

The Classical Moment Problem

An Introduction and
Connections in Analysis

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BIRS Workshop:

**The New Frontier of
Herglotz–Nevanlinna Functions**

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Plan for the hour

The guiding question

How much of a positive measure μ is encoded in the numbers

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- ▶ Third: Connections to Jacobi matrices, orthogonal polynomials, and HN functions.
- ▶ Fourth: Nevanlinna parametrization and important classes of solutions.
- ▶ Fifth: Truncated moment problem.

Moments: shadows of a measure

- ▶ While a measure can be complicated, its moments are simple real numbers

$$s_0 = \mu(\mathbb{R}), \quad s_1 = \int x \, d\mu(x), \quad s_2 = \int x^2 \, d\mu(x), \quad \dots,$$

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Inverse problem

Given only the real sequence

$$s_0, s_1, s_2, \dots,$$

recover/describe all measures with such moments.

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Chebyshev's question, 1874

If a non-negative density ρ satisfies

$$\int_{\mathbb{R}} x^n \rho(x) \, dx = \int_{\mathbb{R}} x^n e^{-x^2} \, dx, \quad \forall n = 0, 1, 2, \dots,$$

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- ▶ For the Gaussian: yes.
- ▶ But in general: no.

Stieltjes' example

- For $\vartheta \in (-1, 1)$, define μ_ϑ on $(0, \infty)$ by

$$d\mu_\vartheta(x) = \frac{1}{\sqrt{\pi}} x^{-\log x} (1 + \vartheta \sin(2\pi \log x)) dx.$$

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Non-uniqueness is not a pathological phenomenon in moment problems.

Three basic questions

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Terminology

- | | |
|---|--|
| ▶ Unique case: <i>determinate</i> | ▶ $I = [0, 1]$: <i>Hausdorff</i> |
| ▶ Non-unique case: <i>indeterminate</i> | ▶ $I = [0, \infty)$: <i>Stieltjes</i> |
| | ▶ $I = \mathbb{R}$: <i>Hamburger</i> |

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Hausdorff moment problem

A sequence $\{s_n\}_{n \geq 0}$ is a moment sequence of a positive measure on $[0, 1]$ if and only if it is *completely monotone*:

$$(-1)^k (\Delta^k s)_n \geq 0, \quad k, n = 0, 1, 2, \dots$$

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- ▶ More subtle phenomena begin on unbounded intervals.

Hamburger MP: existence

- Let $s_n = \int x^n d\mu(x)$. The fact that every square has non-negative integral,

$$\int \left| c_0 + c_1 x + \cdots + c_n x^n \right|^2 d\mu(x) \geq 0,$$

is equivalent to *positive* semi-definiteness of Hankel matrices

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- ▶ Given a sequence s , we denote

$$\mathcal{M}_H(s) := \left\{ \mu \mid \text{supp } \mu \subset \mathbb{R} \text{ infinite and } s_n = \int x^n d\mu(x), \forall n = 0, 1, \dots \right\}.$$

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$$\mathcal{M}_H(s) \neq \emptyset \iff \det H_n(s) > 0, \quad \forall n = 0, 1, \dots$$

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- Let $s_n = \int x^n d\mu(x)$. If $\text{supp } \mu \subset [0, \infty)$, also

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Orthogonal polynomials: moments create a recurrence

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- ▶ Then the moments define an inner product on polynomials $\mathbb{C}[x]$:

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Moment sequence $\longrightarrow$ orthogonal polynomials $\longrightarrow$ recurrence coefficients.

Jacobi matrices: the operator behind the recurrence

- The recurrence is encoded by the *Jacobi matrix*

$$J = \begin{pmatrix} b_0 & a_0 & 0 & \cdots \\ a_0 & b_1 & a_1 & \cdots \\ 0 & a_1 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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Moment sequence $\longleftrightarrow$ orthogonal polynomials $\longleftrightarrow$ Jacobi matrix.

Uniqueness: operator theoretic characterization

Hamburger & Stieltjes MP: determinacy

1. The **Hamburger** moment problem is determinate $\iff J$ is **essentially self-adjoint** (i.e., J has a unique self-adjoint extension).

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- ▶ In one direction, it is not clear that distinct self-adjoint extensions give rise to distinct measures (but they do).
- ▶ The other direction is even less clear. For not only is it not obvious, it is **false** that every solution of the moment problem arises from some measure given by spectral measure of some self-adjoint extension.

The m -function

- ▶ Let J' be a self-adjoint extension of J . Then the m -function

$$m(z) = \langle e_0, (J' - z)^{-1} e_0 \rangle, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

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- ▶ The moment problem is equivalent to asking for Herglotz–Nevanlinna functions with prescribed asymptotic expansion at infinity. Indeterminacy means that the same asymptotic data admit several Herglotz completions.

