Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Discrete Hardy Inequalities

Diskrétní Hardyho nerovnosti

Bachelor's Degree Project

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ČVUT
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ZADÁNÍ BAKALÁŘSKÉ PRÁCE

I. OSOBNÍ A STUDIJNÍ ÚDAJE

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Studijní program: **Matematické inženýrství**

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II. ÚDAJE K BAKALÁŘSKÉ PRÁCI

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Diskrétní Hardyho nerovnosti

Název bakalářské práce anglicky:

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Pokyny pro vypracování:

1) Proveďte rešerši existujících důkazů klasické diskrétní Hardyho nerovnosti [1,2].
2) Proveďte rešerši existujících důkazů optimální diskrétní Hardyho nerovnosti a její optimality [3,4,5].
3) Pokuste se nastudovat a shrnout stav aktuálního výzkumu diskrétních Hardyho nerovností [6,7,8].
4) Promyslete možnosti navázaní na aktuální výzkum a případné řešení otevřených problémů.

Seznam doporučené literatury:

[1] Hardy, G. H.; Littlewood, J. E.; Pólya, G.: Inequalities, Cambridge University Press, Cambridge, 1988.
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III. PŘEVZETÍ ZADÁNÍ

Student bere na vědomí, že je povinen vypracovat bakalářskou práci samostatně, bez cizí pomoci, s výjimkou poskytnutých konzultací. Seznam použité literatury, jiných pramenů a jmen konzultantů je třeba uvést v bakalářské práci.

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Author's declaration:
I declare that this Bachelor's Degree Project is entirely my own work and I have listed all the used sources in the bibliography.

Prague, July 8, 2024 Jakub Waclawek

Název práce:

Diskrétní Hardyho nerovnosti

Autor: Jakub Waclawek

Studijní program: Matematické inženýrství

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Druh práce: Bakalářská práce

Vedoucí práce: Ing. František Štampach, Ph.D., Katedra matematiky

Abstrakt: V této práci nejprve shrneme historický vývoj a známé důkazy klasické diskrétní Hardyho nerovnosti. Dále prozkoumáme podobnosti a rozdíly mezi její diskrétní a spojitou verzí, načež se zaměříme na vylepšenou Hardyho nerovnost pro diskrétní Laplaceův operátor Δ objevenou roku 2018 autory Keller–Pinchover–Pogorzelski. Hlavním výsledkem je pak nalezení obdobných optimálních nerovností pro libovolnou přirozenou mocninu ℓ Laplaceova operátoru. Pro $\ell = 2$ jsme objevili optimální Rellichovu nerovnost, vylepšujíce tak dosavadní nejlepší váhy od autorů Gerhat–Krejčiřík–Štampach a Huang–Ye. Pro $\ell \geq 3$ jsme dokázali hypotézu od Gerhat–Krejčiřík–Štampach a vylepšili klasické Birmanovy váhy obejevené autory Huang–Ye na optimální.

Klíčová slova: diskrétní Birmanovy nerovnosti, diskrétní Hardyho nerovnost, diskrétní Rellichova nerovnost, optimální Birmanovy nerovnosti, diskrétní Laplaceův operátor, optimální Hardyho nerovnost, optimální Rellichova nerovnost

Title:

Discrete Hardy Inequalities

Author: Jakub Waclawek

Abstract: In this thesis, we first summarize the historical development and known proofs of the classical discrete Hardy inequality. Next, we examine similarities and differences between the discrete and continuous versions, focusing on the improved Hardy inequality for the discrete Laplace operator Δ discovered in 2018 by Keller–Pinchover–Pogorzelski. Our main result provides analogous optimal inequalities for an arbitrary integer power ℓ of the Laplace operator. For $\ell = 2$, we find the optimal Rellich weight, improving upon the best-known weights due to Gerhat–Krejčiřík–Štampach and Huang–Ye. For $\ell \geq 3$, we prove a conjecture by Gerhat–Krejčiřík–Štampach and improve the classical discrete weights due to Huang–Ye to optimal weights.

Key words: discrete Birman inequalities, discrete Hardy inequality, discrete Rellich inequality, optimal Birman inequality, discrete Laplace operator, optimal Hardy inequality, optimal Rellichova inequality

Contents

Introduction

During 1915, in pursuit of finding an elementary proof of the *Hilbert inequality* (1.1), Hardy discovered a theorem of great beauty and simplicity which states that whenever the series $\sum_{n=1}^{\infty} a_n^2$ converges, the series of cumulative averages $\sum_{n=1}^{\infty}(A_n/n)^2$, where $A_n \coloneqq \sum_{k=1}^{n} a_k$, also converges. Other great mathematicians, such as E. Landau, G. Pólya, I. Schur, and M. Riesz, also significantly contributed to the subsequent development of the Hardy inequality (we refer the reader to [1] for a thorough historical survey). The final L^p version of the *classical discrete Hardy inequality* reads

$$\sum_{n=1}^{\infty}\left(\frac{A_n}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p, \tag{1}$$

which holds true for all $p > 1$ and for all non-negative sequences $a \in C(\mathbb{N})$, where $C(\mathbb{N})$ denotes the space of all complex sequences indexed by $\mathbb{N}$. Even though Hardy's original work was related to the discrete version, it was mainly the continuous version that found various applications, e.g. in PDEs, mathematical physics, spectral theory, and geometry. The *continuous Hardy inequality* asserts that

$$\int_0^{\infty}\left(\frac{F(x)}{x}\right)^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f(x)^p \, \mathrm{d}x \tag{2}$$

for any non-negative function $f \in L^p((0,\infty))$, where, this time, we utilized the notation $F(x) \coloneqq \int_0^x f(t)\,\mathrm{d}t$. Moreover, in both cases (1) and (2), the constant $(p/(p-1))^p$ is optimal, meaning that it cannot be replaced by a strictly smaller constant.

Both inequalities were first published in the book *Inequalities* [6] by the group of authors Hardy, Littlewood, and Pólya. Since the Hardy inequalities appeared to be of tremendous use in applications and brought up numerous intriguing questions, their research has flourished until today, giving rise to a whole new discipline of modern analysis focused on Hardy-type inequalities.

In this thesis, we first discuss the classical discrete Hardy inequality in Chapter 1. In Section 1.1, we will briefly concentrate on the historical background, while in the rest of the chapter, we provide several possible proofs of (1). At the end of this chapter, in Section 1.4, we examine the similarities (and differences) between the classical discrete and continuous Hardy inequalities, the most important of them being the possible improvement of the discrete version, which will be examined in greater detail in Chapter 2. Note that both Hardy inequalities (1) and (2) can be equivalently understood as lower bounds $-\Delta \geq \rho$ for the Laplace operator on respective spaces in the sense of quadratic forms, see (2.11). In Chapter 3, we focus on the state of the art, which is the problem of finding analogous lower bounds for higher integer powers of the Laplace operator. Our main results are formulated in Section 3.1, where we find optimal discrete Hardy inequalities of higher order and prove the conjecture (3.5) from [15].

Chapter 1

Classical discrete Hardy inequality

In this chapter, we first focus on the historical account of the classical discrete Hardy inequality in Section 1.1 providing the first proof of the weak form of (1). In Sections 1.2 and 1.3, we examine possible proofs of (1), while in Section 1.4 we commentate upon the relation between the discrete and continuous versions of the classical Hardy inequality.

1.1 Historical Background

Hardy's original motivation, when discovering his celebrated inequality, was by no means less famous Hilbert Inequality. In the weak form it asserts that if $\sum_{n=1}^{\infty} a_n^2$ converges, then the double series $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n a_m}{n+m} < \infty$ also converges. More precisely, it claims that the inequality

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n a_m}{n+m} \leq \pi \sum_{n=1}^{\infty} {a_n}^2 \tag{1.1}$$

holds true for all complex sequences $a \in C(\mathbb{N})$. The inequality (1.1) was first deduced by Hilbert in 1906 with constant 2π, utilizing an advanced tool of Fourier's series. Subsequently, additional proofs were given by H. Weyl, F. Weiner, or even I. Schur (1911), who was the first to derive the continuous integral version of (1.1), as well as the optimal constant π. However, Hardy craved for more elementary proof to be found. He accomplished to publish it in the year 1915 in [4]. And it is in this paper, where the weak form of the Hardy inequality implicitly appears for the first time. As a matter of interest, we present two relevant statements from this writing together with Hardy's proofs. We shall start with the following lemma, that connects Hardy's and Hilbert's inequalities in an obvious manner.

Lemma 1.1. *Let $a \in C(\mathbb{N})$ be a non-negative sequence and $A_n := \sum_{k=1}^{n} a_k$. Then the convergence of any of the three series*

$$(i) \ \sum_{n=1}^{\infty} \frac{a_n A_n}{n}, \qquad (ii) \ \sum_{n=1}^{\infty} \left(\frac{A_n}{n}\right)^2, \qquad (iii) \ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n a_m}{n+m}$$

implies the convergence of the other two.

In fact, Hardy proved the integral version of this lemma and mentioned, that the proof for series, which was not included in this work, was analogous. Nevertheless, we will try to outline the ideas of his proof by means of the discrete case, which is perhaps a little more delicate.

Proof. The proof shall be done in two steps.

a) *(i) converges* $\iff$ *(ii) converges*: For the implication ($\Rightarrow$) we use an elementary estimate

$$2a_n A_n = (A_n - A_{n-1})(A_n + A_{n-1}) + a_n^2 = A_n^2 - A_{n-1}^2 + a_n^2 \geq A_n^2 - A_{n-1}^2.$$

Dividing by n and summing over n from 1 to $N \in \mathbb{N}$, we obtain

$$\sum_{n=1}^{N} \frac{2a_n A_n}{n} \geq \sum_{n=1}^{N-1} \left(\frac{A_n^2}{n} - \frac{A_n^2}{n+1} \right) + \frac{A_N^2}{N} \geq \sum_{n=1}^{N-1} \frac{A_n^2}{n(n+1)} \geq \sum_{n=1}^{N-1} \frac{A_n^2}{2n^2}.$$

Since N was arbitrary, we can send $N \to \infty$, thus arriving at

$$\sum_{n=1}^{\infty} \frac{A_n^2}{n^2} \leq 4 \sum_{n=1}^{\infty} \frac{a_n A_n}{n}.$$

The reverse implication ($\Leftarrow$) can be shown in an analogous fashion observing that

$$a_n A_n = (A_n - A_{n-1}) A_n = A_n^2 - A_n A_{n-1} \leq A_n^2 - A_{n-1}^2.$$

Again, after dividing by n and summing over n from 1 to $N \in \mathbb{N}$, we get

$$\sum_{n=1}^{N} \frac{a_n A_n}{n} \leq \sum_{n=1}^{N-1} \frac{A_n^2}{n(n+1)} + \frac{A_N^2}{N} \leq \sum_{n=1}^{N-1} \frac{A_n^2}{n^2} + \frac{A_N^2}{N} \leq \sum_{n=1}^{N-1} \frac{A_n^2}{n^2} + \sum_{n=N}^{\infty} \frac{A_n^2}{n^2} = \sum_{n=1}^{\infty} \frac{A_n^2}{n^2},$$

because

$$\frac{A_N^2}{N} \leq \sum_{n=N}^{\infty} \frac{A_N^2}{n^2} \leq \sum_{n=N}^{\infty} \frac{A_n^2}{n^2},$$

where we used a simple estimate $1/N = \int_N^\infty 1/x^2 \, \mathrm{d}x \leq \sum_{n=N}^\infty 1/n^2$. Sending $N \to \infty$ yields the desired result.

b) *(i) converges* $\iff$ *(iii) converges*: For the implication ($\Rightarrow$) we will make use of the symmetry of the expression (iii) interchanging $n \leftrightarrow m$

$$2 \sum_{n=1}^{N} \sum_{m=1}^{n} \frac{a_n a_m}{n+m} = \sum_{n=1}^{N} \sum_{m=1}^{n} \frac{a_n a_m}{n+m} + \sum_{m=1}^{N} \sum_{n=1}^{m} \frac{a_n a_m}{n+m} = \sum_{n=1}^{N} \sum_{m=1}^{n} \frac{a_n a_m}{n+m} + \sum_{n=1}^{N} \sum_{m=n}^{N} \frac{a_n a_m}{n+m},$$

therefore

$$\sum_{n=1}^{N} \sum_{m=1}^{N} \frac{a_n a_m}{n+m} \leq 2 \sum_{n=1}^{N} \sum_{m=1}^{n} \frac{a_n a_m}{n+m} = 2 \sum_{n=1}^{N} a_n \sum_{m=1}^{n} \frac{a_m}{n+m} \leq 2 \sum_{n=1}^{N} \frac{a_n}{n} \sum_{m=1}^{n} a_m = 2 \sum_{n=1}^{N} \frac{a_n A_n}{n}.$$

Hence the implication is proven after we let $N \to \infty$. Finally, the implication ($\Leftarrow$) is given by the following series of estimates

$$2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n a_m}{n+m} \geq 2 \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{a_n a_m}{n+m} \geq 2 \sum_{n=1}^{\infty} a_n \sum_{m=1}^{n} \frac{a_m}{2n} = \sum_{n=1}^{N} \frac{a_n A_n}{n}. \qquad \square$$

Remark 1.2. At this point, it is beneficial to realise, that it is sufficient to prove Hardy inequality (1) for non-increasing sequences. Indeed, if the series $\sum_{n=1}^\infty a_n$ converges, the sequence $\{a_n\}_{n=1}^\infty$ is bounded and consequently there exists a non-increasing rearrangement $\{\tilde{a}_n\}_{n=1}^\infty$. Evidently, it holds that

$$\sum_{n=1}^{\infty} a_n^p = \sum_{n=1}^{\infty} \tilde{a}_n^p \quad \text{and} \quad A_n \leq \tilde{A}_n \text{ for all } n \in \mathbb{N},$$

thus

$$\sum_{n=1}^{\infty} \left(\frac{A_n}{n} \right)^p \leq \sum_{n=1}^{\infty} \left(\frac{\tilde{A}_n}{n} \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \tilde{a}_n^p = \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p,$$

proving the Hardy inequality without loss of generality.

To complete the proof of the weak form of the discrete Hardy inequality, we employ the following theorem.

Theorem 1.3. *Let $a \in C(\mathbb{N})$ be a non-negative sequence and $A_n \coloneqq \sum_{k=1}^{n} a_k$. Then*

$$\sum_{n=1}^{\infty} a_n^2 < \infty \implies \sum_{n=1}^{\infty} \frac{a_n A_n}{n} < \infty.$$

Remark 1.4. Equipped with Lemma 1.1, we easily observe that Theorem 1.3 implies the weak form of both, Hardy and Hilbert, inequalities.

Proof. Clearly, it is again sufficient to prove the statement for a non-increasing sequence a (the argument is similar as in Remark 1.2).

First, we rewrite the expression $\sum_{n=1}^{\infty} a_n A_n/n = \sum_{n=1}^{\infty}\sum_{m=1}^{n}(a_n a_m)/n = S = \sum_{k=1}^{\infty} S_k$, where we denoted

$$S_k \coloneqq \sum_{k \le \frac{n}{m} < k+1}\sum \frac{a_n a_m}{n}$$

for all $k \in \mathbb{N}$. We infer from the following estimates and the Cauchy–Schwarz inequality that

$$\begin{aligned} S_k &\le \frac{1}{k}\sum_{k \le \frac{n}{m} < k+1}\sum \frac{a_n a_m}{m} = \frac{1}{k}\sum_{m=1}^{\infty}\frac{a_m}{m}\sum_{k \le \frac{n}{m} < k+1} a_n \le \frac{1}{k}\sum_{m=1}^{\infty}\frac{a_m}{m} a_{km} m \\ &\le \frac{1}{k}\left(\sum_{m=1}^{\infty} a_m^2\right)^{1/2}\left(\sum_{m=1}^{\infty} a_{km}^2\right)^{1/2} \le \frac{1}{k}\left(\sum_{m=1}^{\infty} a_m^2\right)^{1/2}\left(\frac{1}{k}\sum_{m=1}^{\infty} a_m^2\right)^{1/2} = \frac{1}{k^{3/2}}\sum_{m=1}^{\infty} a_m^2. \end{aligned}$$

Finally, we get

$$S = \sum_{k=1}^{\infty} S_k \le \sum_{k=1}^{\infty}\frac{1}{k^{3/2}}\sum_{m=1}^{\infty} a_m^2 \le 3\sum_{m=1}^{\infty} a_m^2 < \infty,$$

since $\sum_{k=1}^{\infty} k^{-3/2} \le 1 + \int_1^{\infty} x^{-3/2}\,\mathrm{d}x = 3$. □

Later in the year 1919, Hardy formulated the inequality (2) for $p = 2$ with integration from a positive constant $a > 0$ instead of 0. Moreover, he published a sketch of proof of the discrete version (1). It is important to mention the names of other mathematicians who were involved in the research that followed Hardy's initial discovery. In the year 1920, M. Riesz sent Hardy a letter with a proof of (1) with the constant $(p^2/(p-1))^p$. Based on a letter from I. Schur, Hardy improved the constant to $(p\zeta(p))^p$. In the same letter, Schur pointed out that for $p = 2$ the inequality holds true with the constant 4, therefore Hardy conjectured that the inequality holds even with the optimal constant $(p/(p-1))^p$. In addition, he hypothesised the L^p version of the continuous inequality (2), whilst still integrating from $a > 0$. Although he did not provide a proof of this claim, he showed the optimality of the constant $(p/(p-1))^p$ considering the function $f(x) = x^{-1/p-\varepsilon}$, where $\varepsilon > 0$ is sufficiently small. Finally, one year later, E. Landau proved the final version of (1) and in the year 1925 he pointed out that the discrete version of Hardy inequality can be deduced from the continuous one assuming suitable step function f, see also Section 1.4. This motivated Hardy to formulate and prove the final version of his famous inequality (2).

1.2 Elementary proof

First, we will show an elegant elementary proof from E. B. Elliot from [5], together with the optimality of the constant $(p/(p-1))^p$, which was first proven by Landau, see for example [1]. For this purpose, we will state the following auxiliary claims.

Let $a, b > 0$ and $p, q > 1$, such that $1/p + 1/q = 1$. Then the *Young inequality* reads

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.2}$$

Now, consider $p > 1$ and set $q \coloneqq p/(p-1)$. Then, with the aid of (1.2), we find that

$$a^{p-1}b \leq \frac{(a^{p-1})^q}{q} + \frac{b^p}{p} = (p-1)\frac{a^p}{p} + \frac{b^p}{p},$$

for all $a, b > 0$ and $p > 1$. Moreover, if we put $y \coloneqq b/a$, we observe that

$$pa^{p-1}b \leq (p-1)a^p + b^p \iff py \leq (p-1) + y^p \iff y^p \geq 1 + p(y-1).$$

Denoting $x = y - 1 > -1$, we conclude that the *Bernoulli inequality*

$$(1+x)^p \geq 1 + px \tag{1.3}$$

holds for any $x \geq -1$. Lastly, let us also recall the famous Hölder inequality.

Theorem 1.5 (Hölder inequality)**.** *Let $(X, \mathcal{M}, \mu)$ be a measurable space, $f, g : X \to \mathbb{C}$ be measurable functions and $p, q > 1$ satisfying $1/p + 1/q = 1$. Then it is true that*

$$\int_X |fg| \,\mathrm{d}\mu \leq \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{1/p} \left(\int_X |g|^q \,\mathrm{d}\mu\right)^{1/q}. \tag{1.4}$$

Remark 1.6. If we consider $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$ to be the power set of $\mathbb{N}$, and $\mu = \sum_{n=1}^{\infty} \delta_{\{n\}}$ the counting measure in Theorem 1.5, the Hölder inequality states

$$\sum_{n=1}^{\infty} a_n b_n \leq \left(\sum_{n=1}^{\infty} a_n^p\right)^{1/p} \left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \tag{1.5}$$

for all non-negative sequences $a, b \in C(\mathbb{N})$.

Now, we may proceed to prove the classical discrete Hardy inequality (1).

Proof. First, we show that the inequality (1) holds and then we verify the optimality of the constant $(p/(p-1))^p$. Throughout this proof we will denote $A_n \coloneqq \sum_{k=1}^{n} a_k$, and $\alpha_n \coloneqq \frac{A_n}{n}$ for all $n \in \mathbb{N}$.

a) Since $a_n = n\alpha_n - (n-1)\alpha_{n-1}$, we observe that

$$\alpha_n^p - \frac{p}{p-1}\alpha_n^{p-1}a_n = \alpha_n^p\left(1 - \frac{np}{p-1}\right) + \frac{(n-1)p}{p-1}\alpha_n^{p-1}\alpha_{n-1}.$$

Applying the Young inequality (1.2) to the second term on the right-hand side, we infer that

$$\begin{aligned}\alpha_n^p - \frac{p}{p-1}\alpha_n^{p-1}a_n &\leq \alpha_n^p\left(1 - \frac{np}{p-1}\right) + \frac{n-1}{p-1}\left((p-1)\alpha_n^p + \alpha_{n-1}^p\right)\\ &= \alpha_n^p - \alpha_n^p\frac{np}{p-1} + (n-1)\alpha_n^p + \frac{n-1}{p-1}\alpha_{n-1}^p,\end{aligned}$$

hence

$$\alpha_n^p - \frac{p}{p-1}\alpha_n^{p-1}a_n \leq \frac{1}{p-1}\left((n-1)\alpha_{n-1}^p - n\alpha_n^p\right).$$

By summation by parts form $n = 1$ to N, we get

$$\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p - \frac{p}{p-1}\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^{p-1} a_n \le \frac{-N\alpha_N^p}{p-1} \le 0,$$

therefore, invoking the Hölder inequality (1.5), we have

$$\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p \le \frac{p}{p-1}\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^{p-1} a_n \le \frac{p}{p-1}\left(\sum_{n=1}^{N} a_n^p\right)^{1/p}\left(\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p\right)^{(p-1)/p}.$$

Since it suffices to consider $a \neq 0$, we may divide by the last term on the right-hand side arriving at

$$\left(\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p\right)^{1/p} \le \frac{p}{p-1}\left(\sum_{n=1}^{N} a_n^p\right)^{1/p},$$

which is true for all $N \in \mathbb{N}$. Sending $N \to \infty$ yields the desired result.

b) Secondly, we will show the optimality of the constant $(p/(p-1))^p$, by introducing the sequence

$$a_n = \frac{1}{n^{1/p+\varepsilon}}$$

for all $n \in \mathbb{N}$ and for $0 < \varepsilon < 1 - \frac{1}{p}$. We may estimate

$$A_n = \sum_{k=1}^{n}\frac{1}{k^{1/p+\varepsilon}} > \int_1^n \frac{1}{x^{1/p+\varepsilon}}\,\mathrm{d}x = \frac{1}{1-1/p-\varepsilon}\left(n^{1-1/p-\varepsilon} - 1\right) > \frac{p}{p-1}\left(n^{1-1/p-\varepsilon} - 1\right)$$
$$= \frac{p}{p-1} n^{1-1/p-\varepsilon}\left(1 - \frac{1}{n^{1-1/p-\varepsilon}}\right)$$

It follows from the Bernoulli inequality (1.3) that

$$\left(\frac{A_n}{n}\right)^p > \left(\frac{p}{p-1}\right)^p n^{-1-\varepsilon p}\left(1 - \frac{1}{n^{1-1/p-\varepsilon}}\right)^p \ge \left(\frac{p}{p-1}\right)^p n^{-1-\varepsilon p}\left(1 - \frac{p}{n^{1-1/p-\varepsilon}}\right)$$
$$= \left(\frac{p}{p-1}\right)^p\left(a_n^p - pn^{-2+1/p+\varepsilon-\varepsilon p}\right).$$

Summing from $n = 1$ to $N \in \mathbb{N}$, we get

$$\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p > \left(\frac{p}{p-1}\right)^p\left(\sum_{n=1}^{N} a_n^p - p\sum_{n=1}^{N}\frac{1}{n^{2-1/p-\varepsilon+\varepsilon p}}\right) = \left(\frac{p}{p-1}\right)^p\left(\sum_{n=1}^{N} a_n^p - pC_{N,\varepsilon}\right),$$

where we denoted $C_{N,\varepsilon} \coloneqq \sum_{n=1}^{N} n^{-2+1/p+\varepsilon-\varepsilon p}$. As $2 - 1/p - \varepsilon + \varepsilon p > 1$ and $2 - 1/p > 1$, we have

$$\lim_{N\to\infty} C_{N,\varepsilon} =: C_\varepsilon < \infty \text{ for all } \varepsilon > 0 \quad \text{and} \quad \lim_{\varepsilon\to 0+} C_\varepsilon =: C < \infty.$$

On the other hand, it is true that

$$\sum_{n=1}^{N} a_n^p = \sum_{n=1}^{N}\frac{1}{n^{1+\varepsilon p}} \xrightarrow[\varepsilon\to 0+]{N\to\infty} \infty,$$

thus

$$\frac{\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^p}{\sum_{n=1}^{N} a_n^p} \ge \left(\frac{p}{p-1}\right)^p\left(1 - \frac{pC_{N,\varepsilon}}{\sum_{n=1}^{N} a_n^p}\right) \xrightarrow[\varepsilon\to 0+]{N\to\infty} \left(\frac{p}{p-1}\right)^p,$$

from which the optimality readily follows. □

1.3 Norm of the Cesàro operator

In this section, we shall understand the Hardy inequality (1) as a problem of computing the norm of the *Cesàro* (averaging) *operator*, which is defined as $C : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$:

$$(Ca)_n = \frac{1}{n}\sum_{k=1}^{n} a_k$$

for all $a \in \ell^p(\mathbb{N})$, resp. in the continuous case as $C_\infty : L^p((0,\infty)) \to L^p((0,\infty))$:

$$(C_\infty f)(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t$$

for all $f \in L^p((0,\infty))$. The claim that the operator C, resp. C_∞, maps to $\ell^p(\mathbb{N})$, resp. to $L^p((0,\infty))$, is by no means trivial and it is actually asserted by the weak form of the Hardy inequality. In this section, we further restrict ourselves to real sequences in the Hilbert space $\ell^2(\mathbb{N})$ endowed with the Euclidean inner product defined as $\langle a, b\rangle = \sum_{n=1}^{\infty} a_n b_n$ for all $a, b \in \ell^2(\mathbb{N})$.

