Optimal discrete p-Hardy–Rellich–Birman inequalities

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jointly with

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Workshop on Operator Theory, Complex Analysis, and Applications

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CLASSICAL HARDY INEQUALITIES (1915–25)

MOTIVATION: To prove the HILBERT INEQUALITY (∼1910):

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{u_n\overline{u_m}}{m+n+1}\le\pi\sum_{n=0}^{\infty}|u_n|^2$$

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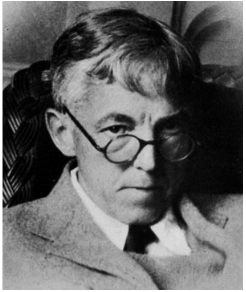
DISCRETE $\qquad \frac{1}{p}+\frac{1}{q}=1 \qquad$ CONTINUOUS



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$$\frac{1}{p} + \frac{1}{q} = 1$$

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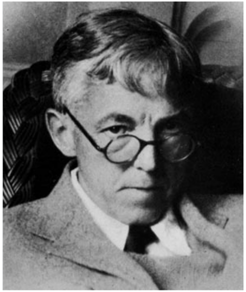

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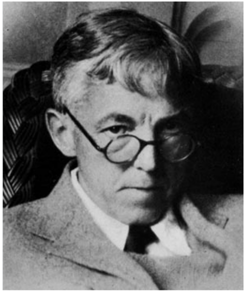

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Classical Hardy inequalities (1915–25)

Motivation: To prove the Hilbert inequality (∼1910):

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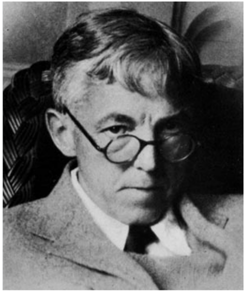

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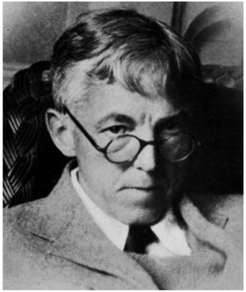

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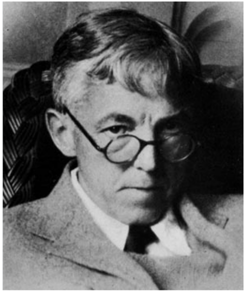

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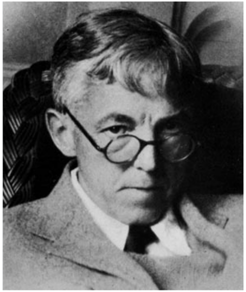

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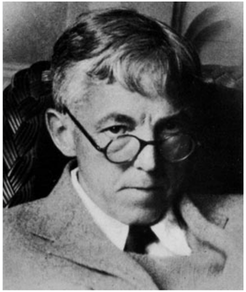

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$$\sum_{n=1}^{\infty} |v_n - v_{n-1}|^p \geq \sum_{n=1}^{\infty} w_n^{\mathrm{FKP}} \, |v_n|^p$$

holds for all $v \in \ell^p(\mathbb{N}_0)$, $v_0 = 0$, with

$$\boxed{w_n^{\mathrm{FKP}} = \left[1 - \left(1 - \frac{1}{n}\right)^{1/q}\right]^{p-1} - \left[\left(1 + \frac{1}{n}\right)^{1/q} - 1\right]^{p-1} > \frac{1}{q^p n^p}.}$$

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- Q: Is there a similar optimal improvement of the classical Hardy inequality?

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- For arbitrary $\ell \in \mathbb{N}$: case $p = 2$ [Huang–Ye, 2024]; case $p > 1$ [Š–Waclawek 2026].

