



Correction: Non-Self-Adjoint Toeplitz Matrices Whose Principal Submatrices Have Real Spectrum

Boris Shapiro¹ · František Štampach¹

Published online: 14 February 2023

© The Author(s) 2023

Abstract

We announce an error in the proof of Theorem 8 of *Constr. Approx.* **48**(2) (2019) 191–226.

Correction to:

Constr Approx (2019) 49:191–226

<https://doi.org/10.1007/s00365-017-9408-0>

In [3, Eq. (7)], the false equality

$$\langle u, T_n(\bar{a})u \rangle = \langle f_u, \bar{a} f_u \rangle_\gamma$$

has been used in the course of the proof of Theorem 8. If the terms are interpreted using the notation from [3], the right-hand side equals the integral

$$\frac{1}{2\pi i} \int_{-\pi}^{\pi} \overline{a(\gamma(t))} f_u(\gamma(t)) \overline{f_u(\gamma^*(t))} \frac{\dot{\gamma}(t)}{\gamma(t)} dt,$$

while the left-hand side coincides with $\langle u, T_n^*(a)u \rangle$ and can be expressed as the integral

$$\frac{1}{2\pi i} \int_{-\pi}^{\pi} \overline{a(\gamma^*(t))} f_u(\gamma(t)) \overline{f_u(\gamma^*(t))} \frac{\dot{\gamma}(t)}{\gamma(t)} dt.$$

The two integrals do not coincide in general. Unfortunately, this error is an essential problem for the idea of the proof of Theorem 8, that was based on a contour integral

The original article can be found online at <https://doi.org/10.1007/s00365-017-9408-0>.

✉ František Štampach
stampach@math.su.se

Boris Shapiro
shapiro@math.su.se

¹ Department of Mathematics, Stockholm University, Kräftriket 5, 106 91 Stockholm, Sweden

representation of the quadratic form of a Toeplitz matrix $T_n(a)$, where the integration path is chosen as the Jordan curve γ on which the symbol a is real-valued.

Recall that [3, Thm. 1], which is the main result of the paper, claims that the following 3 statements are equivalent:

- (i) $\Lambda(b) \subset \mathbb{R}$,
- (ii) $b^{-1}(\mathbb{R})$ contains a Jordan curve,
- (iii) $\text{spec}(T_n(b)) \subset \mathbb{R}$ for all $n \in \mathbb{N}$,

where b is a Laurent polynomial, $T_n(b)$ the $n \times n$ Toeplitz matrix given by the symbol b , and $\Lambda(b)$ is the set of limit points of eigenvalues of $T_n(b)$, as $n \rightarrow \infty$; see [3] for details. The logic of the proof of Theorem 1 was to establish implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). The currently unproven Theorem 8 was used to prove (ii) $\Rightarrow$ (iii). As we are not able to find an alternative proof at the moment, the implication (ii) $\Rightarrow$ (iii) remains unproven.

However, we strongly believe that the currently unproven implication, or at least its weaker form (ii) $\Rightarrow$ (i), is true. The implication (ii) $\Rightarrow$ (iii) is supported by numerous numerical experiments that we have performed. Professors Yi Zhang and Hao-Yan Chen, to whom we are sincerely grateful for noticing the latter error in the proof, expressed a similar opinion.

Recall, by [3, Def. 2], that the Laurent polynomial b is said to belong to the class $\mathcal{R}$, i.e. $b \in \mathcal{R}$, if and only if $\Lambda(b) \subset \mathbb{R}$. We list several places in [3], where arguments based on the claim of [3, Thm. 8] have been used:

- (a) Remark 10.
- (b) Sec. 4.1: Example 1. The symbol $b(z) = z^{-1} + az$, where $a \in \mathbb{C} \setminus \{0\}$, belongs to the class $\mathcal{R}$, if and only if $a > 0$.
- (c) Sec. 4.2: Example 2. The symbol $b(z) = z^{-1} + \alpha z + \beta z^2$, where $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C} \setminus \{0\}$, belongs to the class $\mathcal{R}$, if and only if $\beta \in \mathbb{R} \setminus \{0\}$ and $\alpha^3 \geq 27\beta^2$.
- (d) Sec. 4.3: Example 3. The symbol $b(z) = z^{-r}(1 + az)^{r+s}$, where $r, s \in \mathbb{N}$ and $a \in \mathbb{R} \setminus \{0\}$, belongs to the class $\mathcal{R}$.
- (e) The second paragraph of Sec. 4.4.

