

Spectral Analysis of Jacobi Operators and Related Problems

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- Consider Jacobi operator J acting on vectors from standard basis $\{e_n\}_{n=1}^{\infty}$ of $\ell^2(\mathbb{N})$ as

$$Je_n = w_{n-1}e_{n-1} + \lambda_n e_n + w_n e_{n+1} \quad (w_0 := 0)$$

where $\lambda_n \in \mathbb{C}$, $w_n \in \mathbb{C} \setminus \{0\}$, and $n \in \mathbb{N}$.

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- The matrix representation of J in the standard basis:

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- Objective: Investigation of the spectrum of J when the diagonal sequence dominates the off-diagonal in some sense.

For $z \in \mathbb{C}$ and $\lambda_n > 0$ define

$$A(z) := L^{-1/2}(UW + WU^* - z)L^{-1/2} = \begin{pmatrix} -\frac{z}{\lambda_1} & \frac{w_1}{\sqrt{\lambda_1\lambda_2}} & & & \\ \frac{w_1}{\sqrt{\lambda_1\lambda_2}} & -\frac{z}{\lambda_2} & \frac{w_2}{\sqrt{\lambda_2\lambda_3}} & & \\ & \frac{w_2}{\sqrt{\lambda_2\lambda_3}} & -\frac{z}{\lambda_3} & \frac{w_3}{\sqrt{\lambda_3\lambda_4}} & \\ & & & \ddots & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

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Proposition

Let $A(z)$ be Hilbert-Schmidt operator for some $0 \neq z \in \mathbb{C}$. Then

$$z \in \rho(J) \quad \text{iff} \quad -1 \in \rho(A(z))$$

and it holds

$$(J - z)^{-1} = L^{-1/2}(1 + A(z))^{-1}L^{-1/2}.$$

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To investigate the spectrum of J one can consider operator $A(z)$ instead. Main advantages are:

- $A(z)$ is Hilbert-Schmidt, while J can be unbounded
- one can use function $z \mapsto \det_2(1 + A(z))$ which is well defined as an entire function.

Definition

Let us define $\mathfrak{F} : D \rightarrow \mathbb{C}$ by relation

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \cdots x_{k_m} x_{k_m+1},$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C}; \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables let us identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$.

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- Note that the domain D is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$.

- For all $x \in D$ and $k = 1, 2, \dots$ one has

Recursive relation

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \dots, x_k) \mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \dots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(T^{k+1} x)$$

where T denotes the truncation operator from the left defined on the space of all sequences:

$$T(\{x_k\}_{k=1}^{\infty}) = \{x_{k+1}\}_{k=1}^{\infty}.$$

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- Functions $\mathfrak{F}$ restricted on $\ell^2(\mathbb{N})$ is a continuous functional on $\ell^2(\mathbb{N})$. Further, for $x \in D$, it holds

$$\lim_{n \rightarrow \infty} \mathfrak{F}(x_1, x_2, \dots, x_n) = \mathfrak{F}(x) \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathfrak{F}(T^n x) = 1.$$

- Equivalent definition for $\mathfrak{F}(x_1, x_2, \dots, x_n)$ is:

$$\mathfrak{F}(x_1, x_2, \dots, x_n) = \det X_n = \det \begin{pmatrix} 1 & x_1 & & & \\ x_2 & 1 & x_2 & & \\ & \ddots & \ddots & \ddots & \\ & & x_{n-1} & 1 & x_{n-1} \\ & & & x_n & 1 \end{pmatrix}.$$

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- Function $\mathfrak{F}$ is related to a continued fraction. For a given $x \in D$ such that $\mathfrak{F}(x) \neq 0$, it holds

$$\frac{\mathfrak{F}(Tx)}{\mathfrak{F}(x)} = \frac{1}{1 - \frac{x_1 x_2}{1 - \frac{x_2 x_3}{1 - \frac{x_3 x_4}{1 - \dots}}}}.$$