The classical discrete Hardy inequality (1) together with its optimality, can be understood as the identity

$$\|C\| = 2. \tag{1.6}$$

The inequality $\|C\| \leq 2$ asserts the inequality (1) since

$$\|Ca\|^2 = \sum_{n=1}^{\infty}\left(\frac{A_n}{n}\right)^2 \leq 4\,\|a\|^2 = 4\sum_{n=1}^{\infty} a_n^2,$$

and the second inequality $\|C\| \geq 2$ determines the optimality of the constant 4. In Sections 1.3.1 and 1.3.2, we provide two possible proofs of identity (1.6).

1.3.1 Schur test

In this section, we introduce a useful tool for estimating norms of integral operators, i.e. the *Schur test*, which we will state here in a fully general version, see [7, Theorem 5.2.] for reference.

Theorem 1.7 (Schur test)**.** *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measurable space. Furthermore, let a measurable function $\mathcal{K} : X \times X \to [0,\infty)$ be a non-negative kernel and $p, q > 1$ such that $1/p + 1/q = 1$. If there exist constants $A, B \geq 0$ and a measurable function $h : X \to (0,\infty)$ satisfying*

$$\int_X \mathcal{K}(x,y)h(y)^q\,\mathrm{d}\mu(y) \leq Ah(x)^q \quad \mu\text{-a.e. } x \in X,$$
$$\int_X \mathcal{K}(x,y)h(x)^p\,\mathrm{d}\mu(x) \leq Bh(y)^p \quad \mu\text{-a.e. } y \in X,$$

then for the integral operator T, given by

$$Tf(x) = \int_X \mathcal{K}(x,y)f(y)\,\mathrm{d}\mu(y)$$

for μ-a.e. $x \in X$ and for all $f \in L^p(X,\mu)$, it holds that $T \in \mathcal{B}(L^p(X,\mu))$ and $\|T\| \leq A^{1/q}B^{1/p}$.

Proof. Let $f \in L^p(X,\mu)$, such that $\|f\|_p = 1$, and $x \in X$ arbitrary. We infer from the Hölder inequality (1.4) that

$$\begin{aligned}|Tf(x)| &\leq \int_X \mathcal{K}(x,y)h(y)h(y)^{-1}|f(y)|\,\mathrm{d}\mu(y) = \int_X \mathcal{K}(x,y)^{1/p}\mathcal{K}(x,y)^{1/q}h(y)h(y)^{-1}|f(y)|\,\mathrm{d}\mu(y)\\ &\leq \left(\int_X \mathcal{K}(x,y)h(y)^q\,\mathrm{d}\mu(y)\right)^{1/q}\left(\int_X \mathcal{K}(x,y)h(y)^{-p}|f(y)|^p\,\mathrm{d}\mu(y)\right)^{1/p}\\ &\leq A^{1/q}h(x)\left(\int_X \mathcal{K}(x,y)h(y)^{-p}|f(y)|^p\,\mathrm{d}\mu(y)\right)^{1/p}.\end{aligned}$$

Moreover, using the Tonelli–Fubini theorem we have

$$\begin{aligned}\int_X |Tf(x)|^p\,\mathrm{d}\mu(x) &\leq A^{p/q}\int_X h(x)^p\left(\int_X \mathcal{K}(x,y)h(y)^{-p}|f(y)|^p\,\mathrm{d}\mu(y)\right)\mathrm{d}\mu(x)\\ &= A^{p/q}\int_X h(y)^{-p}|f(y)|^p\left(\int_X \mathcal{K}(x,y)h(x)^p\,\mathrm{d}\mu(x)\right)\mathrm{d}\mu(y)\\ &\leq A^{p/q}B\int_X |f(y)|^p\,\mathrm{d}\mu(y) = A^{p/q}B.\end{aligned}$$

It follows from the definition of the operator norm that $\|T\| \leq A^{1/q}B^{1/p}$. □

Analogously to the situation in vector spaces of finite dimension, bounded operators in $\ell^2(\mathbb{N})$ may also be represented by semi-infinite matrices. Here we consider the matrix of an operator in the standard orthonormal basis $\mathcal{E} = (\delta_n)_{n=1}^{\infty}$, where δ_{nm} is the Kronecker delta symbol for all $m,n \in \mathbb{N}$. If we restrict ourselves to bounded operators only, the operations with semi-infinite matrices will be done similarly to their analogues in spaces of finite dimension and they will exhibit the same properties. In particular, if we denote ${}^{\mathcal{E}}A$ the matrix of the operator A in the standard basis, i.e. $({}^{\mathcal{E}}A)_{m,n} = \langle\delta_m, A\delta_n\rangle$ for all $n,m \in \mathbb{N}$, then it holds that ${}^{\mathcal{E}}(A^*) = ({}^{\mathcal{E}}A)^*$, where * denotes the adjoint operator on the left-hand side and the hermitian adjoint of a matrix on the right-hand side. Furthermore, the composition of operators corresponds to matrix multiplication, which is defined analogously, but with convergent series instead of finite sums. From now on, we will identify the bounded operator $A \in \mathcal{B}(\ell^2(\mathbb{N}))$ with its matrix $A \equiv {}^{\mathcal{E}}A$. With this notation, we can reformulate the Schur test for linear operators on $\ell^2(\mathbb{N})$.

Corollary 1.8 (Schur)**.** *Let $(a_{m,n})_{m,n=1}^{\infty} \equiv A$ be a linear operator in $\ell^2(\mathbb{N})$, and $a_{m,n} \geq 0$ for all $n,m \in \mathbb{N}$. If there exist constants $B, C \geq 0$ and a positive sequence $h \in C(\mathbb{N})$ satisfying*

$$\sum_{n=1}^{\infty} a_{m,n}h_n \leq Bh_m \text{ for all } m \in \mathbb{N},$$

$$\sum_{m=1}^{\infty} a_{m,n}h_m \leq Ch_m \text{ for all } n \in \mathbb{N},$$

then $A \in \mathcal{B}(\ell^2(\mathbb{N}))$ and $\|A\| \leq \sqrt{BC}$.
In particular, if $a_{m,n} = a_{n,m}$ for all $n,m \in \mathbb{N}$ and there exist $D \geq 0$ and a positive sequence $\tilde{h} \in C(\mathbb{N})$ such that

$$\sum_{n=1}^{\infty} a_{m,n}\tilde{h}_n \leq D\tilde{h}_m \text{ for all } m \in \mathbb{N},$$

then $A \in \mathcal{B}(\ell^2(\mathbb{N}))$ and $\|A\| \leq D$.

Proof. The corollary readily follows from the Schur test 1.7 with the choice of $p = q = 2$ and $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(\mathbb{N})$, and $\mu = \sum_{n=1}^{\infty}\delta_{\{n\}}$ the counting measure. □

The matrix representation of the Cesàro operator in the standard basis is

$$C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{i.e.} \quad c_{m,n} = \begin{cases} 1/m & \text{if } m \geq n \\ 0 & \text{if } m < n \end{cases},$$

hence

$$C^* = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 0 & 1/2 & 1/3 & \cdots \\ 0 & 0 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{i.e.} \quad c^*_{m,n} = \begin{cases} 1/n & \text{if } m \leq n \\ 0 & \text{if } m > n \end{cases}$$

for all $m, n \in \mathbb{N}$. For now, we understand C^* only as the hermitian adjoint of the matrix C. In order to show that it is indeed the adjoint operator of C, we have to verify that $C \in \mathcal{B}(\ell^2(\mathbb{N}))$. More precisely we will prove that $\|C\| \leq \sqrt{6}$.

Proof. We will show that with the choice of $h_n = 1/\sqrt{n}$, it is true that

$$\sum_{n=1}^{\infty} c_{m,n} h_n = \sum_{n=1}^{m} c_{m,n} h_n \leq 2h_m \text{ for all } m \in \mathbb{N},$$
$$\sum_{m=1}^{\infty} c_{m,n} h_m = \sum_{m=1}^{\infty} c^*_{n,m} h_m = \sum_{m=n}^{\infty} c^*_{n,m} h_m \leq 3h_n \text{ for all } n \in \mathbb{N}.$$

Indeed, for the first inequality we have

$$\sum_{n=1}^{m} c_{m,n} h_n = \sum_{n=1}^{m} \frac{1}{m} \frac{1}{\sqrt{n}} \leq \frac{1}{m} \int_0^m \frac{1}{\sqrt{x}} \, \mathrm{d}x = \frac{2}{\sqrt{m}} = 2h_m.$$

For the latter inequality, we use an analogous estimate

$$\sum_{m=n}^{\infty} c^*_{n,m} h_m = \sum_{m=n}^{\infty} \frac{1}{m^{3/2}} \leq \frac{1}{n^{3/2}} + \int_n^{\infty} \frac{1}{x^{3/2}} \, \mathrm{d}x = \frac{1}{n^{3/2}} + \frac{2}{\sqrt{n}} \leq \frac{3}{\sqrt{n}} = 3h_n.$$

The proof is complete by the Schur test 1.8. □

It turns out that it is difficult to show that $\|C\| \leq 2$ using the Schur test directly. Nevertheless, it is beneficial to compute the norm of the operator $L := CC^*$, also called the *Hilbert L-matrix*, from which we can easily obtain the norm of the Cesàro operator because we have $\|L\| = \|C\|^2$. The Hilbert L-matrix is given by

$$l_{m,n} = \sum_{k=1}^{\infty} c_{m,k} c^*_{k,n} = \sum_{k=1}^{\min\{m,n\}} \frac{1}{m} \frac{1}{n} = \frac{1}{\max\{m,n\}}$$

for all $n, m \in \mathbb{N}$, or equivalently

$$L = CC^* = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 0 & 1/2 & 1/3 & \cdots \\ 0 & 0 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/2 & 1/3 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.7}$$

We are ready to prove that $\|L\| = 4$ thus $\|C\| = 2$.

Proof. We will show the equality as two inequalities, which shall be proven in two separate steps.

a) $\|L\| \le 4$: This inequality can be verified by the Schur test 1.8. In particular, we will make use of the second claim, since $l_{m,n} = l_{n,m}$ for all $n, m \in \mathbb{N}$. With the same choice of $h_n = 1/\sqrt{n}$ as before we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} l_{m,n} h_n &= \sum_{n=1}^{\infty} \frac{1}{\max\{m,n\}} \frac{1}{\sqrt{n}} = \frac{1}{m} \sum_{n=1}^{m-1} \frac{1}{\sqrt{n}} + \sum_{n=m}^{\infty} \frac{1}{n^{3/2}} \\ &\le \frac{1}{m} + \frac{1}{m} \int_1^m \frac{1}{\sqrt{x}} \,\mathrm{d}x + \frac{1}{m^{3/2}} + \int_m^\infty \frac{1}{x^{3/2}} \,\mathrm{d}x = \frac{1}{m} + \frac{2}{\sqrt{m}} - \frac{2}{m} + \frac{1}{m^{3/2}} + \frac{2}{\sqrt{m}} \\ &= \frac{4}{\sqrt{m}} + \frac{1}{m}\left(\frac{1}{\sqrt{m}} - 1\right) \le \frac{4}{\sqrt{m}},\end{aligned}$$

which yields the inequality.

b) $\|L\| \ge 4$: Since for the norm of a self-adjoint operator $A = A^* \in \mathcal{B}(\mathcal{H})$ acting on an arbitrary Hilbert space $\mathcal{H}$, we have

$$\|A\| = \sup_{\|x\|=1} |\langle x, Ax \rangle| = \sup_{0 \ne x \in \mathcal{H}} \frac{|\langle x, Ax \rangle|}{\|x\|^2},$$

it suffices to find suitable sequences $h(\varepsilon) \in \ell^2(\mathbb{N})$, such that

$$\lim_{\varepsilon \to 0+} \frac{|\langle h(\varepsilon), Lh(\varepsilon) \rangle|}{\|h(\varepsilon)\|^2} = 4.$$

Introducing $h_n(\varepsilon) := n^{-1/2-\varepsilon}$ for all $n \in \mathbb{N}$, we estimate

$$\|h(\varepsilon)\|^2 = \sum_{n=1}^{\infty} \frac{1}{n^{1+2\varepsilon}} = \int_1^\infty \frac{1}{x^{1+2\varepsilon}} \,\mathrm{d}x + \mathcal{O}(1) = \frac{1}{2\varepsilon} + \mathcal{O}(1), \quad \varepsilon \to 0+,$$

where $\mathcal{O}(1)$ as $\varepsilon \to 0+$ is a function $f = \mathcal{O}(1)$ such that $|f(x)| \le K$ for all x sufficiently small and some constant $K \ge 0$. For the numerator, we get

$$\begin{aligned}|\langle h(\varepsilon), Lh(\varepsilon) \rangle| &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-\frac{1}{2}-\varepsilon} \frac{1}{\max\{m,n\}} n^{-\frac{1}{2}-\varepsilon} \\ &= \int_1^\infty \int_1^\infty \frac{1}{x^{1/2+\varepsilon} \max\{x,y\} y^{1/2+\varepsilon}} \,\mathrm{d}x \,\mathrm{d}y + \mathcal{O}(1) \\ &= \int_1^\infty \left(\int_1^x \frac{1}{y^{1/2+\varepsilon}} \,\mathrm{d}y \right) \frac{1}{x^{3/2+\varepsilon}} \,\mathrm{d}x + \int_1^\infty \left(\int_x^\infty \frac{1}{y^{3/2+\varepsilon}} \,\mathrm{d}y \right) \frac{1}{x^{1/2+\varepsilon}} \,\mathrm{d}x + \mathcal{O}(1).\end{aligned}$$

The integrals on the right-hand side can be estimated as

$$\begin{aligned}\int_1^\infty \left(\int_1^x \frac{1}{y^{1/2+\varepsilon}} \,\mathrm{d}y \right) \frac{1}{x^{3/2+\varepsilon}} \,\mathrm{d}x &= \int_1^\infty \left[\frac{y^{1/2-\varepsilon}}{1/2-\varepsilon} \right]_1^x \frac{1}{x^{3/2+\varepsilon}} \,\mathrm{d}x \\ &= \frac{2}{1-2\varepsilon} \int_1^\infty \frac{1}{x^{1+2\varepsilon}} \,\mathrm{d}x - \frac{2}{1-2\varepsilon} \int_1^\infty \frac{1}{x^{3/2+\varepsilon}} \,\mathrm{d}x = \frac{1}{\varepsilon} + \mathcal{O}(1)\end{aligned}$$

and

$$\begin{aligned}\int_1^\infty \left(\int_x^\infty \frac{1}{y^{3/2+\varepsilon}} \,\mathrm{d}y \right) \frac{1}{x^{1/2+\varepsilon}} \,\mathrm{d}x &= \int_1^\infty \left[\frac{y^{-1/2-\varepsilon}}{-1/2-\varepsilon} \right]_x^\infty \frac{1}{x^{1/2+\varepsilon}} \,\mathrm{d}x = \frac{2}{1+2\varepsilon} \int_1^\infty \frac{1}{x^{1+2\varepsilon}} \,\mathrm{d}x \\ &= \frac{1}{\varepsilon} + \mathcal{O}(1).\end{aligned}$$

Altogether, it holds that

$$\lim_{\varepsilon\to 0+}\frac{|\langle h(\varepsilon), Lh(\varepsilon)\rangle|}{\|h(\varepsilon)\|^2} = \lim_{\varepsilon\to 0+}\frac{\frac{2}{\varepsilon}+\mathcal{O}(1)}{\frac{1}{2\varepsilon}+\mathcal{O}(1)} = 4. \qquad \square$$

Remark 1.9. The Schur test 1.7 can be also used to derive the continuous Hardy inequality (2) for $p = 2$. The operator C_∞ is, in fact, an integral operator with the kernel

$$K(x,y) = \begin{cases} 0 & \text{if } x \le y \\ 1/x & \text{if } x > y \end{cases}$$

for all $x, y \in (0,\infty)$. With the choice of $h(x) = x^{-1/4}$ for all $x \in (0,\infty)$ we get

$$\int_0^\infty \mathcal{K}(x,y)h(y)^2\,\mathrm{d}y = \frac{1}{x}\int_0^x \frac{1}{\sqrt{y}}\,\mathrm{d}y = \frac{2}{\sqrt{x}}$$

and

$$\int_0^\infty \mathcal{K}(x,y)h(x)^2\,\mathrm{d}x = \int_y^\infty \frac{1}{x^{3/2}}\,\mathrm{d}x = \frac{2}{\sqrt{y}}.$$

It follows from the Schur test that $\|C_\infty\| \le 2^{1/2}2^{1/2} = 2$.

1.3.2 Matrix factorization

In this section, we show that

$$\|C - I\| = 1, \tag{1.8}$$

together with several other findings, which appeared in [9]. Note, that the identity (1.8) implies the classical discrete Hardy inequality, since

$$\|C\| = \|C - I + I\| \le \|C - I\| + \|I\| = 2.$$

Remark 1.10. The inequality $\ge$ in (1.6) can be shown analogously as in Section 1.2 with the choice of $h_n(\varepsilon) := n^{-1/2-\varepsilon}$ for all $n \in \mathbb{N}$, where $\varepsilon > 0$. Evidently,

$$(C^* h(\varepsilon))_n = \sum_{m=1}^\infty c^*_{n,m} h_m(\varepsilon) = \sum_{m=n}^\infty \frac{1}{m} h_m(\varepsilon) = \sum_{m=n}^\infty \frac{1}{m^{3/2+\varepsilon}},$$

hence

$$\begin{aligned}\|C^* h(\varepsilon)\|^2 &= \sum_{n=1}^\infty \left(\sum_{m=n}^\infty \frac{1}{m^{3/2+\varepsilon}}\right)^2 > \sum_{n=1}^\infty \left(\int_n^\infty \frac{1}{x^{3/2+\varepsilon}}\,\mathrm{d}x\right)^2 = \left(\frac{2}{1+2\varepsilon}\right)^2 \sum_{n=1}^\infty \left(\frac{1}{n^{1/2+\varepsilon}}\right)^2 \\ &= \left(\frac{2}{1+2\varepsilon}\right)^2 \|h(\varepsilon)\|^2.\end{aligned}$$

Altogether, we have

$$\lim_{\varepsilon\to 0+}\frac{\|C^* h(\varepsilon)\|}{\|h(\varepsilon)\|} \ge 2 \quad\Longrightarrow\quad \|C^*\| = \|C\| \ge 2.$$

Now we proceed to show the desired equality $\|C - I\| = 1$. To this end, we will state an elegant lemma regarding matrix factorization.

Lemma 1.11. *Let us denote a diagonal matrix*

$$D := \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1/2 & 0 & \cdots \\ 0 & 0 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $(C-I)(C^*-I) = I - D$.

Proof. Bearing (1.7) in mind, we have

$$CC^* = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/2 & 1/3 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = C + C^* - D,$$

therefore

$$(C-I)(C^*-I) = CC^* - C - C^* + I = I - D. \qquad \square$$

It is easy to check, that the norm of an arbitrary diagonal matrix T can be computed as

$$\|T\| = \sup_{n\in\mathbb{N}} |T_{n,n}|,$$

hence $\|I-D\| = 1$. Since $(C-I)^* = C^* - I$, Lemma 1.11 asserts that $\|I-D\| = \|C-I\|^2$, and therefore we may conclude that the identity (1.8) holds.

Another interesting result can be deduced from this factorization, see [8]. It can be shown, that the supremum in the definition of the operator norm of C is not attained by any non-zero sequence, i.e. the classical discrete Hardy inequality $\|Ca\| < 2\|a\|$ is strict for all $a \in \ell^2(\mathbb{N})$ unless $a \equiv 0$. For this purpose, we will make the following observation.

Proposition 1.12. *Let $A \in \mathcal{B}(\mathcal{H})$, such that $\|A\| \leq 1$. If the exists $0 \neq x \in \mathcal{H}$ satisfying $\|Ax\| = \|x\|$, then there exists $0 \neq y \in \mathcal{H}$ satisfying $\|A^*y\| = \|y\|$.*

Proof. Set $y := Ax \neq 0$, thus $\|x\| = \|y\|$. Furthermore, the inequality $\|A^*A\| = \|A\|^2 \leq 1$ and the Cauchy–Schwarz inequality yield

$$\|x\|^2 = \langle Ax, Ax\rangle = \langle A^*Ax, x\rangle \leq \|A^*Ax\|\,\|x\| \leq \|x\|^2,$$

therefore all inequalities stand as equalities. Dividing both sides by $\|x\| \neq 0$, we obtain

$$\|A^*y\| = \|A^*Ax\| = \|x\| = \|y\|. \qquad \square$$

It is obvious that $\|(I-D)a\| < \|a\|$ whenever $a \neq 0$. Employing Lemma 1.11 and the Cauchy–Schwarz inequality we have

$$\|(C^*-I)a\|^2 = \langle (C^*-I)a, (C^*-I)a\rangle = \langle a, (C-I)(C^*-I)a\rangle \leq \|(I-D)a\|\,\|a\| < \|a\|^2$$

for all $0 \neq a \in \ell^2(\mathbb{N})$. The preceding proposition yields

$$\|Ca\| = \|(C-I)a + a\| \leq \|(C-I)a\| + \|a\| < 2\|a\|.$$

As was further shown in [9], it is possible to derive certain equalities underlying the inequalities $\|(C^*-I)a\|^2 \leq \|a\|^2$ and $\|(C-I)a\|^2 \leq \|a\|^2$.

Proposition 1.13. *For any $a \in \ell^2(\mathbb{N})$ it holds that*

$$\|(C^* - I)a\|^2 = \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) a_n^2.$$

Proof. This is an easy application of Lemma 1.11 that asserts

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) a_n^2 = \langle a, (I - D)a \rangle = \langle a, (C - I)(C^* - I)a \rangle = \|(C^* - I)a\|^2 . \qquad \square$$

Evidently, this equality yields $\|(C^* - I)a\| \leq \|a\|$ for all $a \in \ell^2(\mathbb{N})$, since

$$\|(C^* - I)a\|^2 = \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) a_n^2 \leq \sum_{n=1}^{\infty} a_n^2 = \|a\|^2 .$$

Proposition 1.14. *For any $a \in \ell^2(\mathbb{N})$ it holds that*

$$\sum_{n=2}^{\infty} \frac{n}{n-1}((C - I)a_n)^2 = \sum_{n=1}^{\infty} a_n^2.$$

This proposition can be proven either by direct algebra or by matrix factorization as previously, but perhaps a little more delicately. As a matter of interest, we will show both of these approaches.

