathbb{C}$ (keep $a_n > 0$)
↳ $J \neq J^*$ (still bounded)

Hermitisation : $\begin{pmatrix} 0 & J \\ J^* & 0 \end{pmatrix}$ on $\ell^2(\mathbb{N}_0) \oplus \ell^2(\mathbb{N}_0) \simeq \ell^2(\mathbb{N}_0; \mathbb{C}^2)$

equiv. to $\mathbb{J} = \begin{bmatrix} B_0 & A_0 & & \\ A_0 & B_1 & A_1 & \\ & A_1 & B_2 & A_2 \\ & & \ddots & \ddots \end{bmatrix}$ $\quad A_n = \begin{pmatrix} 0 & a_n \\ a_n & 0 \end{pmatrix}$

$B_n = \begin{pmatrix} 0 & b_n \\ b_n & 0 \end{pmatrix}$

↳ $\mathbb{J} = \mathbb{J}^* \rightsquigarrow$ Spec. meas. of $\mathbb{J}$: $\Sigma := \mathbb{P}_0 E_{\mathbb{J}} \mathbb{P}_0^*$

2×2 matrix-val. meas.

Thm.: There $\exists$ even prob. meas. ν on $\mathbb{R}$ with compact & infinite support and $\exists$ odd $\varphi \in L^\infty(\nu)$, $\|\varphi\|_\infty \leq 1$ such that

$$\boxed{d\Sigma = \begin{pmatrix} 1 & \varphi \\ \bar{\varphi} & 1 \end{pmatrix} d\nu .}$$

Def.: We call (ν, φ) the spectral pair of J.
(We refer to φ as the phase function.)

Properties: 1) $\nu|_{[0,\infty)} = \langle \delta_0, E_{|J|} \delta_0 \rangle$ $\quad (|J| := \sqrt{J^*J})$

2) Spectrum of $|J|$ has multiplicity equal to
$\begin{cases} 1 \text{ on } \{s>0 \mid |\varphi(s)| = 1\} \\ 2 \text{ on } \{s>0 \mid |\varphi(s)| < 1\} \end{cases}$ choosing a Borel repre φ

3) $J = J^* \iff \varphi$ real-valued, and in this case

$$d\mu = (1+\varphi)\, d\nu$$

4) J^*J has simple spectrum $\iff |\varphi| = 1$ ν-a.e.

to stress the relation betw. (ν, φ) & polar decomp. $J = V|J|$

5) $b_n = 0 \iff \psi = 0$

Def. the spectral map : $\Lambda(J) := (\nu, \psi)$.

Thm. (Pushnitski – Š) :

1) Inj. of Λ : J is uniquely determined by its spectr. pair.

2) Sur. of Λ : Given i) even prob. meas. ν on $\mathbb{R}$ with infinite & compact support,

ii) $\psi \in L^\infty(\nu)$, $\|\psi\|_\infty \leq 1$, odd,

then $\exists J$ s.t. $\Lambda(J) = (\nu, \psi)$.

3) SA Schröd. op. ; Assum. : $q \in L^\infty(\mathbb{R}_+)$ real.-valued

$\alpha \in \mathbb{R} \cup \{\infty\}$

$$H := -\frac{d^2}{dx^2} + q \quad \text{in } L^2(\mathbb{R}_+)$$

$f(0) = 0$ ($\alpha = \infty$)

$$\mathrm{Dom}\, H := \{ f \in W^{2,2}(\mathbb{R}_+) \mid f'(0) + \alpha f(0) = 0 \}$$

The m-function : φ, θ sol. of $-f'' + qf = \lambda f$

sat. the Cauchy data : $\varphi(0) = \sin\beta$ $\theta(0) = \cos\beta$

$\beta = \beta(\alpha)$ $\varphi'(0) = -\cos\beta$ $\theta'(0) = \sin\beta$

- q bounded $\Rightarrow \forall \lambda \in \mathbb{C} \setminus \mathbb{R}$ $\exists_1 m_\alpha(\lambda)$ s.t. $\theta(\cdot) - \varphi(\cdot) m_\alpha(\lambda) \in L^2(\mathbb{R}_+)$

- $m_\alpha : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ is Herglotz–Nevanlinna (m_α analytic, $\mathrm{Im}\, m_\alpha(\lambda) \geq 0$, $\mathrm{Im}\, \lambda > 0$, and $m_\alpha(\bar\lambda) = \overline{m_\alpha(\lambda)}$)

Herglotz integral repre:

$$m_\alpha(\lambda) = \mathrm{Re}\, m_\alpha(i) + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\mu(t)$$

μ is the spectral meas. of H ($H \simeq x\cdot$ in $L^2(\mathbb{R}, \mu)$),

Spectral map: $\Lambda : (\alpha, q) \mapsto \mu$

Thm. (Borg[49] - Marchenko[52]): Λ injective (uniqueness), i.e. both q & α uniquely determined by its spectr. meas.