Proposition

Let $\{\lambda_n\}$ be positive and

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \left| \frac{w_n^2}{\lambda_n \lambda_{n+1}} \right| < \infty.$$

Then $A(z)$ is Hilbert-Schmidt for all $z \in \mathbb{C}$ and it holds

$$\det_2(1 + A(z)) = \mathfrak{F} \left(\left\{ \frac{\gamma_n^2}{\lambda_n - z} \right\}_{n=1}^{\infty} \right) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n} \right) e^{z/\lambda_n}$$

where the sequence $\{\gamma_n\}$ can be defined recursively as $\gamma_1 = 1$, $\gamma_{k+1} = w_k/\gamma_k$.

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- In the following we focus just on the function

$$F_J(z) := \mathfrak{F} \left(\left\{ \frac{\gamma_n^2}{\lambda_n - z} \right\}_{n=1}^{\infty} \right).$$

- Function F_J is well defined on $\mathbb{C} \setminus \overline{\{\lambda_n\}}$ if

$$\left\{ \frac{\gamma_n^2}{\lambda_n - z} \right\}_{n=1}^{\infty} \in D \quad \text{for all } z \in \mathbb{C} \setminus \overline{\{\lambda_n\}}$$

which holds if there is at least one $z_0 \in \mathbb{C} \setminus \overline{\{\lambda_n\}}$ such that

$$\sum_{n=1}^{\infty} \left| \frac{w_n^2}{(\lambda_n - z_0)(\lambda_{n+1} - z_0)} \right| < \infty$$

(λ_n and w_n are complex!)

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- This assumptions is assumed everywhere from now.
- F_J is meromorphic function on $\mathbb{C} \setminus \overline{\{\lambda_n\}}$ with poles in $z \in \{\lambda_n\} \setminus \text{der}(\{\lambda_n\})$ of finite order less or equal to the number

$$r(z) := \sum_{n=1}^{\infty} \delta_{z, \lambda_n}.$$

Let us define

$$\mathfrak{Z}(J) := \left\{ z \in \mathbb{C} \setminus \text{der}(\{\lambda_n\}); \lim_{u \rightarrow z} (u - z)^{r(z)} F_J(u) = 0 \right\}$$

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and, for $k \in \mathbb{Z}_+$ and $z \in \mathbb{C} \setminus \text{der}(\{\lambda_n\})$, we put

$$\xi_k(z) := \lim_{u \rightarrow z} (u - z)^{r(z)} \left(\prod_{l=1}^k \frac{w_{l-1}}{u - \lambda_l} \right) \mathfrak{F} \left(\left\{ \frac{\gamma_l^2}{\lambda_l - u} \right\}_{l=k+1}^{\infty} \right)$$

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Theorem

Equalities

$$\text{spec}(J) \setminus \text{der}(\{\lambda_n\}) = \text{spec}_p(J) \setminus \text{der}(\{\lambda_n\}) = \mathfrak{Z}(J)$$

hold and, for $z \in \mathfrak{Z}(J)$,

$$\xi(z) := (\xi_1(z), \xi_2(z), \xi_3(z), \dots)$$

is the eigenvector for eigenvalue z . Moreover, for $z \notin \overline{\{\lambda_n\}}$, vector $\xi(z)$ satisfies the formula

$$\sum_{k=1}^{\infty} (\xi_k(z))^2 = \xi'_0(z) \xi_1(z) - \xi_0(z) \xi'_1(z).$$

- The Green function $G_{ij}(z) = (e_i, (J - z)^{-1} e_j)$ of J is expressible in terms of $\mathfrak{F}$,

$$G_{ij}(z) = -\frac{1}{w_M} \prod_{l=m}^M \left(\frac{w_l}{z - \lambda_l} \right) \frac{\mathfrak{F} \left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=1}^{m-1} \right) \mathfrak{F} \left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=M+1}^{\infty} \right)}{\mathfrak{F} \left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=1}^{\infty} \right)}$$

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where $z \in \rho(J)$, $m := \min(i, j)$, and $M := \max(i, j)$.