Proof. Let us denote $A_n = \sum_{k=1}^{n} a_n$ and $b_n = (C - I)a_n$. Firstly, we examine what the proposition states for sequences satisfying $a_n = 0$ for all $n > N$. In that case $A_n = A_N$ and $b_n = A_N/n$ for all $n \geq N$, hence

$$\sum_{n=2}^{\infty} \frac{n}{n-1} b_n^2 = \sum_{n=2}^{N} \frac{n}{n-1} b_n^2 + \sum_{n=N+1}^{\infty} \frac{n}{n-1} \frac{A_N^2}{n^2} = \sum_{n=2}^{N} \frac{n}{n-1} b_n^2 + A_N^2 \sum_{n=N+1}^{\infty} \frac{1}{n(n-1)}.$$

Moreover,

$$\sum_{n=N+1}^{\infty} \frac{1}{n(n-1)} = \lim_{M \to \infty} \sum_{n=N+1}^{M} \left(\frac{1}{n-1} - \frac{1}{n} \right) = \lim_{M \to \infty} \left(\frac{1}{N} - \frac{1}{M} \right) = \frac{1}{N},$$

therefore altogether, we have

$$\sum_{n=2}^{N} \frac{n}{n-1} b_n^2 + \frac{A_N^2}{N} = \sum_{n=1}^{N} a_n^2. \tag{1.9}$$

This equality follows from mathematical induction. The statement is trivially true for $N = 2$ because

$$2b_2^2 + \frac{A_2^2}{2} = 2\left(\frac{a_1 + a_2}{2} - a_2\right)^2 + \frac{1}{2}(a_1 + a_2)^2 = \frac{1}{2}(a_1 - a_2)^2 + \frac{1}{2}(a_1 + a_2)^2 = a_1^2 + a_2^2.$$

Assuming the statement holds true for $N - 1$, we observe that for N, we have

$$\sum_{n=2}^{N} \frac{n}{n-1} b_n^2 + \frac{A_N^2}{N} = \sum_{n=2}^{N-1} \frac{n}{n-1} b_n^2 + \frac{N}{N-1} b_N^2 + \frac{A_N^2}{N} \pm \frac{A_{N-1}^2}{N-1} \stackrel{?}{=} \sum_{n=1}^{N} a_n^2 = \sum_{n=1}^{N-1} a_n^2 + a_N^2$$

$$\iff \frac{N}{N-1} b_N^2 + \frac{A_N^2}{N} - \frac{A_{N-1}^2}{N-1} = a_N^2.$$

Substituting $b_N = (A_N - Na_N)/N$ and $A_{N-1} = A_N - a_N$, the left-hand side of the preceding equality becomes

$$\frac{1}{N(N-1)}(A_N - Na_N)^2 + \frac{A_N^2}{N} - \frac{(A_N - a_N)^2}{N-1} = \left(\frac{N}{N-1} - \frac{1}{N-1}\right) a_N^2 = a_N^2,$$

which is equal to the right-hand side.

In order to complete the proof, it suffices to show that for all $a \in \ell^2(\mathbb{N})$ the second term on the left-hand side of (1.9) is vanishing for $N \to \infty$, i.e. that

$$\lim_{n\to\infty} \frac{A_n^2}{n} = 0.$$

Indeed, for an arbitrary $a \in \ell^2(\mathbb{N})$ and $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0+1}^{\infty} a_n^2 < \varepsilon$. If we denote $A_n = A_{n_0} + S_n$, i.e. $S_n = \sum_{k=n_0+1}^{n} a_k$, the Cauchy–Schwarz inequality yields

$$S_n^2 \leq (n - n_0) \sum_{k=n_0+1}^{n} a_k^2 \leq \varepsilon(n - n_0) \leq \varepsilon n.$$

For n sufficiently large so that $A_{n_0}^2 \leq \varepsilon n$, it holds that $A_n^2 \leq 2A_{n_0}^2 + 2S_n^2 \leq 4\varepsilon n$, thus

$$\frac{A_n^2}{n} \leq 4\varepsilon.$$

Since ε was arbitrary, the proof is complete. □

In fact, Proposition 1.14 reads

$$\left\langle (C-I)a, \tilde{D}(C-I)a \right\rangle = \left\langle a, (C^* - I)\tilde{D}(C-I)a \right\rangle = \langle a, a \rangle,$$

where we denoted

$$\tilde{D} := \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 2/1 & 0 & \cdots \\ 0 & 0 & 3/2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.10}$$

With this observation, Proposition 1.14 stems from the following lemma.

Lemma 1.15. *In view of the definition* (1.10)*, we have* $(C^* - I)\tilde{D}(C-I) = I$.

Proof. Since $(C^* - I)\tilde{D}(C-I) = C^*\tilde{D}C - \tilde{D}C - C^*\tilde{D} + \tilde{D}$ the lemma can be equivalently formulated as

$$C^*\tilde{D}C = \tilde{D}C + C^*\tilde{D} + I - \tilde{D}.$$

The left multiplication by the matrix $\tilde{D}$ creates zeros in the first row and for $m \geq 2$ it multiplies m-th row by $m/(m-1)$, therefore

$$\tilde{D}C = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ 1/2 & 1/2 & 1/2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ i.e. } (\tilde{D}C)_{m,n} = \begin{cases} 0 & \text{if } m = 1 \\ \dfrac{1}{m-1} & \text{if } m \geq n \text{ and } m \geq 2 \\ 0 & \text{if } m < n \end{cases}$$

for all $m, n \in \mathbb{N}$, and

$$(\tilde{D}C)^* = C^*\tilde{D} = \begin{pmatrix} 0 & 1 & 1/2 & \cdots \\ 0 & 1 & 1/2 & \cdots \\ 0 & 0 & 1/2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ i.e. } (C^*\tilde{D})_{m,n} = \begin{cases} 0 & \text{if } n = 1 \\ \dfrac{1}{n-1} & \text{if } n \geq m \text{ and } n \geq 2 \\ 0 & \text{if } n < m \end{cases}$$

for all $m, n \in \mathbb{N}$. Moreover, whenever $m \neq 1$ or $n \neq 1$

$$(C^*\tilde{D}C)_{m,n} = \sum_{k=1}^{\infty} c^*_{m,k}(\tilde{D}C)_{k,n} = \sum_{k=\max\{m,n\}}^{\infty} \frac{1}{k(k-1)} = \frac{1}{\max\{m,n\}-1},$$

and if $m = n = 1$ we have

$$(C^*\tilde{D}C)_{1,1} = \sum_{k=1}^{\infty} c^*_{1,k}(\tilde{D}C)_{k,1} = 0 + \sum_{k=2}^{\infty} \frac{1}{k(k-1)} = 1.$$

Combining these results, we arrive at

$$C^*\tilde{D}C = \begin{pmatrix} 1 & 1 & 1/2 & 1/3 & \cdots \\ 1 & 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/2 & 1/2 & 1/3 & \cdots \\ 1/3 & 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 1/2 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 1 & 1/2 & 1/3 & \cdots \\ 0 & 1 & 1/2 & 1/3 & \cdots \\ 0 & 0 & 1/2 & 1/3 & \cdots \\ 0 & 0 & 0 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & -1/2 & 0 & \cdots \\ 0 & 0 & 0 & -1/3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \tilde{D}C + C^*\tilde{D} + I - \tilde{D},$$

completing the proof. $\square$

Evidently, Proposition 1.14 asserts inequalities $\|(C-I)a\| \leq \|a\| \leq \sqrt{2}\,\|(C-I)a\|$ for all $a \in \ell^2(\mathbb{N})$. Indeed, $(C-I)a_1 = 0$ and thus

$$\begin{aligned} \|(C-I)a\|^2 &= \sum_{n=1}^{\infty}((C-I)a_n)^2 \leq \sum_{n=2}^{\infty} \frac{n}{n-1}((C-I)a_n)^2 = \sum_{n=1}^{\infty} a_n^2 = \|a\|^2 \\ &\leq 2\sum_{n=1}^{\infty}((C-I)a_n)^2 = 2\,\|(C-I)a\|^2 . \end{aligned}$$

1.4 Relation between continuous and discrete Hardy inequalities

In this section, we will demonstrate that the classical discrete and continuous Hardy inequalities are equivalent in the sense that one implies the other. For completeness, we shall prove the continuous Hardy inequality (2) in a general L^p setting. To this end, we include the general Minkowski inequality, which appears for example in [6, Theorem 202].

Theorem 1.16 (Minkowski)**.** *Let $(X,\mathcal{M},\mu)$, $(Y,\mathcal{N},\nu)$ spaces with σ-finite measures, $p\in[1,\infty]$ and $f: X\times Y\to\mathbb{C}$ measurable. Then*

$$(\forall x\in X)\,(f(x,\cdot)\ \nu\text{-measurable})\ \wedge\ (\forall y\in Y)\,(f(\cdot,y)\ \mu\text{-measurable}),$$
$$\int_Y f(\cdot,y)\,\mathrm{d}\nu(y)\ \mu\text{-measurable}\ \wedge\ \int_X f(x,\cdot)\,\mathrm{d}\mu(x)\ \nu\text{-measurable}$$

and it holds that

$$\left\|\int_Y f(\cdot,y)\,\mathrm{d}\nu(y)\right\|_p \le \int_Y \|f(\cdot,y)\|_p\,\mathrm{d}\nu(y),$$

resp.

$$\left\|\int_X f(x,\cdot)\,\mathrm{d}\mu(x)\right\|_p \le \int_X \|f(x,\cdot)\|_p\,\mathrm{d}\mu(x).$$

Proof. The fact that $f(\cdot,y)$, $f(x,\cdot)$, $\int_Y f(\cdot,y)\,\mathrm{d}\nu(y)$, and $\int_X f(x,\cdot)\,\mathrm{d}\mu(x)$ are measurable is a part of the Tonelli–Fubini theorem. Moreover, it is sufficient to show the first statement since the second could obviously be proven analogously.

a) $p\in(1,\infty)$: Since $\left|\int_Y f(x,y)\,\mathrm{d}\nu(y)\right|\le\int_Y |f(x,y)|\,\mathrm{d}\nu(y)$ for all $x\in X$, it suffices to consider $f\ge 0$ without loss of generality. Let us denote $H(x)\coloneqq\int_Y f(x,y)\,\mathrm{d}\nu(y)$. We will show that $\|H\|_p\le\int_Y\|f(\cdot,y)\|_p\,\mathrm{d}\nu(y)$. Evidently, we can restrict ourselves to the case $\|H\|_p>0$. By the Tonelli–Fubini theorem, we have

$$\begin{aligned}\|H\|_p^p=\int_X H(x)^p\,\mathrm{d}\mu(x)&=\int_X\left(\int_Y f(x,y)\,\mathrm{d}\nu(y)\right)H(x)^{p-1}\,\mathrm{d}\mu(x)\\&=\int_Y\left(\int_X f(x,y)H(x)^{p-1}\,\mathrm{d}\mu(x)\right)\mathrm{d}\nu(y).\end{aligned}$$

Moreover, from the Hölder inequality (1.4) with $q=p/(p-1)$, we infer that

$$\begin{aligned}\|H\|_p^p&\le\int_Y\left(\int_X f(x,y)^p\,\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\left(\int_X H(x)^p\,\mathrm{d}\mu(x)\right)^{\frac{p-1}{p}}\mathrm{d}\nu(y)\\&=\int_Y\left(\int_X f(x,y)^p\,\mathrm{d}\mu(x)\right)^{1/p}\mathrm{d}\nu(y)\,\|H\|_p^{p-1}.\end{aligned}$$

Now, we have to consider two cases, regarding the finiteness of $\|H\|_p$.

(i) $\|H\|_p<\infty$: Dividing the previous inequality by $\|H\|_p^{p-1}$, we obtain the desired result.

(ii) $\|H\|_p=\infty$: It follows from the σ-finiteness of considered spaces that

$$(\exists\{A_n\}_{n=1}^\infty\subset\mathcal{M})(\forall n\in\mathbb{N})(\mu(A_n)<\infty\ \wedge\ \bigcup_{n=1}^\infty A_n=X),$$
$$(\exists\{B_m\}_{m=1}^\infty\subset\mathcal{N})(\forall m\in\mathbb{N})(\nu(B_m)<\infty\ \wedge\ \bigcup_{m=1}^\infty B_m=Y),$$

Without loss of generality, we can assume that $A_n\subset A_{n+1}$ for all $n\in\mathbb{N}$. Otherwise we would define $\tilde{A}_n\coloneqq\bigcup_{k=1}^n A_k$, where $\mu(\tilde{A}_n)\le\sum_{k=1}^n\mu(A_n)<\infty$ and $\bigcup_{n=1}^\infty\tilde{A}_n=X$. Analogously we assume that $B_n\subset B_{n+1}$ for all $n\in\mathbb{N}$. We further denote $f_k\coloneqq\min\{f,k\}$ for all $k\in\mathbb{N}$. From the part (i), we obtain

$$\left(\int_{A_n}\left(\int_{B_m} f_k(x,y)\,\mathrm{d}\nu(y)\right)^p\mathrm{d}\mu(x)\right)^{1/p}\le\int_{B_m}\left(\int_{A_n} f_k(x,y)^p\,\mathrm{d}\mu(x)\right)^{1/p}\mathrm{d}\nu(y).$$

Tending $k\to\infty$ and $m,n\to\infty$, the Monotone Convergence Theorem yields the result.

b) $p = 1$: This case is an easy application of the Tonelli–Fubini theorem

$$\int_X \left| \int_Y f(x,y) \, \mathrm{d}\nu(y) \right| \mathrm{d}\mu(x) \le \int_X \left(\int_Y |f(x,y)| \, \mathrm{d}\nu(y) \right) \mathrm{d}\mu(x) = \int_Y \left(\int_X |f(x,y)| \, \mathrm{d}\mu(x) \right) \mathrm{d}\nu(y).$$

c) $p = \infty$: A simple estimate asserts

$$\left| \int_Y f(\cdot, y) \, \mathrm{d}\nu(y) \right| \le \int_Y |f(\cdot, y)| \, \mathrm{d}\nu(y) \le \int_Y \|f(\cdot, y)\|_\infty \, \mathrm{d}\nu(y),$$

hence

$$\left\| \int_Y f(\cdot, y) \, \mathrm{d}\nu(y) \right\|_\infty \le \int_Y \|f(\cdot, y)\|_\infty \, \mathrm{d}\nu(y).$$

□

Now we may employ an elegant argument from [6, p. 243] to prove the continuous Hardy inequality (2).

Proof. Substituting $t = sx$ we get

$$C_\infty f(x) = \frac{1}{x} \int_0^x f(t) \, \mathrm{d}t = \int_0^1 f(sx) \, \mathrm{d}s.$$

The Minkowski inequality 1.16 and the Tonelli–Fubini theorem yield

$$\begin{aligned} \|C_\infty f\|_p &= \left\| \int_0^1 f(tx) \, \mathrm{d}t \right\|_p \le \int_0^1 \|f(tx)\|_p \, \mathrm{d}t = \int_0^1 \left(\int_0^\infty f(tx)^p \, \mathrm{d}x \right)^{1/p} \mathrm{d}t \\ &= \int_0^1 \left(\int_0^\infty f(s)^p \frac{\mathrm{d}s}{t} \right)^{1/p} \mathrm{d}t = \frac{p}{p-1} \|f\|_p, \end{aligned}$$

where we substituted $tx = s$. □

We are ready to prove the equivalence of the classical discrete and continuous Hardy inequalities. First, we show perhaps a more intuitive implication, which is that the continuous inequality implies the discrete one.

Proof. Having Remark 1.2 in mind, it suffices to consider only $a \in \ell^p(\mathbb{N})$ non-negative and non-increasing. Let us define $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ as $f := \sum_{n=1}^\infty a_n \chi_{[n-1,n)}$. Then for any $x \in [n-1, n)$, where $n \in \mathbb{N}$, we have $F(x) := \int_0^x f(t) \, \mathrm{d}t = \sum_{k=1}^{n-1} a_k + a_n(x - n + 1)$. A simple observation asserts that $(F(x)/x)^p$ is non-increasing on $[n-1, n)$ for all $n \in \mathbb{N}$ since

$$\left(\frac{F(x)}{x} \right)' = \frac{F'(x)x - F(x)}{x^2} = \frac{-\sum_{k=1}^{n-1} a_k + (n-1)a_n}{x^2} \le 0.$$

The last inequality holds because a is non-increasing, thus $\sum_{k=1}^{n-1} a_k \ge (n-1)a_n$. Assuming that the continuous Hardy inequality is true, we may estimate

$$\begin{aligned} \sum_{n=1}^\infty \left(\frac{A_n}{n} \right)^p &\le \sum_{n=1}^\infty \int_{n-1}^n \left(\frac{\sum_{k=1}^{n-1} a_k + a_n(x - n + 1)}{x} \right)^p \mathrm{d}x = \int_0^\infty \left(\frac{F(x)}{x} \right)^p \mathrm{d}x \\ &\le \left(\frac{p}{p-1} \right)^p \int_0^\infty f(x)^p \, \mathrm{d}x = \left(\frac{p}{p-1} \right)^p \sum_{n=1}^\infty a_n^p, \end{aligned}$$

where we denoted $A_n := \sum_{k=1}^n a_k$, hence proving the classical discrete Hardy inequality. □

In order to show the reverse implication, we will rewrite both inequalities into their more renowned forms. Let us denote $a_n =: u_n - u_{n-1}$, where $u_0 := 0$. The classical discrete Hardy inequality can clearly be equivalently written as

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}|^p \geq \left(\frac{p-1}{p}\right)^p \sum_{n=1}^{\infty} \frac{|u_n|^p}{n^p} \tag{1.11}$$

for all finitely supported complex sequences $u \in C_0(\mathbb{N}_0)$ with $u_0 = 0$. Analogously, if we introduce $f =: \varphi'$, the continuous Hardy inequality is equivalent to

$$\int_0^{\infty} |\varphi'(x)|^p \,\mathrm{d}x \geq \left(\frac{p-1}{p}\right)^p \int_0^{\infty} \frac{|\varphi(x)|^p}{x^p} \,\mathrm{d}x \tag{1.12}$$

for all $\varphi \in \mathcal{D}((0,\infty))$, where $\mathcal{D}(U)$ denotes the set of smooth functions with compact support in the subset $U \subset \mathbb{R}$. We shall connect the discrete and continuous versions in an analogous fashion as in [13].

Proof. Firstly, we will show that it suffices to consider only non-negative real function $\varphi \in \mathcal{D}((0,1])$. Indeed, the non-negativity is obvious and if we have $\psi \in \mathcal{D}((0,\infty))$, therefore $\operatorname{supp}\psi \subset [a,b]$ for some $0 < a < b$, we define $\varphi(x) := \psi(bx)$. Consequently, $\operatorname{supp}\varphi \subset [a/b, 1]$ and $\varphi \in \mathcal{D}((0,1])$, hence

$$\begin{aligned}
\int_0^{\infty} \psi'(x)^p \,\mathrm{d}x &= \int_0^{b} \psi'(x)^p \,\mathrm{d}x = b\int_0^{1} \psi'(by)^p \,\mathrm{d}y = b^{1-p}\int_0^{1} \varphi'(y)^p \,\mathrm{d}y \\
&\geq b^{1-p}\left(\frac{p-1}{p}\right)^p \int_0^1 \frac{\varphi(y)^p}{y^p}\,\mathrm{d}y = b^{1-p}\left(\frac{p-1}{p}\right)^p \int_0^1 \frac{\psi(by)^p}{y^p}\,\mathrm{d}y \\
&= b^{1-p}\left(\frac{p-1}{p}\right)^p \int_0^b \frac{\psi(x)^p}{x^p b^{-p}}\frac{\mathrm{d}x}{b} = \left(\frac{p-1}{p}\right)^p \int_0^{\infty} \frac{\psi(x)^p}{x^p}\,\mathrm{d}x,
\end{aligned}$$

where we used the substitution $x = by$ twice.

Next, for $\varphi \in \mathcal{D}((0,1])$, we define sequences $u_n^N := N^{(p-1)/p}\varphi(n/N)$ for all $n \in \mathbb{N}_0$ and $N \in \mathbb{N}$, thus

$$\begin{aligned}
\sum_{n=1}^{\infty} (u_n^N - u_{n-1}^N)^p &= N^{p-1}\sum_{n=1}^{N}\left(\varphi\left(\frac{n}{N}\right) - \varphi\left(\frac{n-1}{N}\right)\right)^p = N^{p-1}\sum_{n=1}^{N}\left(\varphi'\left(\frac{n}{N}\right)\frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right)\right)^p \\
&= N^{p-1}\sum_{n=1}^{N}\varphi'\left(\frac{n}{N}\right)^p \frac{1}{N^p}\left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right)^p = \sum_{n=1}^{N}\varphi'\left(\frac{n}{N}\right)^p \frac{1}{N}\left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right) \\
&= \sum_{n=1}^{N}\varphi'\left(\frac{n}{N}\right)^p \frac{1}{N} + \mathcal{O}\left(\frac{1}{N}\right), \quad N \to \infty,
\end{aligned}$$

where we employed the Taylor-Lagrange formula

$$\varphi\left(\frac{n-1}{N}\right) = \varphi\left(\frac{n}{N}\right) - \varphi'\left(\frac{n}{N}\right)\frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) \quad \text{and} \quad \left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right)^p = 1 + \mathcal{O}\left(\frac{1}{N}\right).$$

On the other hand, we have

$$\left(\frac{p-1}{p}\right)^p \sum_{n=1}^{\infty}\frac{(u_n^N)^p}{n^p} = \left(\frac{p-1}{p}\right)^p \sum_{n=1}^{N}\frac{N^{p-1}\varphi\left(\frac{n}{N}\right)^p}{n^p} = \left(\frac{p-1}{p}\right)^p \sum_{n=1}^{N}\frac{\varphi\left(\frac{n}{N}\right)^p}{\left(\frac{n}{N}\right)^p}\frac{1}{N}.$$

Altogether, with the assumption of the veracity of the classical discrete Hardy inequality, we have

$$\sum_{n=1}^{N}\varphi'\left(\frac{n}{N}\right)^p \frac{1}{N} + \mathcal{O}\left(\frac{1}{N}\right) \geq \left(\frac{p-1}{p}\right)^p \sum_{n=1}^{N}\frac{\varphi\left(\frac{n}{N}\right)^p}{\left(\frac{n}{N}\right)^p}\frac{1}{N}.$$

Since $\varphi \in \mathcal{D}((0,1])$, the function is Riemann integrable, hence after taking the limit $N \to \infty$ in the above expression, we obtain the desired integral inequality

$$\int_0^1 \varphi'(x)^p \, \mathrm{d}x \geq \left(\frac{p-1}{p}\right)^p \int_0^1 \frac{\varphi(x)^p}{x^p} \, \mathrm{d}x. \qquad \square$$

Let us restrict ourselves to the case $p = 2$ once more. We have shown that the constant $1/4$ in (1.11) is optimal. The connection between the discrete and continuous versions thereby implies the optimality of the constant $1/4$ in (1.12) as well. Additionally, the weight $1/(4x^2)$ in (1.12) is known to be *critical*, meaning that if (1.12) holds with $1/(4x^2)$ replaced by a measurable function $\rho(x) \geq 1/(4x^2)$ for a.e. $x > 0$, then $\rho(x) = 1/(4x^2)$ for a.e. $x > 0$. In contrast, the opposite was recently discovered concerning the discrete version of the Hardy inequality, which will be discussed in the following chapter.

Chapter 2

Improved discrete Hardy inequality

In [11], M. Keller, Y. Pinchover, and F. Pogorzelski discovered the improved Hardy inequality which reads

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}|^2 \geq \sum_{n=1}^{\infty} \rho_n^{\mathrm{KPP}} |u_n|^2 \tag{2.1}$$

for all $u \in C_0(\mathbb{N}_0)$ with $u_0 = 0$, where the weight $1/(4n^2)$ in (1.11) was replaced by the point-wise bigger sequence

$$\rho_n^{\mathrm{KPP}} := 2 - \sqrt{\frac{n-1}{n}} - \sqrt{\frac{n+1}{n}} > \frac{1}{4n^2}. \tag{2.2}$$

Additionally, the authors proved in [10] that the weight ρ^{KPP} is *optimal*. The notion of optimality is a rather strong property which was introduced in a discrete graph setting in [14] (see Definition 3.2 below). In particular, the optimality of ρ^{KPP} implies *criticality*, meaning that ρ^{KPP} cannot be further improved by a point-wise bigger sequence such that the analogue of (2.1) would still hold. This chapter will present three approaches for handling the optimal discrete Hardy inequality (2.1).

2.1 Original idea

In [10], M. Keller, Y. Pinchover, and F. Pogorzelski discovered optimal discrete Hardy inequalities for Schrödinger operators in a general graph setting. Later, they published a shortened proof of the optimal discrete Hardy inequality on the half-line [11], considering the standard graph $\mathbb{N}_0$. Let us mention that the notation in this section differs from the rest of this paper as we intend to preserve the *graph notation* from [11]. Namely, the *combinatorial Laplacian* (2.4) shall vary by a sign from the *standard discrete Laplacian* defined by (2.7). We also restrict ourselves to real sequences $u \in C(\mathbb{N})$.

First, we define several terms that will be frequently used in the following proof. A positive sequence $w \in C(\mathbb{N})$, i.e. $w_n > 0$ for all $n \in \mathbb{N}$, is called a *weight* and, in this case, we write $w > 0$. For a weight $w > 0$, we denote the weighted ℓ^2 space

$$\ell^2(w) := \{u \in C(\mathbb{N}) \mid \sum_{n=1}^{\infty} u_n^2 w_n < \infty\}$$

endowed with the scalar product

$$\langle u, v \rangle_w = \sum_{n=1}^{\infty} u_n v_n w_n$$

for all $u, v \in \ell^2(w)$ and the induced norm $\|\cdot\|_w$. We have the following natural unitary transformation between the spaces $\ell^2(\mathbb{N})$ and $\ell^2(w)$.