Practical determinacy tests

Sufficient conditions for uniqueness

1. Carleman's criterion (1922–26): *det* HMP, if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} = \infty \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{1}{a_n} = \infty.$$

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2. *Chihara's condition (1989):* $\det$ HMP, if

$$b_n \rightarrow \infty, \quad \frac{a_n^2}{b_n b_{n+1}} \rightarrow L < \frac{1}{4}, \quad \liminf_{n \rightarrow \infty} \sqrt[n]{b_n} < \frac{1 + \sqrt{1 - 4L}}{1 - \sqrt{1 - 4L}}.$$

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Sufficient conditions for uniqueness

1. *Carleman's criterion (1922–26):* *det* HMP, if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} = \infty \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{1}{a_n} = \infty.$$

2. *Chihara's condition (1989):* *det* HMP, if

$$b_n \rightarrow \infty, \quad \frac{a_n^2}{b_n b_{n+1}} \rightarrow L < \frac{1}{4}, \quad \liminf_{n \rightarrow \infty} \sqrt[n]{b_n} < \frac{1 + \sqrt{1 - 4L}}{1 - \sqrt{1 - 4L}}.$$

- There is a hierarchy of stronger determinacy/self-adjointness criteria extending Carleman's criterion [Petropoulou–Velázquez, 2014].

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$$\int_{\mathbb{R}} \frac{d\mu_\phi(x)}{x - z} = -\frac{A(z)\phi(z) - C(z)}{B(z)\phi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

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$$m(z) \sim -\frac{s_0}{z} - \frac{s_1}{z^2} - \frac{s_2}{z^3} - \cdots, \quad |z| \rightarrow \infty.$$

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- ▶ Nevanlinna's theorem describes exactly "the remaining freedom".

Important class I: Nevanlinna extremal solutions

- For $\phi(z) = t \in \mathbb{R} \cup \{\infty\}$, the measure μ_t is discrete of the form

$$\mu_t = \sum_{x \in \Lambda_t} \rho(x) \delta_x, \quad \rho(x) = \frac{1}{B'(x)D(x) - B(x)D'(x)},$$

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► Take

$$\phi(z) = \begin{cases} \beta + i\gamma, & \Im z > 0, \\ \beta - i\gamma, & \Im z < 0, \end{cases} \quad \beta \in \mathbb{R}, \quad \gamma > 0.$$

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- The solution $\mu_{0,1}$ maximizes Krein's entropy integral; more info in [Gabardo, 1992].

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- ▶ Essentially arbitrary regularly varying growth below order 1 can occur for A, B, C, D [Pruckner–Reiffenstein–Woracek, 2026].

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Existence: even case

A measure μ with given moments $s_0, \dots, s_{2n}$ exists if and only if

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Existence: odd case

A measure μ with given moments $s_0, \dots, s_{2n+1}$ exists if and only if

$$H_n \geq 0 \quad \text{and} \quad (s_{n+1}, \dots, s_{2n+1})^T \in \text{Ran } H_n.$$

(This ensures that s_{2n+2} can be chosen so that $H_{n+1} \geq 0$ and $\text{rank } H_n = \text{rank } H_{n+1}$.)

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- ▶ If $\det H_n \neq 0$, then there are *infinitely many* solutions (including infinitely many with exactly $n + 1$ atoms).
- ▶ Construction in the singular case: Let $r = \text{rank } H_n$. Then $H_{r-1} > 0$ and the polynomial

$$p(x) = x^r - \sum_{j=0}^{r-1} a_j x^j, \quad \text{where } H_{r-1}(a_0, \dots, a_{r-1})^T = (s_r, \dots, s_{2r-1})^T,$$

has distinct real roots $x_1, \dots, x_r$. Then the unique solution is

$$\mu = \sum_{\ell=1}^r w_\ell \delta_{x_\ell}, \quad \text{where } \sum_{\ell=1}^r w_\ell x_\ell^k = s_k, \quad k = 0, \dots, r-1.$$

References for the moment problem

-  J. A. Shohat and J. D. Tamarkin: *The Problem of Moments*, AMS, 1943.
 -  N. I. Akhiezer: *The Classical Moment Problem and Some Related Questions in Analysis*, Oliver & Boyd, 1965.
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Thank you!

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- ▶ The only N-extremal solutions supported within $[0, \infty)$ are μ_t with $\alpha \leq t \leq 0$.
- ▶ For the indeterminate Stieltjes moment problem there is another way how to describe $\mathcal{M}_S$ known as *Krein parametrization*, [Krein, 1967].