The claim of [3, Rem. 10] was drawn as a direct consequence of Theorem 8 and remains open, too. The other points (b)–(e) comprise concrete examples of symbols, which belong to $\mathcal{R}$ for specific restrictions of the parameters. As the main argument for these claims, we found in [3] an explicit parametrization of the Jordan curve γ , for which $b \circ \gamma$ is real-valued in each of the cases. We will prove at least partly the claims without using Theorem 8.

The equivalence in (b) can be verified directly since the eigenvalues of $T_n(b)$ can be computed fully explicitly

$$\lambda_k = 2(-1)^n \sqrt{a} \cos \frac{\pi k}{n+1}, \quad k = 1, 2, \dots, n,$$

with the standard branch of the square root. This example also exhibits (iii), if $a > 0$.

The point (c) is left conjectural as inessential for the paper in its generality. Its relevant particular case, $b \in \mathcal{R}$ for $\alpha = 3a^2$ and $\beta = a^3$, with $a \in \mathbb{R} \setminus \{0\}$, will be checked as a special case of the point (d) in the end of this corrigendum. Lastly, a

description of a possible construction of further examples of not necessarily banded Toeplitz matrices with real spectra from the point (e) is heavily based on the falsely proven Theorem 8 and also remains open at this point.

As a final correction, we briefly indicate the proof of the implication

$$b(z) = \frac{1}{z^r}(1 + az)^{r+s} \text{ with } r, s \in \mathbb{N} \text{ and } a \in \mathbb{R} \setminus \{0\} \Rightarrow b \in \mathcal{R}.$$

without using the fact that b is real on a Jordan curve. This is the claim (d). In fact, one can prove the stronger claim that the eigenvalues of $T_n(b)$ are all real, for all $n \in \mathbb{N}$, i.e. (iii). The argument relies on the theory of oscillatory matrices [2] and has been used for the special case $r = s = 1$ in the proof of [1, Thm. 2.8]. Without loss of generality, we may assume $a = 1$ since two Toeplitz matrices $T_n(b)$ and $T_n(b_a)$, where $b_a(z) := b(az)$, are similar, if $a \neq 0$. Thus, suppose $b(z) = z^{-r}(1 + z)^{r+s}$. Notice the bidiagonal matrix $T_n(1 + z)$ is totally non-negative, see [2, Def. 4, p. 74]. Since $T_n(b)$ is a submatrix of $T_{n+r+s}^{r+s}(1 + z)$, it is also totally non-negative; see [2, p. 74]. Further, without going into details, let us mention the determinant of $T_n(b)$ can be explicitly computed in terms of binomial coefficients as follows:

$$\det T_n(b) = \prod_{j=0}^{r-1} \binom{n+s+j}{s} / \binom{s+j}{j} = \prod_{j=0}^{r-1} \frac{(n+s+j)!j!}{(n+j)!(s+j)!}.$$

The determinant is obviously non-vanishing and therefore $T_n(b)$ is non-singular. By [2, Thm. 10, p. 100], $T_n(b)$ is oscillatory and hence eigenvalues of $T_n(b)$ are all positive (and simple), see [2, Thm. 6, p. 87].

Acknowledgements The authors are grateful to Professors Yi Zhang and Hao-Yan Chen for pointing out the error.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Coussement, E., Coussement, J., Van Assche, W.: Asymptotic zero distribution for a class of multiple orthogonal polynomials. *Trans. Am. Math. Soc.* **360**(10), 5571–5588 (2008)
2. Gantmacher, F.P., Krein, M.G.: Oscillation matrices and kernels and small vibrations of mechanical systems, revised ed. AMS Chelsea Publishing, Providence, RI (2002). Translation based on the 1941 Russian original, Edited and with a preface by Alex Eremenko
3. Shapiro, B., Štampach, F.: Non-self-adjoint Toeplitz matrices whose principal submatrices have real spectrum. *Constr. Approx.* **49**(2), 191–226 (2019)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.