Lemma 2.1. *Let $w \in C(\mathbb{N})$ be a weight. Then the operator $T_w : \ell^2(w^2) \to \ell^2(\mathbb{N})$ given by*

$$T_w u := wu \tag{2.3}$$

for all sequences $u \in \ell^2(w^2)$, where the multiplication of sequences is to be understood point-wise, is unitary.

Proof. The proof readily follows from the observation

$$\|T_w u\|^2 = \|uw\|^2 = \sum_{n=1}^{\infty} u_n^2 w_n^2 = \|u\|_{w^2}. \qquad \square$$

We say that two vertices m, n of the graph $\mathbb{N}_0$ are *connected by an edge* if $|n - m| = 1$, in which case we write $m \sim n$. In this section, we set $u_0 = 0$ for all $u \in C(\mathbb{N})$. Given a weight w, we further define the *combinatorial Laplacian associated with w* as

$$\Delta_w u_n := \frac{1}{w_n^2} \sum_{m \sim n} w_n w_m (u_n - u_m) \tag{2.4}$$

for any $u \in C(\mathbb{N})$ and $n \in \mathbb{N}$. The combinatorial Laplacian Δ_1 is often denoted as Δ, but we will keep the notation Δ_1 in order not to confuse it with the standard Laplacian (2.7). The starting point of the proof of the optimal discrete Hardy inequality is the *ground state transform*, which is the content of the following lemma, see also [10] for a proper definition of the (Agmon) ground state.

Lemma 2.2. *Let $w \in C(\mathbb{N})$ be a weight and $\rho \in C(\mathbb{N})$ a sequence satisfying $(\Delta_1 - \rho)w = 0$ on $\mathbb{N}$. Then*

$$T_w^{-1}(\Delta_1 - \rho)T_w = \Delta_w$$

on $C_0(\mathbb{N})$.

Proof. The statement follows from a direct computation. In view of definitions (2.3) and (2.4), for any $u \in C_0(\mathbb{N})$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}(\Delta_1 T_w u)_n &= \sum_{m \sim n} (w_n u_n - w_m u_m) = u_n \sum_{m \sim n} (w_n - w_m) + \sum_{m \sim n} w_m (u_n - u_m) \\ &= u_n (\Delta_1 - \rho) w_n + u_n \rho_n w_n + \sum_{m \sim n} w_m (u_n - u_m) = 0 + \rho_n (T_w u)_n + (T_w \Delta_w u)_n,\end{aligned}$$

hence

$$\Delta_1 T_w = \rho T_w + T_w \Delta_w.$$

Applying T_w^{-1} from the left to the above equality yields the result. $\square$

Moreover, for a weight $w \in C(\mathbb{N})$ and a sequence $u \in C_0(\mathbb{N})$, we define the *quadratic form h_w associated with w* by the formula

$$h_w(u) := \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m \sim n} w_n w_m (u_n - u_m)^2.$$

The following lemma connects the scalar product and the quadratic form by means of the *Green formula*.

Lemma 2.3. *Let $w \in C(\mathbb{N})$ be a weight. Then for any $u \in C_0(\mathbb{N})$, it holds that*

$$\langle \Delta_w u, u \rangle_{w^2} = h_w(u) \geq 0.$$

Proof. The proof follows trivially from the respective definitions since

$$\begin{aligned}\langle \Delta_w u, u\rangle_{w^2} &= \sum_{n=1}^{\infty} w_n^2 \left(\frac{1}{w_n^2} \sum_{m\sim n} w_n w_m (u_n - u_m) \right) u_n \\ &= \sum_{n=1}^{\infty}\sum_{m\sim n} w_n w_m (u_n - u_m)^2 + \sum_{n=1}^{\infty}\sum_{m\sim n} w_n w_m (u_n - u_m) u_m = 2h_w(u) - \langle \Delta_w u, u\rangle_{w^2}. \quad \square\end{aligned}$$

Proposition 2.4. *Let $w, \rho \in C(\mathbb{N})$ be weights, such that $(\Delta_1 - \rho)w = 0$ on $\mathbb{N}$. Then $h_1(u) \geq \|u\|_\rho^2$ for all $u \in C_0(\mathbb{N})$.*

Proof. Indeed, using Lemmas 2.1, 2.2, and 2.3, we infer that for all $u \in C_0(\mathbb{N})$ it is true that

$$\langle (\Delta_1 - \rho)u, u\rangle = \langle T_w^{-1}(\Delta_1 - \rho)T_w T_w^{-1} u, T_w^{-1} u\rangle_{w^2} = \langle \Delta_w (u/w), u/w\rangle_{w^2} = h_w(u/w) \geq 0,$$

hence

$$\langle (\Delta_1 - \rho)u, u\rangle = \langle \Delta_1 u, u\rangle - \langle \rho u, u\rangle = h_1(u) - \|u\|_\rho^2 \geq 0,$$

by Lemma 2.3 once more. □

With a suitable choice of the weights w and ρ, we may prove the optimal discrete Hardy inequality (2.1). Using the definition (2.2), we set

$$\rho_n := \rho_n^{\mathrm{KPP}} = \frac{\Delta_1 \sqrt{n}}{\sqrt{n}} \quad \text{and} \quad w_n := \sqrt{n} \tag{2.5}$$

for all $n \in \mathbb{N}$. Evidently, with this choice, we have

$$(\Delta_1 - \rho) w_n = \Delta_1 \sqrt{n} - \frac{\Delta_1 \sqrt{n}}{\sqrt{n}} \sqrt{n} = 0$$

for all $n \in \mathbb{N}$, thus Proposition 2.4 implies that

$$h_1(u) = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m\sim n} (u_n - u_m)^2 = \sum_{n=1}^{\infty} (u_n - u_{n-1})^2 \geq \|u\|_\rho^2 = \sum_{n=1}^{\infty} \rho_n^{\mathrm{KPP}} u_n^2$$

for all $u \in C_0(\mathbb{N})$ with $u_0 = 0$. Moreover, the weight can be expanded as

$$\rho_n^{\mathrm{KPP}} = \sum_{k=1}^{\infty} \binom{4k}{2k} \frac{1}{(4k-1)2^{4k-1}} \frac{1}{n^{2k}} > \frac{1}{4n^2} \tag{2.6}$$

for all $n \in \mathbb{N}$ using the generalized binomial theorem.

2.2 Discrete Hardy equality

Looking more carefully at the proof of Proposition 2.4, one may notice that not only we can assert the inequality, but we can even deduce the remainder in the optimal discrete Hardy inequality (2.1). Quick observation provides the equality

$$h_1(u) - \|u\|_\rho^2 = h_w(u/w).$$

With the explicit choice (2.5), we find that

$$\sum_{n=1}^{\infty} (u_n - u_{n-1})^2 - \sum_{n=1}^{\infty} \rho_n^{\mathrm{KPP}} u_n^2 = \sum_{n=2}^{\infty} \left(\sqrt[4]{\frac{n-1}{n}}\, u_n - \sqrt[4]{\frac{n}{n-1}}\, u_{n-1} \right)^2$$

for all real sequences $u \in C_0(\mathbb{N})$.

It turns out, that having the additional information about the remainder is crucial for proving criticality, resp. optimality, of a given Hardy weight. In the note [17] from Krejčiřík–Štampach, the authors found an elementary proof of the above equality for complex sequences. In addition, they proved the criticality of the Hardy weight ρ^{KPP}. Because analogous ideas will be used in the proof of Theorem 3.10 in Chapter 3, we will restate the theorem here, together with its proof for illustrative purposes.

To this end, it is beneficial to introduce our notation of discrete difference operators. From now on, we use the following definition of the *discrete Laplacian*,

$$(\Delta u)_n := \begin{cases} u_{n-1} - 2u_n + u_{n+1} & \text{if } n \geq 2, \\ -2u_1 + u_2 & \text{if } n = 1, \end{cases} \tag{2.7}$$

acting on the space of complex sequences $u \in C(\mathbb{N})$ indexed by $\mathbb{N}$. We will also utilize the *discrete gradient* and *divergence* defined as

$$(\nabla u)_n := \begin{cases} u_n - u_{n-1} & \text{if } n \geq 2, \\ u_1 & \text{if } n = 1, \end{cases} \quad \text{and} \quad (\mathrm{div} u)_n := u_{n+1} - u_n \text{ for } n \in \mathbb{N}. \tag{2.8}$$

Notice that on the Hilbert space $\ell^2(\mathbb{N})$, we have $\nabla^* = -\mathrm{div}$, thus $\Delta = \mathrm{div} \circ \nabla$ and $-\Delta = \nabla^* \circ \nabla$ (we omit $\circ$ when composing difference operators to simplify the notation below).

Theorem 2.5. *For any $u \in C_0(\mathbb{N})$, it holds that*

$$\sum_{n=1}^{\infty} |\nabla u_n|^2 = \sum_{n=1}^{\infty} \rho_n^{\mathrm{KPP}} |u_n|^2 + \sum_{n=2}^{\infty} \left| \sqrt[4]{\frac{n-1}{n}} u_n - \sqrt[4]{\frac{n}{n-1}} u_{n-1} \right|^2, \tag{2.9}$$

where ρ^{KPP} is defined by (2.2). Moreover the Hardy weight ρ^{KPP} is critical.

Proof. a) Set $\mathfrak{g}_n = \sqrt{n}$ and suppose $u \in C_0(\mathbb{N})$. Then for the summands on the right-hand side of (2.9) for any $n \geq 2$, we have

$$\begin{aligned} \left| \sqrt{\frac{\mathfrak{g}_{n-1}}{\mathfrak{g}_n}} u_n - \sqrt{\frac{\mathfrak{g}_n}{\mathfrak{g}_{n-1}}} u_{n-1} \right|^2 &= \frac{\mathfrak{g}_{n-1}}{\mathfrak{g}_n} |u_n|^2 + \frac{\mathfrak{g}_n}{\mathfrak{g}_{n-1}} |u_{n-1}|^2 - 2\,\mathrm{Re}(u_n u_{n-1}) \\ &= |\nabla u_n|^2 - \frac{\nabla \mathfrak{g}_n}{\mathfrak{g}_n} |u_n|^2 + \frac{\nabla \mathfrak{g}_n}{\mathfrak{g}_{n-1}} |u_{n-1}|^2 . \end{aligned}$$

Summing over n from 2 to ∞, we obtain

$$\begin{aligned} \sum_{n=2}^{\infty} \left| \sqrt{\frac{\mathfrak{g}_{n-1}}{\mathfrak{g}_n}} u_n - \sqrt{\frac{\mathfrak{g}_n}{\mathfrak{g}_{n-1}}} u_{n-1} \right|^2 &= \sum_{n=2}^{\infty} |\nabla u_n|^2 - \sum_{n=2}^{\infty} \frac{\nabla \mathfrak{g}_n}{\mathfrak{g}_n} |u_n|^2 + \sum_{n=1}^{\infty} \frac{\nabla \mathfrak{g}_{n+1}}{\mathfrak{g}_n} |u_n|^2 \\ &= \sum_{n=2}^{\infty} |\nabla u_n|^2 + \sum_{n=1}^{\infty} \frac{\mathrm{div} \nabla \mathfrak{g}_n}{\mathfrak{g}_n} |u_n|^2 + \frac{\nabla \mathfrak{g}_1}{\mathfrak{g}_1} |u_1|^2 . \end{aligned}$$

It follows from definitions (2.8), (2.7), and (2.2), that

$$\frac{\nabla \mathfrak{g}_1}{\mathfrak{g}_1} |u_1|^2 = |\nabla u_1|^2 \quad \text{and} \quad \frac{\mathrm{div} \nabla \mathfrak{g}_n}{\mathfrak{g}_n} = \frac{\Delta \sqrt{n}}{\sqrt{n}} = -\rho_n^{\mathrm{KPP}}$$

for all $n \in \mathbb{N}$, hence the above equality coincides with the identity (2.9).

b) Suppose that the sequence ρ, such that $\rho_n \geq \rho_n^{\mathrm{KPP}}$ for all $n \in \mathbb{N}$, satisfies the inequality (2.1). Using identity (2.9) together with the Hardy inequality for ρ, we find that

$$0 \leq \sum_{n=1}^{\infty} (\rho_n - \rho_n^{\mathrm{KPP}}) |u_n|^2 \leq \sum_{n=2}^{\infty} \left| \sqrt[4]{\frac{n-1}{n}} u_n - \sqrt[4]{\frac{n}{n-1}} u_{n-1} \right|^2 \tag{2.10}$$

for all $u \in C_0(\mathbb{N})$. Notice that if we substitute $u_n = \sqrt{n}$, the right-hand side of the above inequality vanishes. However, $\sqrt{n} \notin C_0(\mathbb{N})$, hence we must find a suitable regularization. For $N \geq 2$, set $u_n^N := \xi_n^N \sqrt{n}$, where

$$\xi_n^N := \begin{cases} 1 & \text{if } n \leq N, \\ \frac{2\ln N - \ln n}{\ln N} & \text{if } N < n \leq N^2, \\ 0 & \text{if } n > N^2. \end{cases}$$

Notice that $\xi^N \to 1$ point-wise as $N \to \infty$ and $\xi_n^N \leq \xi_n^{N+1}$ for all $n \in \mathbb{N}$ and $N \geq 2$. Moreover, we have

$$\begin{aligned} \sum_{n=2}^{\infty} \left| \sqrt[4]{\frac{n-1}{n}} u_n^N - \sqrt[4]{\frac{n}{n-1}} u_{n-1}^N \right|^2 &= \sum_{n=2}^{\infty} \sqrt{n(n-1)} \left| \xi_n^N - \xi_{n-1}^N \right|^2 \\ &= \frac{1}{\ln^2 N} \sum_{n=N+1}^{N^2} \sqrt{n(n-1)} \ln^2 \left(\frac{n}{n-1} \right) \\ &\leq \frac{1}{\ln^2 N} \sum_{n=N+1}^{N^2} \frac{\sqrt{n(n-1)}}{(n-1)^2} \leq \frac{2}{\ln^2 N} \int_N^{N^2} \frac{1}{x-1} \,\mathrm{d}x \leq \frac{4}{\ln N}. \end{aligned}$$

Since the last expression tends to 0 as $N \to \infty$, we deduce from the Monotone Convergence Theorem and (2.10) with u^N instead of u, that

$$\sum_{n=1}^{\infty} n(\rho_n - \rho_n^{\mathrm{KPP}}) = 0.$$

Bearing in mind that $\rho_n \geq \rho_n^{\mathrm{KPP}}$ for all $n \in \mathbb{N}$, we conclude that $\rho_n = \rho_n^{\mathrm{KPP}}$ for all $n \in \mathbb{N}$. □

2.3 Factorization of the discrete Laplacian

The proof of Theorem 2.5 relied heavily on the prior knowledge of the remainder term on the right-hand side of (2.9), hence this method might not be viable for deducing more general Hardy-like equalities or inequalities, i.e. the *Rellich* and *Birman inequalities*, see Chapter 3.

Bearing definitions (2.7) and (2.8) in mind, we observe that $-\Delta = \nabla^* \circ \nabla$ is a self-adjoint non-negative operator, which is determined by its quadratic form $\langle \cdot, -\Delta \cdot \rangle$. Therefore, the Hardy inequalities (1.11) and (1.12) can be interpreted in the sense of quadratic forms in $\ell^2(\mathbb{N})$ or $L^2(0,\infty)$ respectively as lower bounds

$$-\Delta \geq \rho \tag{2.11}$$

for the discrete and the continuous Dirichlet Laplacian on the half-line, where ρ stands for the operator of multiplication by either the discrete or the continuous Hardy weight.

In [15], B. Gerhat, D. Krejčiřík, and F. Štampach introduced the idea of factorizing the matrix

$$-\Delta - \rho = R^* R. \tag{2.12}$$

Since $-\Delta - \rho$ is a tridiagonal matrix, it is reasonable to assume that R is a bidiagonal matrix operator of the form

$$\begin{pmatrix} 2-\rho_1 & -1 & 0 & \cdots \\ -1 & 2-\rho_2 & -1 & \cdots \\ 0 & -1 & 2-\rho_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 & \cdots \\ -1/a_1 & a_2 & 0 & \cdots \\ 0 & -1/a_2 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_1 & -1/a_1 & 0 & \cdots \\ 0 & a_2 & -1/a_2 & \cdots \\ 0 & 0 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

i.e. $Ru_n = a_n u_n - u_{n+1}/a_n$ for all $n \in \mathbb{N}$, hence (2.12) takes the form

$$\sum_{n=1}^{\infty} |\nabla u_n|^2 = \sum_{n=1}^{\infty} \rho_n |u_n|^2 + \sum_{n=1}^{\infty} \left| a_n u_n - \frac{1}{a_n} u_{n+1} \right|^2,$$

for all $u \in C_0(\mathbb{N})$. Expanding the above equality with the weight ρ^{KPP} defined by (2.2), we arrive at the set of equations

$$a_1^2 = \sqrt{2} \quad \text{and} \quad a_n^2 + \frac{1}{a_{n-1}^2} = \sqrt{\frac{n+1}{n}} + \sqrt{\frac{n-1}{n}} \quad \text{for all } n \geq 2,$$

which has a unique explicit positive solution

$$a_n = \sqrt[4]{\frac{n+1}{n}} \qquad \Longrightarrow \qquad Ru_n = \sqrt[4]{\frac{n+1}{n}} u_n - \sqrt[4]{\frac{n}{n+1}} u_{n+1}$$

for all $n \in \mathbb{N}$. A shift of the index n in the remainder yields identity (2.9) once again.

In addition, the authors of [15] formulated and proved the optimality of the weight ρ^{KPP}, see Definition 3.2. The methods of this proof are analogous to the ones used in Chapter 3, hence the proof is omitted here.

Chapter 3

Optimal discrete Hardy inequalities of higher order

Lower bounds, analogous to (2.11), were found for higher integer powers of the continuous Dirichlet Laplacian, which we shall call the *continuous Hardy inequalities of higher order.* In 1954, F. Rellich discovered (see [2]) the Hardy inequality of order two (also called the *Rellich inequality*) which reads

$$\int_0^\infty \left|\varphi''(x)\right|^2 \mathrm{d}x \geq \frac{9}{16}\int_0^\infty \frac{|\varphi(x)|^2}{x^4}\,\mathrm{d}x,$$

where φ is from the Sobolev space $H^2(0,\infty)$ with $\varphi(0)=\varphi'(0)=0$. Later, in 1961, Birman generalized the Hardy and Rellich inequalities in [3], deriving the Hardy inequality of an arbitrary order $\ell \in \mathbb{N}$ (also called the *Birman inequality*)

$$\int_0^\infty \left|\varphi^{(\ell)}(x)\right|^2 \mathrm{d}x \geq \frac{((2\ell)!)^2}{16^\ell(\ell!)^2}\int_0^\infty \frac{|\varphi(x)|^2}{x^{2\ell}}\,\mathrm{d}x, \tag{3.1}$$

which holds true for any $\varphi \in H^\ell(0,\infty)$ with $\varphi(0)=\cdots=\varphi^{(\ell-1)}(0)=0$. Moreover, the constant on the right-hand side of (3.1) is known to be the best possible. Thereby arose a natural question whether a discrete analogue of this statement also holds true and whether it admits improvement similarly to the classical Hardy inequality, see Chapter 2. The *classical discrete Hardy inequality of order* ℓ reads

$$\sum_{n=\lceil \ell/2\rceil}^{\infty} \left|(-\Delta)^{\ell/2}u_n\right|^2 \geq \frac{((2\ell)!)^2}{16^\ell(\ell!)^2}\sum_{n=\ell}^{\infty}\frac{|u_n|^2}{n^{2\ell}} \tag{3.2}$$

for any $u \in \ell^2(\mathbb{N})$ satisfying $u_1=\cdots=u_{\ell-1}=0$, where $\lceil x\rceil$ denotes the lowest integer greater or equal to $x \in \mathbb{R}$ and the half-integer powers of discrete Laplacian (2.7) are defined as

$$(-\Delta)^{\ell/2} := \begin{cases} (-\Delta)^m & \text{if } \ell = 2m \in 2\mathbb{N}, \\ \nabla \circ (-\Delta)^m & \text{if } \ell = 2m+1 \in 2\mathbb{N}_0+1. \end{cases} \tag{3.3}$$

The first steps toward proving (3.2) were made in [15], where the authors studied the case of $\ell = 2$, i.e. the *discrete Rellich inequality*. Not only did they prove the inequality, but via a matrix factorization method, similar to the one presented in Section 2.3, they were also able to show that the inequality can be improved to

$$\sum_{n=2}^{\infty} |(-\Delta)u_n|^2 \geq \sum_{n=2}^{\infty} \rho_n^{\mathrm{GKS}}\,|u_n|^2$$

for all $u \in \ell^2(\mathbb{N})$ with $u_1 = 0$, where

$$\rho_n^{\mathrm{GKS}} := \frac{(-\Delta)^2 n^{3/2}}{n^{3/2}} = 6 - 4\left(1-\frac{1}{n}\right)^{3/2} - 4\left(1+\frac{1}{n}\right)^{3/2} + \left(1-\frac{2}{n}\right)^{3/2} + \left(1+\frac{2}{n}\right)^{3/2} > \frac{9}{16n^4} \quad (3.4)$$

for all $n \geq 2$. However, the *optimality* of ρ^{GKS} was not asserted. The authors of [15] also conjectured that the discrete Hardy inequalities of higher order (3.2) can be improved to

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left|(-\Delta)^{\ell/2} u_n\right|^2 \geq \sum_{n=\ell}^{\infty} \frac{(-\Delta)^{\ell} n^{\ell-1/2}}{n^{\ell-1/2}} |u_n|^2 \quad (3.5)$$

for all $\ell \in \mathbb{N}$. Furthermore, the authors demonstrated that the weights in (3.5) improve upon the classical Birman weights on the right-hand side of (3.2).

Later, in [13] X. Huang and D. Ye proved the inequality (3.2) using a weighted analogue of the identity (2.9) and demonstrated that the assumption $u_1 = \cdots = u_{\ell-1} = 0$ is fundamentally necessary for the inequality (3.2) to hold, answering another question from [15], see also [18] in this regard. Moreover, they found yet another discrete Rellich weight

$$\rho_n^{\mathrm{HY}} = \frac{9}{16n^4} + \frac{15}{16n^5} + \frac{213}{128n^6} + \mathcal{O}\left(\frac{1}{n^7}\right), \quad n \to \infty, \quad (3.6)$$

which is asymptotically bigger than the previous best-known weight

$$\rho_n^{\mathrm{GKS}} = \frac{9}{16n^4} + \frac{105}{128n^6} + \mathcal{O}\left(\frac{1}{n^7}\right), \quad n \to \infty,$$

but commented that it can still be improved. A question was raised whether the coefficient 15/16 by the second term on the right-hand side of (3.6) is sharp.

The main result of this work is the discovery of optimal weights for the discrete Hardy inequality of order $\ell \in \mathbb{N}$, establishing an improved optimal version of (3.2). Moreover, we will answer all the questions mentioned above by analyzing properties of concrete discrete Hardy weights of higher order in greater detail.

The chapter is organized as follows. In Section 3.1, we formulate our main results by means of five theorems, all of which shall be proven in Section 3.2. We complement our main results with several remarks on more general families of discrete Hardy weights of higher order addressing their non-uniqueness and optimality in Section 3.3.

3.1 Main Results

In this section, we explain how to construct optimal Hardy weights of any order $\ell \in \mathbb{N}$. Theorems 3.6, 3.8, and 3.10 provide sufficient conditions on a *parameter sequence* $\mathfrak{g}$ to give rise to an optimal discrete Hardy weight of any order. With an explicit choice of $\mathfrak{g} = \mathfrak{g}^{(\ell)}$, depending on the order $\ell \in \mathbb{N}$ of the given inequality, we obtain a concrete Hardy weight of order ℓ in Theorem 3.11. Several properties of these weights will be described in Theorem 3.14.

It turns out to be advantageous, to consider complex sequences indexed by $\mathbb{Z}$, with zero entries up to a certain index, rather than by $\mathbb{N}$. For this reason, we introduce the following subspaces of the space of complex sequences $C(\mathbb{Z})$. Namely, we denote

$$H^{\ell} := \{u \in C(\mathbb{Z}) \mid u_n = 0 \text{ for all } n < \ell\}$$

and

$$\mathcal{H}^{\ell} := H^{\ell} \cap \ell^2(\mathbb{Z})$$

endowed with the Euclidean inner product. Accordingly to our previous notation,

$$\mathcal{H}_0^\ell := \mathcal{H}^\ell \cap C_0(\mathbb{Z})$$

denotes finitely supported sequences in $\mathcal{H}^\ell$. With a slight abuse of notation, we further define the *discrete gradient* and *divergence* acting on complex sequences in $C(\mathbb{Z})$ as

$$(\nabla u)_n := u_n - u_{n-1} \quad \text{and} \quad (\mathrm{div} u)_n := u_{n+1} - u_n \tag{3.7}$$

for all $n \in \mathbb{Z}$. Naturally, we define the *discrete Laplacian* acting on $C(\mathbb{Z})$ as $\Delta := \mathrm{div} \circ \nabla$, i.e.

$$(\Delta u)_n := u_{n-1} - 2u_n + u_{n+1} \tag{3.8}$$

for all $n \in \mathbb{N}$. The half-integer powers of the Laplacian are defined similarly as in (3.3). The main reason for this notation is that ∇, div, and Δ commute on $C(\mathbb{Z})$.