DEFINITION: For a (non-specified) general $\ell \in \mathbb{N}$, any non-negative sequence $\rho = \{\rho_n\}_{n=\ell}^{\infty}$ is called a discrete p-Hardy–Rellich–Birman weight, if and only if the inequality

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(iii) Optimality near infinity: For any $m \geq \ell$ and $\varepsilon > 0$, there exists $u \in C_0(\mathbb{N}_0)$, with $u_n = 0$ for $n < m$, such that

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OPTIMAL DISCRETE p-HARDY–RELLICH–BIRMAN WEIGHT

DEFINITION: For a (non-specified) general $\ell \in \mathbb{N}$, any non-negative sequence $\rho = \{\rho_n\}_{n=\ell}^{\infty}$ is called a discrete p-Hardy–Rellich–Birman weight, if and only if the inequality

$$\boxed{\sum_{n=\ell}^{\infty} \left|\nabla^{\ell} u_n\right|^p \geq \sum_{n=\ell}^{\infty} \rho_n |u_n|^p} \qquad (\star)$$

holds for all $u \in \mathbb{C}_0(\mathbb{N}_0)$ with $u_0 = \cdots = u_{\ell-1} = 0$. The weight ρ is called optimal if it exhibits the following three properties:

(i) Criticality: For any discrete p-Hardy–Rellich–Birman weight $\tilde{\rho}$, such that $\tilde{\rho} \geq \rho$, it follows that $\tilde{\rho} = \rho$.

(ii) Non-attainability: Whenever the equality in $(\star)$ is attained for any u such that the RHS of $(\star)$ is finite, then $u \equiv 0$.

(iii) Optimality near infinity: For any $m \geq \ell$ and $\varepsilon > 0$, there exists $u \in C_0(\mathbb{N}_0)$, with $u_n = 0$ for $n < m$, such that

$$\sum_{n=\ell}^{\infty} \left|\nabla^{\ell} u_n\right|^p < (1+\varepsilon) \sum_{n=\ell}^{\infty} \rho_n |u_n|^p.$$

- Motivated by [Devyver–Fraas–Pinchover, 2014] and [Keller–Pinchover–Pogorzelski, 2018].

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- The definition of $-\Delta_p^{(\ell)}$ stems from a polarization of the ℓ^p-norm of $\nabla^\ell u$:

$$\sum_{n=\ell}^{\infty} \left|\nabla^\ell u_n\right|^p = \sum_{n=\ell}^{\infty} \overline{u_n} \left(-\Delta_p^{(\ell)} u\right)_n$$

for $u \in C_0(\mathbb{N}_0)$ with $u_0 = \cdots = u_{\ell-1} = 0$.

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Assumptions: Given $\mathfrak{g} = \{\mathfrak{g}_n\}_{n=0}^{\infty}$ with $\mathfrak{g}_0 = \cdots = \mathfrak{g}_{\ell-1} = 0$. [parameter sequence]

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- If (MON) and (SUPHAR) with non-strict inequalities hold, then $\rho(\mathfrak{g})$ is a discrete p-Hardy –Rellich–Birman weight which is non-attainable in a weaker sense.

COMMENTS ON THE PROOF

- Two sided control:

$$\sum_{k=1}^{\ell} R_k^{(\ell)}(q;\mathfrak{g},u) \lesssim \sum_{n=\ell}^{\infty} \left|\nabla^{\ell} u_n\right|^p - \sum_{n=\ell}^{\infty} \frac{-\Delta_p^{(\ell)} \mathfrak{g}_n}{\mathfrak{g}_n^{p-1}} |u_n|^p \lesssim \sum_{k=1}^{\ell} R_k^{(\ell)}(p;\mathfrak{g},u),$$

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- Here we are cheating because $\mathfrak{g} \notin C_0(\mathbb{N}_0)$, but one can make it rigorous by an approximation argument.

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- $\mathfrak{g}^{(\ell,p)}$ obtained by an experienced guess.

THEOREM [Š-Waclawek]: The sequence

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- Important ingredient for the proof is that the function

$$g(x) := \frac{\Gamma(x+1/q)}{\Gamma(x+1)}$$

is logarithmically completely monotone on $\mathbb{R}_+$. It means that, for $h := -g'/g$, it holds $(-1)^k h^{(k)} > 0$ on $\mathbb{R}_+$ for all $k \in \mathbb{N}_0$.