- Especially, we get a compact formula for the Weyl m -function $m(z) = G_{11}(z)$,

$$m(z) = \frac{\mathfrak{F} \left(\left\{ \frac{\gamma_j^2}{\lambda_j - z} \right\}_{j=2}^{\infty} \right)}{(\lambda_1 - z) \mathfrak{F} \left(\left\{ \frac{\gamma_j^2}{\lambda_j - z} \right\}_{j=1}^{\infty} \right)}.$$

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- Basic Hypergeometric Functions ${}_1\phi_1$.

Let $w, \alpha \in \mathbb{C}$, $z - r\alpha \notin \alpha\mathbb{N}$, and $r \in \mathbb{Z}_+$ then it holds

$$\mathfrak{F}\left(\left\{\frac{w}{\alpha k - z}\right\}_{k=r+1}^{\infty}\right) = \left(\frac{w}{\alpha}\right)^{-r+z/\alpha} \Gamma\left(1 + r - \frac{z}{\alpha}\right) J_{r-z/\alpha}\left(\frac{2w}{\alpha}\right).$$

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For $r = 0$, the above function is characteristic function for Jacobi operator J of the form

$$J = \begin{pmatrix} \alpha & w & & & \\ w & 2\alpha & w & & \\ & w & 3\alpha & w & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

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Hence we have

$$\text{spec}(J) = \{z \in \mathbb{C}; J_{-\frac{z}{\alpha}}\left(\frac{2w}{\alpha}\right) = 0\},$$

and the formula for the k th entry of the respective eigenvector is

$$v_k(z) = (-1)^k J_{k-\frac{z}{\alpha}}\left(\frac{2w}{\alpha}\right).$$

For $w, \nu \in \mathbb{C}$ and $q \in (0, 1)$, a q -analogue of the previous formula reads

$$\mathfrak{F}\left(\left\{\frac{w}{[\nu+k]_q}\right\}_{k=1}^{\infty}\right) = q^{-\nu/4} \Gamma_q(\nu+1) w^{-\nu} J_{\nu}^{(2)}(2q^{-1/4}(1-q)w; q).$$

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$$v_n(z) = q^{(n-1)(n-2)/4} (q^n z^{-1}; q)_{\infty} \left(\frac{\beta}{z}\right)^n {}_0\phi_1(; q^n z^{-1}; q, -q^n z^{-2}\beta^2).$$

- For $\mu > 0$, $\nu \in \mathbb{R}$, and $k = 1, 2, \dots$, let us put

$$\lambda_k = \frac{\nu}{(\mu + k - 1)(\mu + k)}, \quad w_k = \frac{1}{\mu + k} \sqrt{\frac{(\mu + k)^2 + \nu^2}{4(\mu + k)^2 - 1}}.$$

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For $\delta, a \in \mathbb{C}$, and $n \in \mathbb{Z}_+$, it holds

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- For $a > 0$, the operator J is not hermitian, however, $\text{spec}(J)$ is real!

- For α, β, γ , with $\beta > 0, \gamma > 0$ and $\alpha + \beta > 0$, let us consider matrix J of the form

$$J(\alpha, \beta, \gamma) = \begin{pmatrix} \gamma & \sqrt{\alpha+\beta} & & \\ \sqrt{\alpha+\beta} & 2\gamma & \sqrt{\alpha+2\beta} & \\ & \sqrt{\alpha+2\beta} & 3\gamma & \sqrt{\alpha+3\beta} \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

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$$F_J(\alpha, \beta, \gamma; z) = {}_1F_1\left(1 - \frac{\alpha}{\beta} - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}; 1 - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}; \frac{\beta}{\gamma^2}\right) / \Gamma\left(1 - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}\right)$$

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- For $\alpha = 0$, the spectrum can be found fully explicitly,

$$\text{spec } J(0, \beta, \gamma) = \left\{ -\frac{\beta}{\gamma} + \gamma j; j = 1, 2, 3, \dots \right\}.$$

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Thank you!