Remark 3.1. It is important to point out the differences between positive powers of the discrete Laplacian $L := -\Delta|_{\ell^2(\mathbb{N})}$ on the half-line, which coincides with the definition (2.7) and which we studied in Sections 2.2 and 2.3, and between positive powers of the discrete Laplacian $-\Delta$ acting on $C(\mathbb{Z})$ defined by (3.8). In a recent study [18], the authors analyzed positive powers L^α for $\alpha > 0$, defined by the standard functional calculus using spectral resolution of L. While L coincides with $-\Delta|_{\mathcal{H}^1}$ after an obvious identification of spaces $\ell^2(\mathbb{N}) \equiv \mathcal{H}^1$, their integer powers differ by a finite rank operator. This can be readily seen from their matrix representations. For example, for $\ell = 3$, operators $(-\Delta)^3$ (restricted to $\mathcal{H}^1$) and L^3 are determined by the semi-infinite matrices

$$(-\Delta)^3 = \begin{pmatrix} 20 & -15 & 6 & -1 & & & & \\ -15 & 20 & -15 & 6 & -1 & & & \\ 6 & -15 & 20 & -15 & 6 & -1 & & \\ -1 & 6 & -15 & 20 & -15 & 6 & -1 & \\ & -1 & 6 & -15 & 20 & -15 & 6 & -1 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

and

$$L^3 = \begin{pmatrix} 14 & -14 & 6 & -1 & & & & \\ -14 & 20 & -15 & 6 & -1 & & & \\ 6 & -15 & 20 & -15 & 6 & -1 & & \\ -1 & 6 & -15 & 20 & -15 & 6 & -1 & \\ & -1 & 6 & -15 & 20 & -15 & 6 & -1 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Notice that the matrix of L^3 is not Toeplitz and differs from the matrix of $(-\Delta)^3$ by the upper-left 2×2 matrix. In general, matrices of L^ℓ and $(-\Delta)^\ell$ differ by an upper-left $(\ell - 1) \times (\ell - 1)$ matrix, hence $L^\ell|_{\mathcal{H}^\ell(\mathbb{N})} = (-\Delta)^\ell|_{\mathcal{H}^\ell}$, where we denoted and naturally identified the space

$$\mathcal{H}^\ell(\mathbb{N}) := \{u \in \ell^2(\mathbb{N}) \mid u_1 = \cdots = u_{\ell-1} = 0\} \equiv \mathcal{H}^\ell.$$

In fact, the matrix of $(-\Delta)^\ell$ is a sub-matrix of L^ℓ after removing the first $\ell - 1$ rows and columns. Both $(-\Delta)^\ell$ and L^ℓ determine non-negative operators on $\ell^2(\mathbb{N})$. However, it is proven in [18] that L^α is critical if and only if $\alpha \geq 3/2$ meaning that, if $L^\alpha \geq \rho \geq 0$ with $\alpha \geq 3/2$, then ρ must be trivial. Hence non-trivial Hardy-like inequalities exist for T^α only if $\alpha \in (0, 3/2)$, and some non-trivial weights (although not optimal) were also found in [18]. Clearly, this contrasts with the operator $(-\Delta)^\ell$ considered here since $(-\Delta)^\ell$ is sub-critical on $\mathcal{H}^\ell$ for every $\ell \in \mathbb{N}$.

Let us recall the definition of *optimality* adopted from [14], [10], and also [15], formulated for higher integer powers of the Laplacian acting on the space H^ℓ.

Definition 3.2. Let $\ell \in \mathbb{N}$. A positive sequence $\{\rho_n\}_{n=\ell}^\infty$ is called a *discrete Hardy weight of order* ℓ if and only if the *discrete Hardy inequality of order* ℓ

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left|(-\Delta)^{\ell/2} u_n\right|^2 \geq \sum_{n=\ell}^{\infty} \rho_n \left|u_n\right|^2 \tag{3.9}$$

holds for any $u \in \mathcal{H}_0^\ell$. In addition, the weight ρ is said to be *optimal* if it exhibits the following three properties:

(i) *Criticality*: The weight ρ is called *critical* if for any Hardy weight $\tilde{\rho}$ of order ℓ such that $\tilde{\rho}_n \geq \rho_n$ for all $n \geq \ell$ it follows that $\tilde{\rho} = \rho$.

(ii) *Non-attainability*: The weight ρ is called *non-attainable* if whenever the equality in (3.9) is attained for $u \in H^\ell$ such that the right-hand side is finite, i.e. $\sqrt{\rho} u \in \mathcal{H}^\ell$, necessarily $u \equiv 0$.

(iii) *Optimality near infinity*: The weight ρ is called *optimal near infinity* if for any $M \geq \ell$ and $\varepsilon > 0$ there exists $u \in \mathcal{H}_0^M$ such that

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left|(-\Delta)^{\ell/2} u_n\right|^2 < (1+\varepsilon) \sum_{n=\ell}^{\infty} \rho_n \left|u_n\right|^2. \tag{3.10}$$

Remark 3.3. We complement the definition with several explanatory remarks.

(i) Criticality means that the Hardy weight ρ of order ℓ cannot be further improved by a point-wise greater weight. However, in [16] it was shown that there exist infinitely many critical Hardy weights of order 1. Analogous property will be observed in Section 3.3 for Hardy weights of an arbitrary order ℓ.

(ii) Non-attainability of a Hardy weight ρ of order ℓ means that in the weighted ℓ^2-space of complex sequences such that $\sqrt{\rho} u \in \mathcal{H}^\ell$, the inequality (3.9) is strict unless $u \equiv 0$.

(iii) Optimality of ρ near infinity means that the constant 1 on the right-hand side of (3.9) cannot be further improved, even if the space $\mathcal{H}_0^\ell$ is replaced by $\mathcal{H}_0^M$ for an arbitrary $M \geq \ell$. Equivalently, it can be formulated as

$$\inf_{u \in H_0^M(\mathbb{N}) \setminus \{0\}} \frac{\sum\limits_{n=\lceil \ell/2 \rceil}^{\infty} \left|(-\Delta)^{\ell/2} u_n\right|^2}{\sum\limits_{n=\ell}^{\infty} \rho_n \left|u_n\right|^2} = 1. \tag{3.11}$$

Remark 3.4. The inequality (3.9) can be straightforwardly extended to all $u \in \mathcal{H}^\ell$ (or even to all $u \in H^\ell$) and therefore to all $u \in \mathcal{H}^\ell(\mathbb{N})$ with $(-\Delta)^{\ell/2}$ also acting on the respective spaces. Since

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left|(-\Delta)^{\ell/2} u_n\right|^2 = \sum_{n=\ell}^{\infty} \overline{u_n} (-\Delta)^\ell u_n = \left\langle u, (-\Delta)^\ell u \right\rangle,$$

the inequality (3.9) can be equivalently written as a lower bound for the integer power of the discrete Laplacian

$$(-\Delta)^\ell \geq \rho$$

on the space $\mathcal{H}^\ell$, where we again identify ρ with the corresponding multiplication operator.

In [16], a generalization of identity (2.9) was derived for generating Hardy weights of order 1. Our starting point shall be a generalized weighted analogue of this identity, which first appeared in [13]. For the reader's convenience, we will restate the equality here allowing complex sequences together with its proof, which is analogous to the proof of Theorem 2.5 but will be listed in Section 3.2.1 nonetheless.

Theorem 3.5. *Let $V \in C(\mathbb{Z})$ and $\mathfrak{g} \in H^1$ such that $\mathfrak{g}_n > 0$ for all $n \geq 1$. Then for any $u \in \mathcal{H}_0^1$, we have the identity*

$$\sum_{n=1}^{\infty} V_n \left|\nabla u_n\right|^2 + \sum_{n=1}^{\infty} \frac{\operatorname{div}(V\nabla\mathfrak{g})_n}{\mathfrak{g}_n} \left|u_n\right|^2 = \sum_{n=1}^{\infty} V_{n+1} \left|\sqrt{\frac{\mathfrak{g}_n}{\mathfrak{g}_{n+1}}} u_{n+1} - \sqrt{\frac{\mathfrak{g}_{n+1}}{\mathfrak{g}_n}} u_n\right|^2.$$

Via iteration of this identity, we will obtain our first main result, which is an analogous identity for the quadratic form of $(-\Delta)^\ell$ with $\ell \in \mathbb{N}$. The key idea for deriving such identities lies in a suitable choice of the parametric sequences $\mathfrak{g}$ in the individual steps of the iterative process, which generate convenient weights V in terms of $\mathfrak{g}$. For this reason, it is necessary to impose some positivity assumptions on the parameter sequence $\mathfrak{g}$. In fact, in Theorems 3.6, 3.8, and 3.10 the assumptions on $\mathfrak{g}$ will be gradually strengthened, because of which they will be numbered separately.

Theorem 3.6. *Let $\ell \in \mathbb{N}$. Suppose that*

$$\mathfrak{g} \in H^\ell \text{ such that } \operatorname{div}^k \mathfrak{g}_n > 0 \text{ for all } n \geq \ell - k \text{ and } k \in \{0, \ldots, \ell - 1\}. \tag{A1}$$

Then for any $u \in \mathcal{H}_0^\ell$, we have the identity

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left|(-\Delta)^{\ell/2} u_n\right|^2 = \sum_{n=\ell}^{\infty} \frac{(-\Delta)^\ell \mathfrak{g}_n}{\mathfrak{g}_n} \left|u_n\right|^2 + \sum_{k=0}^{\ell-1} \mathcal{R}_k^{(\ell)}(\mathfrak{g}; u), \tag{3.12}$$

where

$$\mathcal{R}_k^{(\ell)}(\mathfrak{g}; u) := \sum_{n=\ell-k}^{\infty} \frac{(-\Delta)^{\ell-1-k} \operatorname{div}^{k+1} \mathfrak{g}_n}{\operatorname{div}^{k+1} \mathfrak{g}_n} \left|\sqrt{\frac{\operatorname{div}^k \mathfrak{g}_n}{\operatorname{div}^k \mathfrak{g}_{n+1}}} \operatorname{div}^k u_{n+1} - \sqrt{\frac{\operatorname{div}^k \mathfrak{g}_{n+1}}{\operatorname{div}^k \mathfrak{g}_n}} \operatorname{div}^k u_n\right|^2. \tag{3.13}$$

(For $k = \ell - 1$, the coefficient in front of the absolute value in (3.13) is to be understood as 1.)

Remark 3.7. In fact, Theorem 3.6 can be seen as a corollary of a weighted analogue of the identity (3.12), which may be of independent interest (we refer the reader to [13] and [19], where the authors studied weighted Hardy and Rellich inequalities). Namely, if $\ell \in \mathbb{N}$ and $\mathfrak{g}$ satisfies the assumptions (A1), then for any weight $V \in C(\mathbb{Z})$, we have the identity

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} V_{n-\lceil \ell/2 \rceil+1} \left|(-\Delta)^{\ell/2} u_n\right|^2 = (-1)^\ell \sum_{n=\ell}^{\infty} \frac{\operatorname{div}\nabla^{\ell-1}(V(\operatorname{div}^{\ell-1}\nabla\mathfrak{g}))_n}{\mathfrak{g}_n} \left|u_n\right|^2 + \sum_{k=0}^{\ell-1} \mathcal{R}_k^{(\ell)}(V; \mathfrak{g}; u), \tag{3.14}$$

where $\mathcal{R}_k^{(\ell)}(V; \mathfrak{g}; u)$ is defined as

$$(-1)^{\ell+k+1} \sum_{n=\ell-k}^{\infty} \frac{\operatorname{div}\nabla^{\ell-2-k}(V(\operatorname{div}^{\ell-2-k}\nabla\operatorname{div}^{k+1}\mathfrak{g}))}{\operatorname{div}^{k+1}\mathfrak{g}_n} \left|\sqrt{\frac{\operatorname{div}^k \mathfrak{g}_n}{\operatorname{div}^k \mathfrak{g}_{n+1}}} \operatorname{div}^k u_{n+1} - \sqrt{\frac{\operatorname{div}^k \mathfrak{g}_{n+1}}{\operatorname{div}^k \mathfrak{g}_n}} \operatorname{div}^k u_n\right|^2; \tag{3.15}$$

for $k = \ell - 1$, the coefficient in front of the absolute value in (3.15) is to be understood as V_{n+1}. Notice that for $V \equiv 1$ this statement becomes Theorem 3.6. The differences between the first term on the right-hand side of (3.12) and (3.14) and the coefficients in front of the absolute value in (3.13) and (3.15) is due to the fact, that the multiplication operator V does not commute with ∇ nor div. Similarly, the reason for shifting the indices of V on the left-hand side of (3.14) emerged from the non-commutativity of V and the *shift operator* (3.30), in order to obtain more concise formulas. The proof of this generalization is analogous to the proof of Theorem 3.6 and is therefore omitted.

By imposing additional assumptions on the parameter sequence $\mathfrak{g}$, we may ensure non-negativity of the remainders on the right-hand side of (3.12) obtaining an abstract discrete Hardy inequality of order ℓ.

Theorem 3.8. *Let $\ell \in \mathbb{N}$. Suppose (A1) and, in addition,*

$$(-\Delta)^{\ell-k}\mathrm{div}^k \mathfrak{g}_n \geq 0 \text{ for all } n \geq \ell+1-k \text{ and } k \in \{1, \dots, \ell-1\}. \tag{A2}$$

Then for all $u \in \mathcal{H}_0^\ell$, we have the inequality

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left|(-\Delta)^{\ell/2} u_n\right|^2 \geq \sum_{n=\ell}^{\infty} \rho_n(\mathfrak{g}) \left|u_n\right|^2, \tag{3.16}$$

where $\rho(\mathfrak{g}) := (-\Delta)^\ell \mathfrak{g}/\mathfrak{g}$. If moreover,

$$(-\Delta)^\ell \mathfrak{g}_n > 0 \text{ for all } n \geq \ell, \tag{A3}$$

then $\rho(\mathfrak{g}) > 0$, i.e. $\rho(\mathfrak{g})$ is a discrete Hardy weight of order ℓ.

Remark 3.9. If the parameter sequence $\mathfrak{g}$ satisfies the assumptions (A1) and (A2), the remainder terms on the right-hand side of (3.12) can be interpreted as a norm

$$\mathcal{R}_k^{(\ell)}(\mathfrak{g}; u) = \left\| R_k^{(\ell)}(\mathfrak{g}) u \right\|^2,$$

where

$$R_k^{(\ell)}(\mathfrak{g}) u_n := \sqrt{\frac{(-\Delta)^{\ell-1-k}\mathrm{div}^{k+1}\mathfrak{g}_n}{\mathrm{div}^{k+1}\mathfrak{g}_n}} \left(\sqrt{\frac{\mathrm{div}^k \mathfrak{g}_n}{\mathrm{div}^k \mathfrak{g}_{n+1}}} \mathrm{div}^k u_{n+1} - \sqrt{\frac{\mathrm{div}^k \mathfrak{g}_{n+1}}{\mathrm{div}^k \mathfrak{g}_n}} \mathrm{div}^k u_n \right) \tag{3.17}$$

for all $k \in \{0, \dots, \ell-1\}$ and $n \geq \ell - k$ (for $k = \ell - 1$ the coefficient in front of the parentheses is to be understood as 1). Since $(-\Delta)^\ell$ is a bounded operator, the identity (3.12) can be extended to all $u \in \mathcal{H}^\ell$ (in fact, it can be extended even further, see Proposition 3.19). Recalling Remark 3.4, equality (3.12) yields an algebraic identity on the level of semi-infinite matrices. Namely, restricting indices of the respective matrices to $\ell, \ell+1, \dots,$ we obtain the identity

$$(-\Delta)^\ell - \rho(\mathfrak{g}) = \sum_{k=0}^{\ell-1} \left(\tilde{R}_k^{(\ell)}(\mathfrak{g})\right)^* \tilde{R}_k^{(\ell)}(\mathfrak{g}) \tag{3.18}$$

on the space $\mathcal{H}^\ell$, where $\rho(\mathfrak{g}) = (-\Delta)^\ell \mathfrak{g}/\mathfrak{g}$ is identified with the corresponding multiplication operator and $\tilde{R}_k^{(\ell)}(\mathfrak{g}) := \mathrm{S}^{-k} R_k^{(\ell)}(\mathfrak{g})$ and S^{-k} acts as the *backward shift* of the index by k, see (3.30) below. The shift S^{-k} is present as a consequence of the range of the summation index n in (3.13), which is starting from $\ell - k$.

For $\ell = 1$, such factorization has been used to provide an alternative proof of the optimal discrete Hardy inequality in [15]. For $\ell = 2$, the authors of [15] factorized the matrix $(-\Delta)^2 - \rho(\mathfrak{g})$, with $\mathfrak{g}_n = n^{3/2}$, into a single remainder matrix of a form R^*R, where R is a tridiagonal matrix. As the remainder were sought in terms of a single matrix rather than two matrices, its entries could not be found explicitly. The idea was to decompose the pentadiagonal matrix $(-\Delta)^2 - \rho(\mathfrak{g})$ into a product of a tridiagonal matrix and its adjoint reducing the order of the corresponding difference operators. The idea behind the factorization (3.18) is similar, however, its main novelty is that the order is reduced successively giving rise to more remainder terms on the right expressed explicitly in terms of the parameter sequence $\mathfrak{g}$. In addition, the diagonal term $\rho(\mathfrak{g})$, i.e. the actual Hardy weight of order ℓ, is also identified in terms of $\mathfrak{g}$. It turns out, that such a concrete description is crucial for the proof of optimality of the weight $\rho(\mathfrak{g})$ in Theorem 3.10.

The matrix identity (3.18) can be viewed as a factorization of particular banded Toeplitz matrices. Indeed, the non-vanishing matrix elements of $(-\Delta)^\ell$ are

$$(-\Delta)^\ell_{m,n} = \left\langle \delta_m, (-\Delta)^\ell \delta_n \right\rangle = (-1)^{n-m} \binom{2\ell}{\ell + n - m}$$

for $m, n \geq \ell$ with $|n - m| \leq \ell$, hence the matrix representation of $(-\Delta)^\ell$ with respect to the standard basis of $\ell^2(\mathbb{Z})$, here denoted as $\{\delta_n \mid n \geq \ell\}$, is a semi-infinite Hermitian banded Toeplitz matrix with diagonals given by the binomial coefficients. On the other hand, by inspection of matrix entries of remainder matrices (3.17), we observe that $\tilde{R}_k^{(\ell)}(\mathfrak{g})$ are semi-infinite $(k+2)$-diagonal lower Hessenberg matrices, i.e. the (m,n)-th entry of $\tilde{R}_k^{(\ell)}(\mathfrak{g})$ vanishes if $n - m > 1$ or $m - n > k$.

Under additional requirements on the asymptotic behaviour of $\mathfrak{g}_n$, as $n \to \infty$, and strict positivity in assumption (A2) for $k = 1$, we can assert optimality of the discrete Hardy weight $\rho(\mathfrak{g})$.

Theorem 3.10. *Let $\ell \in \mathbb{N}$. Suppose (A1), (A2), (A3), and, in addition, suppose that $\mathfrak{g}$ admits the asymptotic expansion*

$$\mathfrak{g}_n = \sum_{j=0}^{2\ell} \alpha_j n^{\ell - 1/2 - j} + \mathcal{O}\left(n^{-\ell - 3/2}\right) \text{ for some } \alpha_j \in \mathbb{R} \text{ with } \alpha_0 \neq 0, \tag{A4}$$

as $n \to \infty$. Then the discrete Hardy weight $\rho(\mathfrak{g}) = (-\Delta)^\ell \mathfrak{g}/\mathfrak{g}$ of order ℓ is critical and optimal near infinity. If moreover,

$$\ell = 1 \text{ or } (-\Delta)^{\ell-1} \operatorname{div} \mathfrak{g}_n > 0 \text{ for all } n \geq \ell \geq 2, \tag{A5}$$

then $\rho(\mathfrak{g})$ is also non-attainable and therefore optimal.

It remains to find a concrete parameter sequence $\mathfrak{g}$, which satisfies all the assumptions of the above theorems. To this end, for given $\ell \in \mathbb{N}$, we define

$$\mathfrak{g}_n^{(\ell)} := \sqrt{n} \prod_{j=1}^{\ell-1} (n - j) \tag{3.19}$$

for all $n \in \mathbb{N}_0$ and $\mathfrak{g}_n := 0$ for all $n < 0$. Additional (but more complicated) examples will be discussed in Section 3.3.

Theorem 3.11. *Let $\ell \in \mathbb{N}$. The sequence $\rho^{(\ell)}$, given by*

$$\rho_n^{(\ell)} := \frac{(-\Delta)^\ell \mathfrak{g}_n^{(\ell)}}{\mathfrak{g}_n^{(\ell)}} \tag{3.20}$$

for all $n \geq \ell$, where $\mathfrak{g}^{(\ell)}$ is given by (3.19), is an optimal discrete Hardy weight of order ℓ.

Our final theorem summarizes properties of the optimal weight $\rho^{(\ell)}$ in greater detail. A remarkable property is that $\rho_n^{(\ell)}$ has a convergent series representation in negative powers of n with all coefficients being non-negative (in fact, for $\ell \geq 2$ the coefficients are positive) for all $n \geq \ell$. Consequently, using more terms of the truncated series representation always produces a tighter inequality.

In order to formulate this theorem, we introduce several combinatorial numbers. First, we will make use of the *binomial coefficient* and the *Pochhammer symbol* defined by standard formulas

$$\binom{\nu}{n} := \frac{\nu(\nu-1)\dots(\nu-n+1)}{n!} \quad \text{and} \quad (\nu)_n := \nu(\nu+1)\dots(\nu+n-1)$$

for all $\nu \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Next, we denote the *Stirling numbers of the first kind* as

$$s(n,k) := (-1)^{n+k} \sum_{1 \leq i_1 < \dots < i_{n-k} < n} i_1 i_2 \dots i_{n-k} \tag{3.21}$$

and the *Stirling numbers of the second kind* as

$$S(n,k) := \sum_{\substack{j_1,\dots,j_k \geq 0 \\ j_1+\dots+j_k = n-k}} 1^{j_1} 2^{j_2} \dots k^{j_k} \tag{3.22}$$

for all $n \in \mathbb{N}$ and $k \in \{0, \dots, n-1\}$, see [20, § 26.8]. By convention, we also set $s(n,n) = S(n,n) := 1$ for all $n \in \mathbb{N}_0$ and $s(n,k) := 0$ whenever $k < 0$. Moreover, we employ the numbers

$$X_m^{(\ell)} := \sum_{j=-\ell}^{\ell} \binom{2\ell}{\ell+j} (-1)^j j^m \tag{3.23}$$

for all $m, \ell \in \mathbb{N}$ with $X_0^{(\ell)} := 0$ for all $\ell \in \mathbb{N}$.

Remark 3.12. For $m \in \mathbb{N}$ odd and $\ell \in \mathbb{N}$, we have

$$X_m^{(\ell)} = \sum_{j=-\ell}^{\ell} \binom{2\ell}{\ell+j} (-1)^j j^m = \sum_{j=-\ell}^{\ell} \binom{2\ell}{\ell-j} (-1)^{-j} (-j)^m = -\sum_{j=-\ell}^{\ell} \binom{2\ell}{\ell+j} (-1)^j j^m = -X_m^{(\ell)},$$

hence $X_m^{(\ell)} = 0$. Furthermore, in [15] the authors found that

$$X_m^{(\ell)} = 0 \text{ for all } m < 2\ell \tag{3.24}$$

and that for the remaining values, we have

$$X_{2\ell}^{(\ell)} = (-1)^\ell (2\ell)! \quad \text{and} \quad X_{2\ell+2m}^{(\ell)} = (-1)^\ell (2\ell)! \sum_{1 \leq k_1 \leq \dots \leq k_m \leq \ell} (k_1 k_2 \dots k_m)^2 \tag{3.25}$$

for all $m \in \mathbb{N}$. The expressions in (3.25) reveal the nontrivial fact that $(-1)^\ell X_{2\ell+2m}^{(\ell)}$ is a positive integer for all $m \in \mathbb{N}_0$.

Lastly, we define the numbers

$$r_m^{(\ell)} := \sum_{j=2\ell}^{m} \binom{\ell+j-m-1/2}{j} s(\ell, \ell+j-m) X_j^{(\ell)} \tag{3.26}$$

for all $\ell \in \mathbb{N}$ and $m \geq 2\ell$.