Characterisation; $\mathrm{Ran}\,\Lambda$ = "no answer"

- first works by Gelfand-Levitan, Krein (50s)
- Remling '02; $q \in L^1_{loc}(\mathbb{R}_+)$
- Simon-etal; $q \in L^1(\mathbb{R}_+)$, $q \in L^2(\mathbb{R}_+)$

4) NSA Schröd. op: Assum.: relax q real-valued to complex-val
& $\alpha \in \mathbb{R} \cup \{\infty\}$ to $\alpha \in \mathbb{C} \cup \{\infty\}$
$\Rightarrow H \neq H^*$

Hermitization: $\begin{pmatrix} 0 & H \\ H^* & 0 \end{pmatrix}$ in $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+) \simeq L^2(\mathbb{R}_+; \mathbb{C}^2)$

[21]
$$H := -\mathcal{E} \frac{d^2}{dx^2} + Q \qquad \mathcal{E} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Q = \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}$$

Fund. sys.: Φ, Θ $\mathbb{C}^{2\times 2}$-val. sol. to $-\mathcal{E}F'' + QF = \lambda F$
sat. Cauchy data:

M-func.: $\forall \alpha \in \mathbb{C}\setminus\mathbb{R}$ $\exists_1$ $M_\alpha(\lambda) \in \mathbb{C}^{2,2}$ s.t.

$$\Theta(\cdot) - \phi(\cdot) M_\alpha(\lambda) \in L^2(\mathbb{R}_+; \mathbb{C}^{2,2}).$$

↳ $M_\alpha : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{2,2}$ is H-N, i.e. analytic

- $\operatorname{Im} M_\alpha(\lambda) = \frac{1}{2i}\left(M_\alpha(\lambda) - M_\alpha(\lambda)^*\right) \geq 0$ for $\operatorname{Im}\lambda > 0$
- $M_\alpha(\lambda)^* = M_\alpha(\bar\lambda)$

↳ Herglotz inf. repre:

$$M_\alpha(\lambda) = \operatorname{Re} M_\alpha(i) + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2}\right) d\Sigma(t).$$

The $\mathbb{C}^{2,2}$-valued meas. Σ is the spectral meas. of H

The same thm. as in the Jacobi case:

$$\boxed{d\Sigma = \begin{pmatrix} 1 & \bar\psi \\ \psi & 1 \end{pmatrix} d\nu}$$

(ν, ψ) is the spec. pair of H

Λ:

Thm. (à la Borg-Marchenko): The spectr. map $(\alpha, q) \in (\mathbb{C}\cup\{\infty\}) \times L^\infty(\mathbb{R}_+) \longmapsto (\nu, \psi)$ is injective.

Open prob.: $\operatorname{Ran}\Lambda = ?$ for various classes of potentials

Properties :

1) $\Lambda(H) = (\nu, \psi) \Rightarrow \Lambda(H^*) = (\nu, \overline{\psi})$

In part., $H = H^* \Leftrightarrow \psi$ is real-valued

2) $H = H^* \Rightarrow d\mu = (1+\psi)\, d\nu$

3) Marchenko's type asympt. formulas :

- $\nu([0,r]) = \begin{cases} C_\alpha r^{1/2} + o(r^{1/2}), & \alpha \neq 0 \\ C_0 r^{3/2} + o(r^{3/2}), & \alpha = 0 \end{cases}$,

- $\int_0^r \psi(s)\, d\nu(s) = \begin{cases} - \text{ " } - \end{cases}$

($\psi(s) \xrightarrow[s\to\infty]{} 1$ " in an average weak sense ")

4) If $\lambda > 0$ is simple evl. of $|H|$ $\Rightarrow \exists f$ up to a sign $\pm$ norm. sol. $e \in L^2(\mathbb{R}_+)$ est. $-e'' + qe = \lambda e$,

$e'(0) + \alpha e(0) = 0$.

We have formulas

$\nu([\lambda]) =$ " in terms of $\alpha, e(0), e'(0)$ ",

$\psi(\lambda) =$ " - " - ".