- If $p = 2$, we have the new discrete Hardy–Rellich–Birman inequality

$$\boxed{\sum_{n=\ell}^{\infty} |\nabla^{\ell} u_n|^2 \geq \left(\frac{1}{2}\right)_{\ell}^{2} \sum_{n=\ell}^{\infty} \frac{|u_n|^2}{n(n-1/2)(n-1)\cdots(n-\ell+1/2)},}$$

for $u \in C_0(\mathbb{N}_0)$, $u_0 = \cdots = u_{\ell-1} = 0$, with optimal weight.

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- Compare the weight $\quad \rho_n^{(1,2)} = \dfrac{1}{4n(n-1/2)} \quad$ to $\quad \begin{cases} w_n^{\mathrm{H}} = \dfrac{1}{4n^2}, \\ w_n^{\mathrm{KPP}} = 2 - \sqrt{1 - \dfrac{1}{n}} - \sqrt{1 + \dfrac{1}{n}}. \end{cases}$

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- $\hat{\mathfrak{g}}_n = n^{\ell-1/2}$ - conjectured by [Gerhat–Krejčiřík–Š, 2025]; it produces a non-optimal improvement of the classical discrete Birman weight (notice that $\hat{\mathfrak{g}}_1 \neq 0, \dots, \hat{\mathfrak{g}}_{\ell-1} \neq 0$).

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- If $\hat{\mathfrak{g}}(x) = x^{\ell-1/2}$, then

$$\hat{\rho}_n = (-1)^\ell \frac{\hat{g}^{(2\ell)}(x)}{\hat{g}(x)} = \left(\frac{1}{2}\right)_\ell^2 \frac{1}{x^{2\ell}}$$

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CONJECTURE [Š-Waclawek]: The last claim is true for all $p > 1$ and $\ell \in \mathbb{N}$.

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$$\boxed{\tilde{\mathfrak{g}}_n^{(\ell,p)} := n^{1-1/p}(n-1)(n-2)\dots(n-\ell+1)}$$

- For $\ell = 1$, we get the optimal weight of [Fischer–Keller–Pogorzelski, 2023]
- For $p = 2$, we get the optimal weight of [Š–Waclawek, 2024]

THEOREM [Š-Waclawek]: If $p > 1$ and $\ell = 1$ or $p \in \mathbb{N}, p > 1$ and $\ell \in \mathbb{N}$, then

$$\tilde{\rho}^{(\ell,p)} := \frac{-\Delta_p^{(\ell)} \tilde{\mathfrak{g}}^{(\ell,p)}}{\left(\tilde{\mathfrak{g}}^{(\ell,p)}\right)^{p-1}}$$

is an optimal discrete p-Hardy–Rellich–Birman weight.

CONJECTURE [Š-Waclawek]: The last claim is true for all $p > 1$ and $\ell \in \mathbb{N}$.

- Sequence $\tilde{\mathfrak{g}}^{(\ell,p)}$ does not behave nicely w.r.t. the discrete differentiation.

LITERATURE

BASED ON:

- F. Š., J. Waclawek: *Optimal diescrete p-Hardy–Rellich–Birman inequalities*, preprint.
- F. Š., J. Waclawek: *Optimal diescrete Hardy–Rellich–Birman inequalities*, accepted in J. Anal. Math.

RELEVANT:

- F. Fischer, M. Keller, F. Pogorzelski: *An improved discrete p-Hardy inequality*, Integr. Equ. Oper. Theory 95 (2023).
- B. Gerhat, D. Krejčiřík, F. Š.: *An improved discrete Rellich inequality on the half-line*, Israel J. Math. 268 (2025).
- X. Huang, D. Ye: *One-dimensional sharp discrete Hardy-Rellich inequalities*, J. Lond. Math. Soc. (2) 109 (2024).
- M. Keller, Y. Pinchover, F. Pogorzelski: *Optimal Hardy inequalities for Schrödinger operators on graphs*, Comm. Math. Phys. 358 (2018).
- F. Š., J. Waclawek: *The p-Hardy–Rellich–Birman inequalities on the half-line*, preprint.
- F. Š., J. Waclawek: *A Herglotz–Nevanlinna function from the optimal discrete p-Hardy weight*, Proc. Amer. Math. Soc. 154 (2026).

Thank you!