Remark 3.13. Since $s(\ell,j)=0$ whenever $j\leq 0$ and $X_j^{(\ell)}=0$ whenever j is odd, we may restrict the range of the summation index j in formula (3.26) even further. It turns out, that after this restriction, each summand is positive and the formula reads

$$r_m^{(\ell)}=\sum_{\substack{j=\max(2\ell,m-\ell+1)\\ j\equiv 0 \bmod 2}}^{m}\binom{\ell+j-m-1/2}{j}s(\ell,\ell+j-m)X_j^{(\ell)}.$$

Consequently, it holds true that for all $\ell\geq 2$ and $m\geq 2\ell$, we have $r_m^{(\ell)}>0$ and if $\ell=1$, then $r_{2m+1}^{(1)}=0$ and $r_{2m}^{(1)}>0$ for all $m\geq 1$, which will be shown in the proof of claim (ii) of Theorem 3.14 in Section 3.2.6.

Theorem 3.14. *Let $\ell\in\mathbb{N}$ and $\rho^{(\ell)}$ defined by the formulas (3.20) and (3.19), then the following properties hold true.*

(i) The weight sequence $\rho^{(\ell)}$ admits the convergent series expansion

$$\rho_n^{(\ell)}=\sum_{k=2\ell}^{\infty}\frac{A_k^{(\ell)}}{n^k} \tag{3.27}$$

for all $n\geq\ell$, where the coefficients are defined as

$$A_k^{(\ell)}:=\sum_{m=2\ell}^{k}S(k-m+\ell-1,\ell-1)r_m^{(\ell)}. \tag{3.28}$$

(ii) For all $\ell\geq 2$ and $k\geq 2\ell$, we have $A_k^{(\ell)}>0$.
For $\ell=1$ and $k\geq 1$, we have $A_{2k+1}^{(1)}=0$ and $A_{2k}^{(1)}>0$.

(iii) For all $n\geq\ell\geq 2$, it is true that

$$\rho_n^{(\ell)}>\frac{(-\Delta)^{\ell}n^{\ell-1/2}}{n^{\ell-1/2}}>\left(\frac{1}{2}\right)_{\ell}^{2}\frac{1}{n^{2\ell}}. \tag{3.29}$$

(For $\ell=1$, the first inequality in (3.29) holds as equality.)

Remark 3.15. We complement Theorem 3.14 with several remarks.

(i) For the first two coefficients in the expansion (3.27), we have

$$A_{2\ell}^{(\ell)}=\left(\frac{1}{2}\right)_{\ell}^{2}\quad\text{and}\quad A_{2\ell+1}^{(\ell)}=\frac{2\ell^2(\ell-1)}{2\ell-1}\left(\frac{1}{2}\right)_{\ell}^{2}$$

for all $\ell\in\mathbb{N}$, which can be computed directly from the formulas (3.28) and (3.26), realizing that $X_{2\ell+1}^{(\ell)}=0$ and $X_{2\ell}^{(\ell)}=(-1)^{\ell}(2\ell)!$.

(ii) The first inequality in (3.29) shows that the weight $\rho^{(\ell)}$ improves upon the weights suggested by Gerhat–Krejčiřík–Štampach, see (3.5), and proves the conjecture in [15]. Noticing that

$$\left(\frac{1}{2}\right)_{\ell}^{2}=\frac{((2\ell)!)^2}{16^{\ell}\,(\ell!)^2},$$

the second inequality in (3.29) shows that $\rho^{(\ell)}$ improves upon the classical discrete Hardy weights (3.2).

(iii) For $\ell = 1$, we rediscovered the optimal Hardy weight (2.2), whose expansion (2.6) is explicitly known, see [11]. The expansions in the first few terms for $\ell = 2, 3, 4, 5$, as $n \to \infty$, read

$$\begin{aligned}
\rho_n^{(2)} &= \frac{9}{16n^4} + \frac{3}{2n^5} + \frac{297}{128n^6} + \mathcal{O}\left(\frac{1}{n^7}\right), \\
\rho_n^{(3)} &= \frac{225}{64n^6} + \frac{405}{16n^7} + \frac{114975}{1024n^8} + \mathcal{O}\left(\frac{1}{n^9}\right), \\
\rho_n^{(4)} &= \frac{11025}{256n^8} + \frac{4725}{8n^9} + \frac{4879665}{1024n^{10}} + \mathcal{O}\left(\frac{1}{n^{11}}\right), \\
\rho_n^{(5)} &= \frac{893025}{1024n^{10}} + \frac{2480625}{128n^{11}} + \frac{4023077625}{16384n^{12}} + \mathcal{O}\left(\frac{1}{n^{13}}\right).
\end{aligned}$$

As was mentioned earlier, in [13], the authors ask whether the constant $15/16$ by the second term in (3.6) is sharp. The above expansion of $\rho_n^{(2)}$ shows that it is not the case.

3.2 Proofs

In the course of the proofs worked out below, we will frequently use (besides the operators ∇, div, and Δ) the *forward shift* operator S acting on $C(\mathbb{Z})$ defined as

$$\mathrm{S}u_n := u_{n+1} \tag{3.30}$$

for all $n \in \mathbb{Z}$. Obviously, for any $k \in \mathbb{Z}$ it holds that $\mathrm{S}^k u_n = u_{n+k}$ for all $n \in \mathbb{Z}$. Recalling definition (3.7), we see that $\mathrm{div} = \mathrm{S} - \mathrm{I}$ and $\nabla = \mathrm{I} - \mathrm{S}^{-1}$, where I stands for the identity operator. In particular, we have identities $\mathrm{div} = \mathrm{S} \circ \nabla = \nabla \circ \mathrm{S}$, that will be used several times below. Note also that subspaces H^ℓ are not preserved under the action of S and hence neither under the action of div (S is a bijection of H^ℓ onto $H^{\ell-1}$).

3.2.1 Proof of Theorem 3.5

Suppose $\mathfrak{g} \in H^1$, $\mathfrak{g}_n > 0$ for all $n \geq 1$, and $u \in \mathcal{H}_0^1$. For any $n \geq 2$, we have

$$\begin{aligned}
\left| \sqrt{\frac{\mathfrak{g}_{n-1}}{\mathfrak{g}_n}} u_n - \sqrt{\frac{\mathfrak{g}_n}{\mathfrak{g}_{n-1}}} u_{n-1} \right|^2 &= \frac{\mathfrak{g}_{n-1}}{\mathfrak{g}_n} |u_n|^2 + \frac{\mathfrak{g}_n}{\mathfrak{g}_{n-1}} |u_{n-1}|^2 - 2\,\mathrm{Re}(u_n u_{n-1}) \\
&= |\nabla u_n|^2 - \frac{\nabla \mathfrak{g}_n}{\mathfrak{g}_n} |u_n|^2 + \frac{\nabla \mathfrak{g}_n}{\mathfrak{g}_{n-1}} |u_{n-1}|^2 .
\end{aligned}$$

Multiplying both sides by V_n and summing over n from 2 to ∞, we obtain

$$\begin{aligned}
\sum_{n=2}^{\infty} V_n \left| \sqrt{\frac{\mathfrak{g}_{n-1}}{\mathfrak{g}_n}} u_n - \sqrt{\frac{\mathfrak{g}_n}{\mathfrak{g}_{n-1}}} u_{n-1} \right|^2 &= \sum_{n=2}^{\infty} V_n |\nabla u_n|^2 - \sum_{n=2}^{\infty} V_n \frac{\nabla \mathfrak{g}_n}{\mathfrak{g}_n} |u_n|^2 + \sum_{n=1}^{\infty} V_{n+1} \frac{\nabla \mathfrak{g}_{n+1}}{\mathfrak{g}_n} |u_n|^2 \\
&= \sum_{n=2}^{\infty} V_n |\nabla u_n|^2 + \sum_{n=1}^{\infty} \frac{\mathrm{div}(V \nabla \mathfrak{g})_n}{\mathfrak{g}_n} |u_n|^2 + V_1 \frac{\nabla \mathfrak{g}_1}{\mathfrak{g}_1} |u_1|^2
\end{aligned}$$

By assumptions, $u_0 = \mathfrak{g}_0 = 0$, thus

$$V_1 \frac{\nabla \mathfrak{g}_1}{\mathfrak{g}_1} |u_1|^2 = V_1 |\nabla u_1|^2 .$$

A shift of the summation index n by one on the left-hand side now yields the claim of Theorem 3.5. $\square$

3.2.2 Proof of Theorem 3.6

The proof proceeds by a two-step induction in $\ell \in \mathbb{N}$:

a) We verify Theorem 3.6 for $\ell = 1, 2$.

b) Assuming Theorem 3.6 to hold for all $\ell \leq 2m$, where $m \in \mathbb{N}$, we prove it for $\ell = 2m + 1$.

c) Assuming Theorem 3.6 to hold for all $\ell \leq 2m + 1$, we prove it also for $\ell = 2m + 2$.

The reason to treat even and odd indices ℓ separately stems from the fact that, when lowering a half-integer power of the discrete Laplacian, ∇ or div emerge depending on the parity of ℓ because we have

$$(-\Delta)^{\ell/2} = \begin{cases} \nabla(-\Delta)^{(\ell-1)/2} & \text{if } \ell \in 2\mathbb{N}_0 + 1, \\ -\mathrm{div}(-\Delta)^{(\ell-1)/2} & \text{if } \ell \in 2\mathbb{N}. \end{cases}$$

Since the resulting differences are subtle, parts b) and c) of the proof are analogical, and therefore we only briefly indicate the proof of c). Note also that the case $\ell = 2$ is shown only for clarity and illustrative purposes since it could be omitted after interchanging the order of the steps a) and b).

a) For $\ell = 1$, Theorem 3.6 coincides with the special case of Theorem 3.5 with $V \equiv 1$. Next, we suppose $\ell = 2$ and $\mathfrak{g}$ to satisfy the assumption (A1), i.e. $\mathfrak{g} \in H^2$, $\mathfrak{g}_n > 0$ for all $n \geq 2$, and $\mathrm{div}\mathfrak{g}_n > 0$ for all $n \geq 1$. Clearly, $\mathrm{div}\mathfrak{g} \in H^1$ hence we may apply Theorem 3.5 with $\mathfrak{g}$ replaced by $\mathrm{div}\mathfrak{g}$, u replaced by $\mathrm{div}u$, and $V \equiv 1$, from which it follows that

$$\sum_{n=1}^{\infty} |(-\Delta)u_n|^2 = \sum_{n=1}^{\infty} |\mathrm{div}\nabla u_n|^2 = \sum_{n=1}^{\infty} |\nabla(\mathrm{div}u)_n|^2 = \sum_{n=1}^{\infty} \frac{(-\Delta)\mathrm{div}\mathfrak{g}_n}{\mathrm{div}\mathfrak{g}_n} |\mathrm{div}u_n|^2 + \mathcal{R}_1^{(2)}(\mathfrak{g}; u)$$

for all $u \in \mathcal{H}_0^2$, where $\mathcal{R}_1^{(2)}(\mathfrak{g}; u)$ is defined by (3.13). Bearing in mind that $\mathrm{div}u = \nabla \mathrm{S}u$, we apply Theorem 3.5 once more, this time with $\mathrm{S}\mathfrak{g} \in H^1$, $\mathrm{S}u \in \mathcal{H}_0^1$, and $V_n := (-\Delta)\mathrm{div}\mathfrak{g}_n / \mathrm{div}\mathfrak{g}_n$ to the first term on the right-hand side, obtaining the identity

$$\sum_{n=1}^{\infty} |(-\Delta)u_n|^2 = -\sum_{n=1}^{\infty} \frac{\mathrm{div}(V\nabla \mathrm{S}\mathfrak{g})_n}{\mathrm{S}\mathfrak{g}_n} |\mathrm{S}u_n|^2 + \sum_{n=1}^{\infty} V_{n+1} \left| \sqrt{\frac{\mathrm{S}\mathfrak{g}_n}{\mathrm{S}\mathfrak{g}_{n+1}}} \mathrm{S}u_{n+1} - \sqrt{\frac{\mathrm{S}\mathfrak{g}_{n+1}}{\mathrm{S}\mathfrak{g}_n}} \mathrm{S}u_n \right|^2 + \mathcal{R}_1^{(2)}(\mathfrak{g}; u)$$

for all $u \in \mathcal{H}_0^2$. Taking also into account that

$$-\mathrm{div}(V\nabla \mathrm{S}\mathfrak{g}) = \mathrm{div}\left(\frac{\Delta \mathrm{div}\mathfrak{g}}{\mathrm{div}\mathfrak{g}} \mathrm{div}\mathfrak{g}\right) = \mathrm{S}(-\Delta)^2 \mathfrak{g},$$

we arrive at the desired result

$$\sum_{n=1}^{\infty} |(-\Delta)u_n|^2 = \sum_{n=2}^{\infty} \frac{(-\Delta)^2 \mathfrak{g}_n}{\mathfrak{g}_n} |u_n|^2 + \mathcal{R}_0^{(2)}(\mathfrak{g}; u) + \mathcal{R}_1^{(2)}(\mathfrak{g}; u)$$

for all $u \in \mathcal{H}^2$, with $\mathcal{R}_0^{(2)}(\mathfrak{g}; u)$ given again by the general definition (3.13).

b) Suppose $m \in \mathbb{N}$ and assume that Theorem 3.6 holds true for all $\ell \leq 2m$. Let us consider sequences $u \in \mathcal{H}_0^{2m+1}$ and $\mathfrak{g} \in H^{2m+1}$ satisfying the assumption (A1) for $\ell = 2m + 1$. We shall verify (3.12) for $\ell = 2m + 1$. We have

$$\sum_{n=\lceil \frac{2m+1}{2} \rceil}^{\infty} \left| (-\Delta)^{(2m+1)/2} u_n \right|^2 = \sum_{n=m+1}^{\infty} |\nabla(-\Delta)^m u_n|^2 = \sum_{n=m+1}^{\infty} |(-\Delta)^m \nabla u_n|^2 = \sum_{n=m}^{\infty} |(-\Delta)^m \mathrm{div} u_n| .$$

Since $\mathrm{div}u \in \mathcal{H}_0^{2m}$ and $\mathrm{div}\mathfrak{g}$ satisfies assumption (A1) for $\ell = 2m$, we may apply the induction hypothesis and obtain

$$\sum_{n=m}^{\infty} |(-\Delta)^m \mathrm{div}u_n|^2 = \sum_{n=2m}^{\infty} \frac{(-\Delta)^{2m}\mathrm{div}\mathfrak{g}_n}{\mathrm{div}\mathfrak{g}_n} |\mathrm{div}u_n|^2 + \sum_{k=0}^{2m-1} \mathcal{R}_k^{(2m)}(\mathrm{div}\mathfrak{g};\mathrm{div}u).$$

Recalling the general definition of remainders (3.13), we observe that

$$\mathcal{R}_k^{(2m)}(\mathrm{div}\mathfrak{g};\mathrm{div}u) = \mathcal{R}_{k+1}^{(2m+1)}(\mathfrak{g};u) \quad \Longrightarrow \quad \sum_{k=0}^{2m-1} \mathcal{R}_k^{(2m)}(\mathrm{div}\mathfrak{g};\mathrm{div}u) = \sum_{k=1}^{2m} \mathcal{R}_k^{(2m+1)}(\mathfrak{g};u).$$

Next, we shift the index in the first sum on the right-hand side finding that

$$\sum_{n=\lceil \frac{2m+1}{2} \rceil}^{\infty} \left|(-\Delta)^{(2m+1)/2} u_n\right|^2 = \sum_{n=1}^{\infty} \frac{\mathrm{S}^{2m-1}(-\Delta)^{2m}\mathrm{div}\mathfrak{g}_n}{\mathrm{S}^{2m-1}\mathrm{div}\mathfrak{g}_n} \left|\nabla \mathrm{S}^{2m} u_n\right|^2 + \sum_{k=1}^{2m} \mathcal{R}_k^{(2m+1)}(\mathfrak{g};u). \tag{3.31}$$

Furthermore, we apply Theorem 3.5 with $\mathrm{S}^{2m}u \in \mathcal{H}_0^1$, together with $\mathrm{S}^{2m}\mathfrak{g} \in H^1$, and the weight

$$V_n := \frac{\mathrm{S}^{2m-1}(-\Delta)^{2m}\mathrm{div}\mathfrak{g}_n}{\mathrm{S}^{2m-1}\mathrm{div}\mathfrak{g}_n},$$

to the first term on the right-hand side of (3.31) obtaining

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\mathrm{S}^{2m-1}(-\Delta)^{2m}\mathrm{div}\mathfrak{g}_n}{\mathrm{S}^{2m-1}\mathrm{div}\mathfrak{g}_n} \left|\nabla \mathrm{S}^{2m} u_n\right|^2 = &- \sum_{n=1}^{\infty} \frac{\mathrm{div}(V\nabla \mathrm{S}^{2m}\mathfrak{g})_n}{\mathrm{S}^{2m}\mathfrak{g}_n} \left|\mathrm{S}^{2m}u_n\right|^2 \\ &+ \sum_{n=1}^{\infty} V_{n+1} \left| \sqrt{\frac{\mathrm{S}^{2m}\mathfrak{g}_n}{\mathrm{S}^{2m}\mathfrak{g}_{n+1}}} \mathrm{S}^{2m}u_{n+1} - \sqrt{\frac{\mathrm{S}^{2m}\mathfrak{g}_{n+1}}{\mathrm{S}^{2m}\mathfrak{g}_n}} \mathrm{S}^{2m}u_n \right|^2.\end{aligned}$$

Again, bearing in mind that

$$\begin{aligned}-\mathrm{div}(V\nabla \mathrm{S}^{2m}\mathfrak{g}) &= -\mathrm{div}\left(\frac{\mathrm{S}^{2m-1}(-\Delta)^{2m}\mathrm{div}\mathfrak{g}}{\mathrm{S}^{2m-1}\mathrm{div}\mathfrak{g}} \mathrm{S}^{2m-1}\mathrm{div}\mathfrak{g}\right) = -\mathrm{S}^{2m-1}\mathrm{div}(-\Delta)^{2m}\mathrm{div}\mathfrak{g} \\ &= \mathrm{S}^{2m}(-\Delta)^{2m+1}\mathfrak{g},\end{aligned}$$

we arrive, after shifting the indices, at the formula

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\mathrm{S}^{2m-1}(-\Delta)^{2m}\mathrm{div}\mathfrak{g}_n}{\mathrm{S}^{2m-1}\mathrm{div}\mathfrak{g}_n} \left|\nabla \mathrm{S}^{2m} u_n\right|^2 = &\sum_{n=2m+1}^{\infty} \frac{(-\Delta)^{2m+1}\mathfrak{g}_n}{\mathfrak{g}_n} |u_n|^2 \\ &+ \sum_{n=2m+1}^{\infty} \frac{(-\Delta)^{2m}\mathrm{div}\mathfrak{g}_n}{\mathrm{div}\mathfrak{g}_n} \left| \sqrt{\frac{\mathfrak{g}_n}{\mathfrak{g}_{n+1}}} u_{n+1} - \sqrt{\frac{\mathfrak{g}_{n+1}}{\mathfrak{g}_n}} u_n \right|^2.\end{aligned}$$

By (3.13), the last term coincides with $\mathcal{R}_0^{(2m+1)}(\mathfrak{g};u)$. Hence, when combined with (3.31), we obtain the identity (3.12) for $\ell = 2m+1$.

c) Lastly, we assume that Theorem 3.6 holds for all $\ell \leq 2m+1$ and verify it for $\ell = 2m+2$. If we have $u \in \mathcal{H}_0^{2m+2}$ and $\mathfrak{g} \in H^{2m+2}$ satisfying the assumption (A1) for $\ell = 2m+2$, we can write

$$\sum_{n=\lceil \frac{2m+2}{2} \rceil}^{\infty} \left|(-\Delta)^{(2m+2)/2} u_n\right|^2 = \sum_{n=m+1}^{\infty} \left|(-\Delta)^{m+1} u_n\right|^2 = \sum_{n=m+1}^{\infty} |\nabla(-\Delta^m)(\mathrm{div}u)_n|^2$$

and apply Theorem 3.6 for $\ell = 2m+1$, i.e. the induction hypothesis, with $\mathrm{div}u \in \mathcal{H}_0^{2m+1}$ and $\mathrm{div}\mathfrak{g} \in H^{2m+1}$ satisfying the assumption (A1) for $\ell = 2m+1$. Using the same methods as in b) we would obtain that the identity (3.12) holds for $\ell = 2m+2$, thus the proof of Theorem 3.6 is complete. □

3.2.3 Proof of Theorem 3.8

The claim is an immediate consequence of Theorem 3.6. It suffices to notice that the assumptions (A2) together with (A1) guarantee non-negativity of all the remainders on the right-hand side of (3.12). Assumption (A3) then implies positivity of the weight $\rho(\mathfrak{g})$. The proof of Theorem 3.8 is complete. $\square$

3.2.4 Proof of Theorem 3.10

We check that, under the respective assumptions, the discrete Hardy weight $\rho(\mathfrak{g}) := (-\Delta)^{\ell}\mathfrak{g}/\mathfrak{g}$ possesses all the three properties from Definition 3.2: criticality, non-attainability, and optimality near infinity. The properties will be gradually worked out in three separate steps.

Similarly to the proof of Theorem 2.5, the core idea of this proof is that all the remainders on the right-hand side of (3.12) vanish if we consider $u = \mathfrak{g}$. However, $g \notin \mathcal{H}_0^{\ell}$ so we must find a suitable regularization of $\mathfrak{g}$. Furthermore, the asymptotic properties of $\operatorname{div}^k \mathfrak{g}$ turn out to be important to assert optimality of the weight $\rho(\mathfrak{g})$. For this purpose, we will state the following lemmas. Additionally, for more concise formulas, we introduce the *middle operator* acting on $C(\mathbb{Z})$ defined as $\mathrm{M} := (\mathrm{S}+\mathrm{I})/2$, i.e.

$$\mathrm{M}u_n := \frac{u_n + u_{n+1}}{2}.$$

Lemma 3.16. *Let $k \in \mathbb{N}$, $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, and g be a sequence admitting the asymptotic expansion*

$$g_n = \sum_{j=0}^{k} a_j n^{\alpha-j} + \mathcal{O}\left(n^{\alpha-k-1}\right), \quad n \to \infty, \tag{3.32}$$

for some $a_j \in \mathbb{R}$ and $a_0 \neq 0$. Then the following claims hold true.

(i) For all $m \in \mathbb{Z}$, there exist coefficients $a_j^{(m)} \in \mathbb{R}$ such that

$$\mathrm{S}^m g_n = a_0 n^{\alpha} + \sum_{j=1}^{k} a_j^{(m)} n^{\alpha-j} + \mathcal{O}\left(n^{\alpha-k-1}\right), \quad n \to \infty.$$

(ii) For all $m \in \mathbb{N}_0$, there exist coefficients $b_j^{(m)} \in \mathbb{R}$ with $b_0^{(m)} \neq 0$ such that

$$\mathrm{M}^m g_n = \sum_{j=0}^{k} b_j^{(m)} n^{\alpha-j} + \mathcal{O}\left(n^{\alpha-k-1}\right), \quad n \to \infty.$$

(iii) For all $m \in \{0, \ldots, k\}$, there exist coefficients $c_j^{(m)} \in \mathbb{R}$ with $c_m^{(m)} \neq 0$ such that

$$\operatorname{div}^m g_n = \sum_{j=m}^{k} c_j^{(m)} n^{\alpha-j} + \mathcal{O}\left(n^{\alpha-k-1}\right), \quad n \to \infty.$$

Proof. In order to prove Lemma 3.16, it suffices to show the first point for $m = \pm 1$, which can be seen from the Taylor series expansion

$$(n \pm 1)^{\beta} = n^{\beta}\left(1 \pm \frac{1}{n}\right)^{\beta} = n^{\beta} \sum_{j=0}^{\infty} \binom{\beta}{j} \left(\frac{\pm 1}{n}\right)^{j} = n^{\beta} + \sum_{j=1}^{k} c_j^{(\pm)} n^{\beta-j} + \mathcal{O}\left(n^{\alpha-k-1}\right), \quad n \to \infty.$$

The rest readily follows by induction, but we leave the technical details as an easy exercise for the reader. $\square$

The third statement of Lemma 3.16 asserts that, to some extent, the sequence $\mathrm{div}^k\mathfrak{g}$, for $\mathfrak{g} \in C(\mathbb{Z})$ of the form (3.32), can be treated as a *derivative* $\mathfrak{g}^{(k)}$. We will need another auxiliary claim of this sort – a generalization of the Mean Value Theorem, which is most likely known, but we will prove it here for the reader's convenience nonetheless.

Lemma 3.17. *Let $n \in \mathbb{Z}$, $N \in \mathbb{N}$ and $g \in C([n, n+N])$ such that $g \in C^N((n, n+N))$. Then there exists $\xi \in (n, n+N)$ such that*

$$\mathrm{div}^N g_n = g^{(N)}(\xi),$$

where we denoted $g_n \coloneqq g(n)$.

Proof. Let p be a polynomial of degree less or equal to N such that $p_{n+j} = g_{n+j}$ for all $j \in \{0, \dots, N\}$. Here and below, we use the notation $p_n \coloneqq p(n)$ and $g_n \coloneqq g(n)$. Notice that

$$\mathrm{div}^N g_n = \sum_{j=0}^{N} \binom{N}{j} (-1)^{N-j} g_{n+j} = \sum_{j=0}^{N} \binom{N}{j} (-1)^{N-j} p_{n+j} = \mathrm{div}^N p_n.$$

Next, we will show that

$$\mathrm{div}^N p_n = p^{(N)}(n).$$

First, we prove that if the degree of p is less or equal to N, then the polynomial $\mathrm{div}p(x) = p(x+1) - p(x)$ is of degree less or equal to $N-1$. Indeed, for $p(x) = \sum_{k=0}^{n} a_k x^k$, where $a_k \in \mathbb{R}$, we have

$$\mathrm{div}p(x) = \sum_{k=0}^{n} a_k \left((x+1)^k - x^k \right) = \sum_{k=0}^{n} a_k \left(\sum_{j=0}^{k} \binom{k}{j} x^j - x^k \right) = \sum_{k=0}^{n} a_k \sum_{j=0}^{k-1} \binom{k}{j} x^j.$$

Consequently, $\mathrm{div}^N p = c$ for some constant $c \in \mathbb{R}$ and therefore $\mathrm{div}^M p = 0$ for all $M > N$. Because both div and $\mathrm{d}/\mathrm{d}x$ are linear operators on respective spaces, it suffices to show that $(\mathrm{div}^N x^N) = (x^N)^{(N)} = N!$. We will prove this by induction once again. For $N = 0$ the statement holds trivially. Assuming the claim holds for $N-1$ we have

$$\begin{aligned}
\mathrm{div}^N x^N &= \mathrm{div}^{N-1}(\mathrm{div} x^N) = \mathrm{div}^{N-1}((x+1)^N - x^N) = \sum_{k=0}^{N-1} \binom{N}{k} (\mathrm{div}^{N-1} x^k) \\
&= N(\mathrm{div}^{N-1} x^{N-1}) = N(N-1)! = N!.
\end{aligned}$$

As $p_{n+j} - g_{n+j} = 0$ for all $j \in \{0, \dots, N\}$, we infer from Rolle's theorem, that for all $j \in \{0, \dots, N-1\}$ there exists $c_j \in (n+j, n+j+1)$ such that $p'(c_j) - g'(c_j) = 0$. By iteration of Rolle's theorem, we obtain $\xi \in (n, n+N)$ such that

$$g^{(N)}(\xi) = p^{(N)}(\xi) = p^{(N)}(n) = \mathrm{div}^N p_n = \mathrm{div}^N g_n. \qquad \square$$

The last auxiliary identity is the *Leibnitz formula* for the discrete divergence. The multiplication of sequences is to be understood point-wise.

Lemma 3.18. *For all $m \in \mathbb{N}_0$ and sequences $f, g \in C(\mathbb{Z})$, we have*

$$\mathrm{div}^m(fg) = \sum_{j=0}^{m} \binom{m}{j} \left(\mathrm{div}^j \mathrm{M}^{m-j} f \right) \left(\mathrm{div}^{m-j} \mathrm{M}^j g \right).$$

Proof. The proof of Lemma 3.18 proceeds by induction. The statement is trivially true for $m = 0$. For $m = 1$, we have

$$\operatorname{div}(fg) = (\mathrm{S}-\mathrm{I})(fg) = \begin{cases} (\mathrm{S}f)(\mathrm{S}g) - fg \pm (\mathrm{S}f)g = (\mathrm{S}f)(\operatorname{div}g) + (\operatorname{div}f)g \\ (\mathrm{S}f)(\mathrm{S}g) - fg \pm f(\mathrm{S}g) = f(\operatorname{div}g) + (\operatorname{div}f)(\mathrm{S}g) \end{cases}$$

Adding the two lines above and diving by 2, we verify the statement for $m = 1$. Next, for the case $m+1$ we infer from the induction hypothesis that

$$\begin{aligned}
\operatorname{div}^{m+1}(fg) =& \operatorname{div}(\operatorname{div}^m(fg)) = \operatorname{div}\left(\sum_{j=0}^{m}\binom{m}{j}(\operatorname{div}^j \mathrm{M}^{m-j} f)(\operatorname{div}^{m-j}\mathrm{M}^j g)\right) \\
=& \sum_{j=0}^{m}\binom{m}{j}(\operatorname{div}^{j+1}\mathrm{M}^{m-j}f)(\operatorname{div}^{m-j}\mathrm{M}^{j+1}g) + \sum_{j=0}^{m}\binom{m}{j}(\operatorname{div}^j\mathrm{M}^{m+1-j}f)(\operatorname{div}^{m+1-j}\mathrm{M}^j g) \\
=& (\operatorname{div}^{m+1}f)(\mathrm{M}^{m+1}g) + (\mathrm{M}^{m+1}f)(\operatorname{div}^{m+1}g) \\
& + \sum_{j=1}^{m}\left[\binom{m}{j-1} + \binom{m}{j}\right](\operatorname{div}^j\mathrm{M}^{m+1-j}f)(\operatorname{div}^{m+1-j}\mathrm{M}^j g) \\
=& \sum_{j=0}^{m+1}\binom{m+1}{j}(\operatorname{div}^j\mathrm{M}^{m+1-j}f)(\operatorname{div}^{m+1-j}\mathrm{M}^j g),
\end{aligned}$$

hence the proof is complete. □

Now, we are ready to prove Theorem 3.10.

a) *Proof of criticality*: Suppose that the sequence $\tilde{\rho} = \{\tilde{\rho}_n\}_{n=\ell}^{\infty}$ is a discrete Hardy weight of order ℓ satisfying $\tilde{\rho}_n \geq \rho_n(\mathfrak{g})$ for all $n \geq \ell$. Using identity (3.12) for the weight $\rho(\mathfrak{g})$ together with the discrete Hardy inequality of order ℓ for the weight $\tilde{\rho}$, we find that

$$0 \leq \sum_{n=\ell}^{\infty}(\tilde{\rho}_n - \rho_n(\mathfrak{g}))\,|u_n|^2 \leq \sum_{k=0}^{\ell-1}\mathcal{R}_k^{(\ell)}(\mathfrak{g};u) \tag{3.33}$$

for all $u \in \mathcal{H}_0^{\ell}$.

As was mentioned earlier, an important fact is that all the remainders are simultaneously annihilated if we set $u = \mathfrak{g}$, i.e. $R_k^{(\ell)}(\mathfrak{g};\mathfrak{g}) = 0$ for all $k \in \{0,\dots,\ell-1\}$. However, we cannot directly substitute $u = \mathfrak{g}$ into (3.33) and conclude from here that $\tilde{\rho} = \rho(\mathfrak{g})$ since $\mathfrak{g} \notin \mathcal{H}_0^{\ell}$. This is an issue which is to be overcome by a suitable regularization of $\mathfrak{g}$.

Fix an arbitrary $\varepsilon \in (0,1/2)$ and a smooth function η such that $\eta \equiv 0$ on $(-\infty,\varepsilon)$ and $\eta \equiv 1$ on $(1-\varepsilon,\infty)$. Then for any $N \geq 2$, we introduce $u^N := \xi^N\mathfrak{g}$, where $\xi_n^N := \xi^N(n)$ is a cut-off sequence define as

$$\xi^N(x) := \begin{cases} 1 & \text{if } x \leq N, \\ \eta\left(\frac{2\ln N - \ln x}{\ln N}\right) & \text{if } N < x \leq N^2, \\ 0 & \text{if } x > N^2. \end{cases} \tag{3.34}$$

Notice that $\xi^N \to 1$ and hence $u^N \to \mathfrak{g}$ point-wise as $N \to \infty$. With this choice of $u^N \in \mathcal{H}_0^{\ell}$, we will show that for all $k \in \{0,\dots,\ell-1\}$ we may estimate

$$0 \leq \mathcal{R}_k^{(\ell)}\left(\mathfrak{g};u^N\right) \lesssim \frac{1}{\ln N}, \tag{3.35}$$

where $\lesssim$ means the inequality $\leq$ up to a N-independent multiplicative constant. Invoking Fatou's lemma, we infer from (3.33) and (3.35) that

$$\sum_{n=\ell}^{\infty}(\tilde{\rho}_n-\rho_n(\mathfrak{g}))\,\mathfrak{g}_n^2=\sum_{n=\ell}^{\infty}(\tilde{\rho}_n-\rho_n(\mathfrak{g}))\lim_{N\to\infty}\left|u_n^N\right|^2\leq\liminf_{N\to\infty}\sum_{n=\ell}^{\infty}(\tilde{\rho}_n-\rho_n(\mathfrak{g}))\left|u_n^N\right|^2=0.$$

Bearing in mind, that all the terms in the sum on the left are non-negative and $\mathfrak{g}_n>0$ for all $n\geq\ell$ by (A1), we may conclude that $\tilde{\rho}_n=\rho_n(\mathfrak{g})$ for all $n\geq\ell$, and the proof of criticality of $\rho(\mathfrak{g})$ will be complete.

It remains to verify the inequality (3.35). Recalling Remark 3.9, we use the notation

$$\mathcal{R}_k^{(\ell)}(\mathfrak{g};u)=\left\|R_k^{(\ell)}(\mathfrak{g})u\right\|^2=\sum_{n=\ell-k}^{\infty}\left|R_k^{(\ell)}(\mathfrak{g})u_n\right|^2$$

for all $k\in\{0,\dots,\ell-1\}$, where the operators $R_k^{(\ell)}(\mathfrak{g})$ are defined by (3.17). Next, we substitute $u=u^N=\xi^N\mathfrak{g}$ into (3.17) and inspect the two factors of

$$\left|R_k^{(\ell)}(\mathfrak{g})u_n^N\right|=\sqrt{\frac{(-\Delta)^{\ell-1-k}\mathrm{div}^{k+1}\mathfrak{g}_n}{\mathrm{div}^{k+1}\mathfrak{g}_n}}\left|\sqrt{\frac{\mathrm{div}^k\mathfrak{g}_n}{\mathrm{div}^k\mathfrak{g}_{n+1}}}\mathrm{div}^k(\xi^N\mathfrak{g})_{n+1}-\sqrt{\frac{\mathrm{div}^k\mathfrak{g}_{n+1}}{\mathrm{div}^k\mathfrak{g}_n}}\mathrm{div}^k(\xi^N\mathfrak{g})_n\right|$$

separately.

For the first factor, by assumption (A3) and claims (i) and (iii) of Lemma 3.16, we observe that

$$\sqrt{\frac{(-\Delta)^{\ell-1-k}\mathrm{div}^{k+1}\mathfrak{g}_n}{\mathrm{div}^{k+1}\mathfrak{g}_n}}=\sqrt{\frac{(-1)^{\ell-1-k}\mathrm{div}^{2\ell-k-1}\mathrm{S}^{k+1-\ell}\mathfrak{g}_n}{\mathrm{div}^{k+1}\mathfrak{g}_n}}\lesssim\sqrt{\frac{n^{\ell-1/2-(2\ell-k-1)}}{n^{\ell-1/2-(k+1)}}}=n^{k+1-\ell}$$

for any $n\geq\ell$ and $k\in\{0,\dots,\ell-1\}$, where the unspecified constant is n-independent but may depend on k or ℓ.

Similarly, we find that

$$\sqrt{\frac{\mathrm{div}^k\mathfrak{g}_n}{\mathrm{div}^k\mathfrak{g}_{n+1}}}\lesssim 1,$$

and therefore the second factor can be estimated as

$$\sqrt{\frac{\mathrm{div}^k\mathfrak{g}_n}{\mathrm{div}^k\mathfrak{g}_{n+1}}}\left|\mathrm{div}^k(\xi^N\mathfrak{g})_{n+1}-\frac{\mathrm{div}^k\mathfrak{g}_{n+1}}{\mathrm{div}^k\mathfrak{g}_n}\mathrm{div}^k(\xi^N\mathfrak{g})_n\right|\lesssim\left|\mathrm{div}^k(\xi^N\mathfrak{g})_{n+1}-\frac{\mathrm{div}^k\mathfrak{g}_{n+1}}{\mathrm{div}^k\mathfrak{g}_n}\mathrm{div}^k(\xi^N\mathfrak{g})_n\right|.$$

Applying Lemma 3.18 in the last expression, we can rewrite it as

$$\left|\sum_{j=0}^{k}\binom{k}{j}(\mathrm{div}^j\mathrm{M}^{k-j}\mathfrak{g})_{n+1}(\mathrm{div}^{k-j}\mathrm{M}^j\xi^N)_{n+1}-\frac{\mathrm{div}^k\mathfrak{g}_{n+1}}{\mathrm{div}^k\mathfrak{g}_n}\sum_{j=0}^{k}\binom{k}{j}(\mathrm{div}^j\mathrm{M}^{k-j}\mathfrak{g})_n(\mathrm{div}^{k-j}\mathrm{M}^j\xi^N)_n\right|$$
$$\leq\sum_{j=0}^{k}\binom{k}{j}(\mathrm{div}^j\mathrm{M}^{k-j}\mathfrak{g})_{n+1}\left|(\mathrm{div}^{k-j}\mathrm{M}^j\xi^N)_{n+1}-\frac{\mathrm{div}^k\mathfrak{g}_{n+1}}{\mathrm{div}^k\mathfrak{g}_n}\frac{\mathrm{div}^j\mathrm{M}^{k-j}\mathfrak{g}_n}{\mathrm{div}^j\mathrm{M}^{k-j}\mathfrak{g}_{n+1}}(\mathrm{div}^{k-j}\mathrm{M}^j\xi^N)_n\right|.$$

Yet another application of (A3) and formulas from Lemma 3.16 reveals that

$$(\mathrm{div}^j\mathrm{M}^{k-j}\mathfrak{g})_{n+1}\lesssim n^{\ell-1/2-j}\quad\text{and}\quad\frac{\mathrm{div}^k\mathfrak{g}_{n+1}}{\mathrm{div}^k\mathfrak{g}_n}\frac{\mathrm{div}^j\mathrm{M}^{k-j}\mathfrak{g}_n}{\mathrm{div}^j\mathrm{M}^{k-j}\mathfrak{g}_{n+1}}=\frac{1+\mathcal{O}(1/n)}{1+\mathcal{O}(1/n)}\frac{1+\mathcal{O}(1/n)}{1+\mathcal{O}(1/n)}=1+p_n^{(k,j)},$$

where $p_n^{(k,j)} = \mathcal{O}(1/n)$, as $n \to \infty$, i.e. $p_n^{(k,j)} \lesssim 1/n$. Altogether, we deduce the upper bound

$$\left|R_k^{(\ell)}(\mathfrak{g})u_n^N\right| \lesssim n^{k+1/2} \sum_{j=0}^{k} \binom{k}{j} n^{-j} \left(\left|(\mathrm{div}^{k+1-j} \mathrm{M}^j \xi^N)_n\right| + \left|p_n^{(k,j)} (\mathrm{div}^{k-j} \mathrm{M}^j \xi^N)_n\right| \right) \tag{3.36}$$

for all $N \geq 2$, $n \geq \ell$, and $k \in \{0, \ldots, \ell-1\}$, where the unspecified multiplicative constant does not depend on n nor N. In order to complete the proof, we have to estimate also the terms with $\mathrm{div}^m \xi$ in (3.36). To this end, note that for all $x \in [N, N^2]$, we have

$$(\xi^N)'(x) = -\eta' \left(\frac{2 \ln N - \ln x}{\ln N} \right) \frac{1}{x \ln N},$$

thus $\left|(\xi^N)'(x)\right| \lesssim 1/(x \ln N)$, where we estimated η' with $\max_{x \in [0,1]} |\eta'(x)|$. By induction, it is straightforward to generalize this bound to higher-order derivatives, i.e.

$$\left|(\xi^N)^{(m)}(x)\right| \lesssim \frac{1}{x^m \ln N} \tag{3.37}$$

for all $m \in \mathbb{N}$ and $x > 0$. Consequently, Lemma 3.17 implies that

$$\left|\mathrm{div}^m \xi_n^N\right| \lesssim \frac{1}{n^m \ln N},$$

which is true for any $m, n \in \mathbb{N}$ and $N \geq 2$. As the right-hand side is a decreasing function of n, we also have

$$\left|\mathrm{div}^m \mathrm{M}^j \xi^N\right| \lesssim \frac{1}{n^m \ln N}$$

for any $j \in \mathbb{N}_0$, from which we infer the needed estimates in (3.36) getting

$$\left|R_k^{(\ell)}(\mathfrak{g})u_n^N\right| \lesssim n^{k+1/2} \sum_{j=0}^{k} \binom{k}{j} n^{-j} \left(\frac{1}{n^{k+1-j} \ln N} + \frac{1}{n} \frac{1}{n^{k-j} \ln N} \right) \lesssim \frac{1}{\sqrt{n} \ln N} \tag{3.38}$$

for all $N \geq 2$, $n \geq \ell$, and $k \in \{0, \ldots, \ell-1\}$. Finally, for sufficiently large N we may estimate

$$\mathcal{R}_k^{(\ell)}(\mathfrak{g}; u) = \sum_{n=N-k}^{N^2} \left|R_k^{(\ell)}(\mathfrak{g})u_n^N\right|^2 \lesssim \frac{1}{\ln^2 N} \sum_{n=N-k}^{N^2} \frac{1}{n} \lesssim \frac{1}{\ln^2 N} \int_N^{N^2} \frac{1}{n} \, \mathrm{d}n = \frac{1}{\ln N}$$

for all $k \in \{0, \ldots, \ell-1\}$ arriving at the desired result (3.35). The proof of criticality is complete.

b) *Proof of non-attainability*: For the proof of non-attainability, we first state the following auxiliary claim which asserts that under certain assumptions, the identity from Theorem 3.6 extends from $\mathcal{H}_0^\ell$ to all sequences in H^ℓ for which the left-hand side of (3.12) is finite.

Proposition 3.19. *Let $\ell \in \mathbb{N}$. Suppose that assumptions* (A1), (A2), *and* (A3) *hold. Then the identity* (3.12) *extends from $\mathcal{H}_0^\ell$ to all sequences from the space*

$$\mathcal{D}^\ell := \left\{ u \in H^\ell \,\middle|\, \|(-\Delta)^{\ell/2} u\| < \infty \right\}.$$

Proof. We will prove Proposition 3.19 in three steps.

a) First, we show that the range of $(-\Delta)^{\ell/2}|_{\mathcal{H}_0^\ell}$ is dense in $\mathcal{H}^{\lceil \ell/2 \rceil}$. It follows from the definition of $(-\Delta)^{\ell/2}$ that for any $u \in \mathcal{H}_0^\ell$, we have $(-\Delta)^{\ell/2} u_n = 0$ for all $n < \lceil \ell/2 \rceil$, i.e. $(-\Delta)^{\ell/2} u \in H^{\lceil \ell/2 \rceil}$. Moreover, from the boundedness of the operator $(-\Delta)^{\ell/2}$, we infer that $(-\Delta)^{\ell/2} u \in \ell^2(\mathbb{Z})$, hence $(-\Delta)^{\ell/2} u \in \mathcal{H}^{\lceil \ell/2 \rceil}$. In order to prove the density, we check that if $v \in \mathcal{H}^{\lceil \ell/2 \rceil}$ satisfies

$$\langle v, (-\Delta)^{\ell/2} u \rangle = 0$$

for all $u \in \mathcal{H}_0^\ell$, then $v \equiv 0$. According to the parity of ℓ, we distinguish two cases. Let $\ell = 2m \in 2\mathbb{N}$, then

$$\langle v, (-\Delta)^{\ell/2} u \rangle = \langle v, (-\Delta)^m u \rangle = \langle (-\Delta)^m v, u \rangle$$

for all $u \in \mathcal{H}_0^{2m}$. In particular, by taking $u = \delta_n$, i.e. $u_k = \delta_{nk}$, where δ is the Kronecker delta, for $n \geq 2m$, we observe that v solves the linear difference equation of order $2m$ with constant coefficients

$$(-\Delta)^m v_n = \sum_{j=-m}^{m} \binom{2m}{m+j} (-1)^j v_{n+j} = 0$$

for all $n \geq 2m$. It is easy to show that the fundamental system of the above linear difference equation consists of sequences $1, n, \dots, n^{2m-1}$. Therefore, we find that

$$v_n = \sum_{j=0}^{2m-1} c_j n^j$$

for all $n \geq m$ and some $c_j \in \mathbb{C}$. Taking also into account that $v \in \ell^2(\mathbb{Z})$, necessarily, we have $c_j = 0$ for all $j \in \{0, \dots, 2m-1\}$ and since $v \in H^m$, we conclude that $v \equiv 0$. The argument for $l = 2m+1 \in 2\mathbb{N}_0 + 1$ is analogous. In this case, we have

$$\langle v, (-\Delta)^{\ell/2} u \rangle = \langle v, \nabla(-\Delta)^m u \rangle = -\langle (-\Delta)^m \mathrm{div} v, u \rangle = 0$$

for all $u \in \mathcal{H}_0^{2m+1}$. Again, we consider $u = \delta_n$ for $n \geq 2m+1$, from which we infer that v solves the linear difference equation

$$-(-\Delta)^m \mathrm{div} v_n = \sum_{j=-m}^{m+1} \binom{2m+1}{m+j} (-1)^j v_{n+j} = 0$$

for all $n \geq 2m+1$, whose general solution is given by the linear combination

$$v_n = \sum_{j=0}^{2m} c_j n^j$$

for all $n \geq m+1$ and some $c_j \in \mathbb{C}$. The same argumentation as in the case where ℓ is even then implies $v \equiv 0$.

b) Pick arbitrary $v \in \mathcal{H}^{\lceil \ell/2 \rceil}$. By a), there exists a sequence $\{u^N\}_{N=1}^\infty \subset \mathcal{H}_0^\ell$ such that $(-\Delta)^{\ell/2} u^N \to v$, as $N \to \infty$, in the norm of $\ell^2(\mathbb{Z})$. In particular, the sequence $\{(-\Delta)^{\ell/2} u^N\}_{N=1}^\infty$ is Cauchy in $\ell^2(\mathbb{Z})$. Recalling Remark 3.9, assumptions (A1) and (A2) imply that the remainders in (3.12) can be interpreted as a (non-negative) norm of $R_k^{(\ell)}(\mathfrak{g}) u^N$ and assumption (A3) guarantees positivity of the weight $\rho_n(\mathfrak{g})$. Consequently, identity (3.12) asserts inequalities

$$\left\| \sqrt{\rho(\mathfrak{g})} (u^N - u^M) \right\| \leq \left\| (-\Delta)^{\ell/2} (u^N - u^M) \right\| \quad \text{and} \quad \left\| R_k^{(\ell)}(\mathfrak{g}) (u^N - u^M) \right\| \leq \left\| (-\Delta)^{\ell/2} (u^N - u^M) \right\|$$

for all $k \in \{0, \dots, \ell-1\}$ and $N, M \in \mathbb{N}$, hence the sequences $\{\sqrt{\rho(\mathfrak{g})} u^N\}_{N=1}^\infty$ and $\{R_k^{(\ell)}(\mathfrak{g}) u^N\}_{N=1}^\infty$ are also Cauchy and therefore convergent in $\ell^2(\mathbb{Z})$. Let us denote w the ℓ^2-limit of $\sqrt{\rho(\mathfrak{g})} u^N$, as $N \to \infty$. Clearly, $w \in \mathcal{H}^\ell$ and $\sqrt{\rho_n(\mathfrak{g})} u_n^N \to w_n$, as $N \to \infty$, for every $n \in \mathbb{Z}$. Since $\rho_n(\mathfrak{g}) > 0$ for all $n \geq \ell$, there exists $u \in H^\ell$ such that $w = \sqrt{\rho(\mathfrak{g})} u$ and $u_n^N \to u_n$, as $N \to \infty$, for all $n \in \mathbb{Z}$. Moreover, the limits of all the above-mentioned ℓ^2-convergent sequences must coincide with their point-wise limits, i.e.

$$(-\Delta)^{\ell/2} u^N \to (-\Delta)^{\ell/2} u, \qquad \sqrt{\rho(\mathfrak{g})} u^N \to \sqrt{\rho(\mathfrak{g})} u, \qquad R_k^{(\ell)}(\mathfrak{g}) u^N \to R_k^{(\ell)}(\mathfrak{g}) u$$

in $\ell^2(\mathbb{Z})$ as $N \to \infty$. In particular, we have $v = (-\Delta)^{\ell/2} u$, hence $u \in \mathcal{D}^\ell$. Tending $N \to \infty$ in the identity (3.12) for u_N, i.e. in

$$\left\| (-\Delta)^{\ell/2} u^N \right\|^2 = \left\| \sqrt{\rho(\mathfrak{g})} u^N \right\|^2 + \sum_{k=0}^{\ell-1} \left\| R_k^{(\ell)}(\mathfrak{g}) u^N \right\|^2,$$

yields

$$\left\| (-\Delta)^{\ell/2} u \right\|^2 = \left\| \sqrt{\rho(\mathfrak{g})} u \right\|^2 + \sum_{k=0}^{\ell-1} \left\| R_k^{(\ell)}(\mathfrak{g}) u \right\|^2.$$

c) Now, pick arbitrary $\tilde{u} \in \mathcal{D}^\ell$. By definition, $(-\Delta)^{\ell/2}\tilde{u} \in \mathcal{H}^{\lceil \ell/2 \rceil}$. Part b) applied to $(-\Delta)^{\ell/2}\tilde{u}$ implies the existence of a sequence $u \in H^\ell$, such that the above identity holds and $(-\Delta)^{\ell/2}\tilde{u} = (-\Delta)^{\ell/2} u$. In order to finish the proof, it remains to show that for two sequences $u, \tilde{u} \in H^\ell$, we have the implication

$$(-\Delta)^{\ell/2}\tilde{u} = (-\Delta)^{\ell/2} u \quad \implies \quad \tilde{u} = u.$$

By linearity, it suffices to prove that for any $w \in H^\ell$ such that $(-\Delta)^{\ell/2} w = 0$, we have $w \equiv 0$. This can be easily seen from the fact that w solves the linear difference equation

$$(-\Delta)^{\ell/2} w_n = 0$$

for all $n \in \mathbb{Z}$ with the boundary condition $w_n = 0$ for all $n < \ell$. Recursively, we obtain $w_n = 0$ also for all $n \geq \ell$, thus proving Proposition 3.19. □

Now, suppose that $u \in H^\ell$ fulfils (3.16) as equality whose (both) sides are finite. In view of Proposition 3.19, assumptions (A1), (A2), and (A3) imply that $\mathcal{R}_k^{(\ell)}(\mathfrak{g}; u) = 0$ for all $k \in \{0, \dots, \ell-1\}$. In particular, for $k = 0$, we derive a necessary condition (which is in fact sufficient)

$$\frac{(-\Delta)^{\ell-1} \mathrm{div} \mathfrak{g}_n}{\mathrm{div} \mathfrak{g}_n} \left| \sqrt{\frac{\mathfrak{g}_n}{\mathfrak{g}_{n+1}}} u_{n+1} - \sqrt{\frac{\mathfrak{g}_{n+1}}{\mathfrak{g}_n}} u_n \right|^2 = 0$$

for all $n \geq \ell$, see (3.13). By the additional assumption (A5) and also (A1), the prefactor is strictly positive and therefore the sequence u is a solution of the difference equation of the first order

$$\sqrt{\frac{\mathfrak{g}_n}{\mathfrak{g}_{n+1}}} u_{n+1} - \sqrt{\frac{\mathfrak{g}_{n+1}}{\mathfrak{g}_n}} u_n = 0$$

for all $n \geq \ell$. However, this equation has the only solution $\mathfrak{g}$ up to a multiplicative constant, i.e. $u_n = c\mathfrak{g}_n$ for some constant $c \in \mathbb{C}$ and $n \geq \ell$. Obviously, for such u it holds that $\mathcal{R}_k^{(\ell)}(\mathfrak{g}; u) = 0$ for all $k \in \{0, \dots, \ell-1\}$. By assumption (A3) and Lemma 3.16, we have

$$\rho_n(\mathfrak{g}) \gtrsim \frac{n^{\ell-1/2-2\ell}}{n^{\ell-1/2}} = n^{-2\ell}$$

and due to the asymptotic behaviour of $\mathfrak{g}_n$, we may conclude that

$$\rho_n(\mathfrak{g})\mathfrak{g}_n^2 \gtrsim \frac{1}{n^{2\ell}} n^{2\ell-1} = \frac{1}{n} \tag{3.39}$$

for all $n \geq \ell$. Consequently, the right-hand side of (3.16) can be estimated as

$$\sum_{n=\ell}^{\infty} \rho_n(\mathfrak{g}) \left| u_n \right|^2 \gtrsim \sum_{n=\ell}^{\infty} \frac{1}{n} = \infty,$$

whenever $c \neq 0$, hence the proof of non-attainability is complete.

c) *Proof of optimality near infinity*: Fix $M \in \mathbb{N}$. As follows form (3.11), this property can be proven by finding a sequence $\{u^N\}_{N\geq M} \subset H_0^M(\mathbb{N}) \setminus \{0\}$ such that

$$\lim_{N\to\infty} \frac{\sum_{k=0}^{\ell-1} \mathcal{R}_k^{(\ell)}(\mathfrak{g}; u^N)}{\sum_{n=\ell}^{\infty} \rho_n(\mathfrak{g}) \, |u_n^N|^2} = 0. \tag{3.40}$$

Such sequence can be obtained by slightly modifying the regularization (3.34) used in the proof of criticality a). We define $u^N := \xi^N \mathfrak{g}$, where, this time, ξ^N is given by

$$\xi^N(x) := \begin{cases} 0 & \text{if } x \leq N, \\ \eta\left(\frac{\ln x - \ln N}{\ln N}\right) & \text{if } N < x \leq N^2, \\ 1 & \text{if } N^2 < x \leq 2N^2, \\ \eta\left(\frac{\ln(2N^3) - \ln x}{\ln N}\right) & \text{if } 2N^2 < x \leq 2N^3, \\ 0 & \text{if } x > 2N^3, \end{cases}$$

where function η has the same properties as in (3.34). Then, indeed, $u^N \in \mathcal{H}_0^M \setminus \{0\}$ for all $N \geq M$. Note that the inequality (3.37) still holds for any $m \in \mathbb{N}$ and $x > 0$. Therefore, the same estimates as in the proof of criticality a) apply and we get

$$\mathcal{R}_k^{(\ell)}(\mathfrak{g}; u^N) = \sum_{n=N-k}^{N^2} \left| R_k^{(\ell)}(\mathfrak{g}) u_n^N \right|^2 + \sum_{n=2N^2-k}^{2N^3} \left| R_k^{(\ell)}(\mathfrak{g}) u_n^N \right|^2 \lesssim \frac{1}{\ln^2 N} \left(\sum_{n=N-k}^{N^2} \frac{1}{n} + \sum_{n=2N^2-k}^{2N^3} \frac{1}{n} \right) \lesssim \frac{1}{\ln N}$$

for all N sufficiently large and $k \in \{0, \dots, \ell - 1\}$.

On the other hand, having (3.39) in mind, we may estimate the denominator in (3.40) from below as

$$\sum_{n=\ell}^{\infty} \rho_n(\mathfrak{g}) \, |u_n^N|^2 \geq \sum_{n=N^2}^{2N^2} \rho_n(\mathfrak{g}) \mathfrak{g}_n^2 \gtrsim \sum_{n=N^2}^{2N^2} \frac{1}{n} \geq \int_{N^2}^{2N^2} \frac{1}{n} \, \mathrm{d}n = \ln 2$$

for all N sufficiently large. Altogether, the above estimates immediately imply (3.40), completing the proof of optimality near infinity and subsequently also the proof of Theorem 3.10. □

3.2.5 Proof of Theorem 3.11

For the parameter sequence $\mathfrak{g}^{(\ell)}$ defined by (3.19), we verify below that

a) $\mathrm{div}^k \mathfrak{g}_n^{(\ell)} > 0$ for all $n \geq \ell - k$ and $k \in \{0, \dots, \ell - 1\}$;

b) $(-\Delta)^{\ell-k} \mathrm{div}^k \mathfrak{g}_n^{(\ell)} > 0$ for all $n \geq \ell - k$ and $k \in \{0, \dots, \ell - 1\}$.

Claim a) together with the obvious fact $\mathfrak{g}^{(\ell)} \in H^\ell$ means that $\mathfrak{g}^{(\ell)}$ fulfils assumption (A1). Claim b) implies that $\mathfrak{g}^{(\ell)}$ satisfies assumptions (A2), (A3), and (A5). As it follows trivially from definition (3.19) that $\mathfrak{g}^{(\ell)}$ satisfies also (A4), Theorem 3.10 asserts that the sequence $\rho^{(\ell)}$ given by (3.20) is an optimal discrete Hardy weight of order ℓ.

a) First we show that $\mathrm{div}^\ell \mathfrak{g}_n^{(\ell)} > 0$ for all $n \geq 0$. For $x \geq 0$, let us denote

$$\mathfrak{g}^{(\ell)}(x) := \sqrt{x} \prod_{j=1}^{\ell-1} (x - j).$$

Then $\mathfrak{g}_n^{(\ell)} = \mathfrak{g}^{(\ell)}(n)$ for all $n \in \mathbb{N}_0$. By [20, Eq. (26.8.7)], we have

$$\mathfrak{g}^{(\ell)}(x) = \sum_{j=1}^{\ell} s(\ell,j)x^{j-1/2}, \tag{3.41}$$

where $s(\ell,j)$ are the Stirling numbers of the first kind defined by formula (3.21). Evidently, we have $(-1)^{\ell+j}s(\ell,j) > 0$ for all $j \in \{1,\dots,\ell\}$. It follows from Lemma 3.17, that for any $n \in \mathbb{N}_0$ there exists $\xi \in (n, n+\ell)$, such that

$$\operatorname{div}^{\ell}\mathfrak{g}_n^{(\ell)} = \frac{\mathrm{d}^{\ell}\mathfrak{g}^{(\ell)}}{\mathrm{dx}^{\ell}}(\xi).$$

Moreover, for any $x > 0$, we have

$$\frac{\mathrm{d}^{\ell}\mathfrak{g}^{(\ell)}}{\mathrm{dx}^{\ell}}(x) = \sum_{j=1}^{\ell} s(\ell,j)\frac{\mathrm{d}^{\ell}}{\mathrm{dx}^{\ell}}x^{j-1/2} = \sum_{j=1}^{\ell} b_j^{(\ell)}x^{j-1/2-\ell},$$

where

$$b_j^{(\ell)} := (-1)^{\ell+j}s(\ell,j)\prod_{k=1}^{\ell}\left|j+\frac{1}{2}-k\right| > 0$$

for all $j \in \{1,\dots,\ell\}$. Therefore, we infer that $\operatorname{div}^{\ell}\mathfrak{g}_n^{(\ell)} > 0$ for all $n \in \mathbb{N}_0$, indeed.

Next, notice that, since $\mathfrak{g}^{(\ell)} \in H^{\ell}$, we have

$$\operatorname{div}^{k}\mathfrak{g}_{\ell-k}^{(\ell)} = \mathfrak{g}_{\ell}^{(\ell)} = \sqrt{\ell}(\ell-1)! > 0 \tag{3.42}$$

for every $k \in \{0,\dots,\ell\}$. By definition of the discrete divergence, $\operatorname{div}^{\ell}\mathfrak{g}^{(\ell)} > 0$ on $\mathbb{N}_0$ means that the sequence $\operatorname{div}^{\ell-1}\mathfrak{g}^{(\ell)}$ is strictly increasing on $\mathbb{N}_0$. Since $\operatorname{div}^{\ell-1}\mathfrak{g}_1^{(\ell)} > 0$ by (3.42), we conclude that $\operatorname{div}^{\ell-1}\mathfrak{g}_n^{(\ell)} > 0$ for all $n \geq 1$. Iteration of this argument yields claim a).

b) We make use of Lemma 3.17 once more. Since

$$(-\Delta)^{\ell-k}\operatorname{div}^{k}\mathfrak{g}_n = (-1)^{\ell-k}\operatorname{div}^{2\ell-k}\mathfrak{g}_{n-\ell+k}^{(\ell)},$$

Lemma 3.17 implies that, for any $n \geq \ell-k$ and $k \in \{0,\dots,\ell\}$, there is $\xi > 0$ such that

$$(-\Delta)^{\ell-k}\operatorname{div}^{k}\mathfrak{g}_n = (-1)^{\ell-k}\frac{\mathrm{d}^{2\ell-\mathrm{k}}\mathfrak{g}^{(\ell)}}{\mathrm{dx}^{2\ell-\mathrm{k}}}(\xi)$$

Similarly as in part a) of this proof, we find, this time, that

$$(-1)^{\ell-k}\frac{\mathrm{d}^{2\ell-\mathrm{k}}\mathfrak{g}^{(\ell)}}{\mathrm{dx}^{2\ell-\mathrm{k}}}(x) = (-1)^{\ell-k}\sum_{j=1}^{\ell} s(\ell,j)\frac{\mathrm{d}^{2\ell-\mathrm{k}}}{\mathrm{dx}^{2\ell-\mathrm{k}}}x^{j-1/2} = \sum_{j=1}^{\ell} c_j^{(\ell)}x^{j-1/2-2\ell+k},$$

where

$$c_j^{(\ell)} := (-1)^{\ell-k}s(\ell,j)(-1)^{2\ell-k-j}\prod_{k=1}^{2\ell-k}\left|j+\frac{1}{2}-k\right| = (-1)^{\ell+j}s(\ell,j)\prod_{k=1}^{2\ell-k}\left|j+\frac{1}{2}-k\right| > 0$$

for every $j \in \{1,\dots,\ell\}$. Consequently,

$$(-1)^{\ell-k}\frac{\mathrm{d}^{2\ell-\mathrm{k}}\mathfrak{g}^{(\ell)}}{\mathrm{dx}^{2\ell-\mathrm{k}}}(x) > 0$$

for all $x > 0$ and $k \in \{0,\dots,\ell\}$, from which claim b) readily follows. The proof of Theorem 3.11 is complete. □

3.2.6 Proof of Theorem 3.14

First, with the aid of the generalized binomial theorem, we find for all $\nu \in \mathbb{R}$ and $n > \ell$ that

$$\begin{aligned}(-\Delta)^\ell n^\nu &= \sum_{j=-\ell}^{\ell} \binom{2\ell}{\ell+j}(-1)^j (n+j)^\nu = n^\nu \sum_{j=-\ell}^{\ell}\binom{2\ell}{\ell+j}(-1)^j\left(1+\frac{j}{n}\right)^\nu \\ &= n^\nu \sum_{j=-\ell}^{\ell}\binom{2\ell}{\ell+j}(-1)^j \sum_{m=0}^{\infty}\binom{\nu}{m}\frac{j^m}{n^m} = n^\nu \sum_{m=0}^{\infty}\binom{\nu}{m}\left[\sum_{j=-\ell}^{\ell}\binom{2\ell}{\ell+j}(-1)^j j^m\right]\frac{1}{n^m}.\end{aligned}$$

Recalling the definition (3.23) and the property (3.24), we arrive at the identity

$$(-\Delta)^\ell n^\nu = \sum_{m=2\ell}^{\infty}\binom{\nu}{m}\frac{X_m^{(\ell)}}{n^{m-\nu}} \tag{3.43}$$

for all $\nu \in \mathbb{R}$ and $n > \ell$. Moreover, if $\nu > 0$, the convergence of the series in (3.43) can be extended to all $n \geq \ell$ by inspection of the asymptotic behaviour of the summand. Indeed, one deduces from (3.23) and the Stirling formula, see [21], that

$$X_{2m}^{(\ell)} \sim 2(-1)^\ell \ell^{2m} \quad \text{and} \quad \binom{\nu}{m} \sim \frac{1}{\Gamma(-\nu)}\frac{(-1)^m}{m^{\nu+1}}$$

as $m \to \infty$. Therefore, the non-vanishing even summands of (3.43) behave as

$$\binom{\nu}{2m}\frac{X_{2m}^{(\ell)}}{n^{2m-\nu}} \sim \frac{2n^\nu}{\Gamma(-\nu)}\frac{(-1)^\ell}{(2m)^{\nu+1}}\left(\frac{\ell}{n}\right)^{2m},$$

as $m \to \infty$ and the series in (3.43) converges even for $n = \ell$.

a) *Proof of claim (i)*: Recalling that

$$\mathfrak{g}_n^{(\ell)} = \sum_{j=1}^{\ell} s(\ell, j) n^{j-1/2}$$

for all $n \geq 0$, see (3.41), and invoking the expansion (3.43), we find that

$$\begin{aligned}(-\Delta)^\ell \mathfrak{g}_n^{(\ell)} &= \sum_{j=1}^{\ell} s(\ell,j)(-\Delta)^\ell n^{j-1/2} = \sum_{j=1}^{\ell} s(\ell,j)\sum_{m=2\ell}^{\infty}\binom{j-1/2}{m}\frac{X_m^{(\ell)}}{n^{m-j+1/2}} \\ &= n^{\ell-1/2}\sum_{m=2\ell}^{\infty}\sum_{j=0}^{\ell-1}\binom{\ell-j-1/2}{m} s(\ell,\ell-j)\frac{X_m^{(\ell)}}{n^{m+j}}\end{aligned}$$

for all $n \geq \ell$. Therefore, for the weight $\rho^{(\ell)}$, we have

$$\begin{aligned}\rho_n^{(\ell)} = \frac{(-\Delta)^\ell \mathfrak{g}_n^{(\ell)}}{\mathfrak{g}_n^{(\ell)}} &= \frac{n^{\ell-1}}{(n-1)\dots(n-\ell+1)}\sum_{m=2\ell}^{\infty}\sum_{j=0}^{\ell-1}\binom{\ell-j-1/2}{m} s(\ell,\ell-j)\frac{X_m^{(\ell)}}{n^{m+j}} \\ &= \frac{n^{\ell-1}}{(n-1)\dots(n-\ell+1)}\sum_{m=2\ell}^{\infty}\sum_{j=m}^{\infty}\binom{\ell+m-j-1/2}{m} s(\ell,\ell+m-j)\frac{X_m^{(\ell)}}{n^j} \\ &= \frac{n^{\ell-1}}{(n-1)\dots(n-\ell+1)}\sum_{j=2\ell}^{\infty}\left[\sum_{m=2\ell}^{j}\binom{\ell+m-j-1/2}{m} s(\ell,\ell+m-j)X_m^{(\ell)}\right]\frac{1}{n^j}\end{aligned}$$

for all $n \geq \ell$. Using also the definition (3.26), we infer that

$$\rho_n^{(\ell)} = \frac{n^{\ell-1}}{(n-1)\dots(n-\ell+1)} \sum_{j=2\ell}^{\infty} \frac{r_j^{(\ell)}}{n^j}$$

for all $n \geq \ell$. The final step is to expand the prefactor in front of the sum in the above expression in terms of negative powers of n, which is to be done using the Stirling numbers of the second kind (3.22). Namely, by [20, Eq. (26.8.11)], we have

$$\frac{n^{\ell-1}}{(n-1)\dots(n-\ell+1)} = \sum_{j=0}^{\infty} \frac{S(j+\ell-1,\ell-1)}{n^j}$$

for all $n \geq \ell \geq 1$. Altogether, we find that

$$\begin{aligned}\rho_n^{(\ell)} &= \sum_{j=0}^{\infty} \frac{S(j+\ell-1,\ell-1)}{n^j} \sum_{j=0}^{\infty} \frac{r_{j+2\ell}^{(\ell)}}{n^{j+2\ell}} = \sum_{k=0}^{\infty} \left[\sum_{m=0}^{k} S(k-m+\ell-1,\ell-1) r_{m+2\ell}^{(\ell)} \right] \frac{1}{n^{k+2\ell}} \\ &= \sum_{k=2\ell}^{\infty} \left[\sum_{m=2\ell}^{k} S(k-m+\ell-1,\ell-1) r_m^{(\ell)} \right] \frac{1}{n^k}\end{aligned}$$

for all $n \geq \ell$, from which we extract the formula (3.28) for the coefficients $A_k^{(\ell)}$, thus completing the proof of claim (i).

b) *Proof of claim (ii)*: Since $\rho^{(1)} = \rho^{\mathrm{KPP}}$, see (2.2), for $\ell = 1$, the claim is an immediate consequence of the explicitly known expansion (2.6), from which we deduce that

$$A_{2k+1}^{(1)} = 0 \quad \text{and} \quad A_{2k}^{(1)} = \binom{4k}{2k} \frac{1}{(4k-1)2^{4k-1}} > 0$$

for all $k \in \mathbb{N}$.

Suppose $\ell \geq 2$. Recalling the definition (3.22), we have $S(k-m+\ell-1,\ell-1) > 0$ for all $\ell \geq 2$ and $k \geq m$, hence in order to prove that $A_k^{(\ell)} > 0$ for all $k \geq 2\ell$, it suffices to show, that $r_m^{(\ell)} > 0$ for all $m \geq 2\ell$ which is done in the rest of this part. Recalling formulas (3.21) and (3.25), we find by inspection of the sign of each of the three terms in the sum from (3.26) that

$$(-1)^\ell X_j^{(\ell)} \geq 0, \quad (-1)^{j+m} s(\ell,\ell+j-m) \geq 0, \quad (-1)^{\ell+m} \binom{\ell+j-m-1/2}{j} \geq 0 \tag{3.44}$$

for all $m \geq 2\ell$ and $m \geq j \geq 2\ell$. Taking also into account that $X_j^{(\ell)} = 0$ whenever j is odd, we see that each summand from the sum for coefficients $r_m^{(\ell)}$ in (3.26) is non-negative. Consequently, $r_m^{(\ell)} \geq 0$ for all $m \geq 2\ell$. Moreover, we can estimate $r_m^{(\ell)}$ from below by the last non-vanishing summand which corresponds to index $j = m$ if m is even and $j = m-1$ if m is odd. If $m \geq 2\ell$ is even, then

$$r_m^{(\ell)} \geq \binom{\ell-1/2}{m} s(\ell,\ell) X_m^{(\ell)} = \left| \binom{\ell-1/2}{m} X_m^{(\ell)} \right| > 0,$$

by (3.25). If $m \geq 2\ell$ is odd, then

$$r_m^{(\ell)} \geq \binom{\ell-3/2}{m-1} s(\ell,\ell-1) X_{m-1}^{(\ell)} = \left| \binom{\ell-3/2}{m-1} \binom{\ell}{2} X_{m-1}^{(\ell)} \right| > 0,$$

by (3.25) again and the fact that $s(\ell,\ell-1) = -\ell(\ell-1)/2$. In total, we verify that $r_m^{(\ell)} > 0$ for all $m \geq 2\ell$ and the proof of claim (ii) is complete.

c) *Proof of claim (iii)*: The case for $\ell = 1$ is an immediate consequence of the expansion (2.6). Suppose $n \geq \ell \geq 2$. The inequality

$$\frac{(-\Delta)^{\ell} n^{\ell-1/2}}{n^{\ell-1/2}} > \left(\frac{1}{2}\right)_{\ell}^{2} \frac{1}{n^{2\ell}}$$

has been already proven in [15]. It can be also deduced by using (3.43) with $\nu = \ell - 1/2$ and noticing that each summand corresponding to m odd is vanishing while each summand corresponding to m even is positive, see (3.44). Then we may estimate

$$\frac{(-\Delta)^{\ell} n^{\ell-1/2}}{n^{\ell-1/2}} > \binom{\ell - 1/2}{2\ell} \frac{X_{2\ell}^{(\ell)}}{n^{2\ell}} = \left(\frac{1}{2}\right)_{\ell}^{2} \frac{1}{n^{2\ell}}$$

by (3.25) and a little algebra.

Next, we verify the inequality

$$\rho_n^{(\ell)} > \frac{(-\Delta)^{\ell} n^{\ell-1/2}}{n^{\ell-1/2}}.$$

We show that the summands in

$$(-\Delta)^{\ell} \mathfrak{g}_n^{(\ell)} = \sum_{j=1}^{\ell} s(\ell, j)(-\Delta)^{\ell} n^{j-1/2}$$

are all positive. First, recall that $(-1)^{\ell+j} s(\ell, j) > 0$ for all $j \in \{1, \dots, \ell\}$, see (3.21). Second, Lemma 3.17 implies the existence of $\xi \in (n - \ell, n + \ell)$, hence $\xi > 0$, such that

$$(-\Delta)^{\ell} n^{j-1/2} = (-1)^{\ell} \frac{\mathrm{d}^{2\ell}}{\mathrm{d}x^{2\ell}}\bigg|_{x=\xi} x^{j-1/2} = (-1)^{\ell} \prod_{k=1}^{2\ell} \left(j + \frac{1}{2} - k\right) \xi^{j-2\ell-1/2}.$$

It follows that $(-1)^{j+\ell}(-\Delta)^{\ell} n^{j-1/2} > 0$ for all $j \in \{1, \dots, \ell\}$. Consequently, we may estimate

$$(-\Delta)^{\ell} \mathfrak{g}_n^{(\ell)} > s(\ell, \ell)(-\Delta)^{\ell} n^{\ell-1/2} = (-\Delta)^{\ell} n^{\ell-1/2},$$

from which we conclude that

$$\rho_n^{(\ell)}(\mathfrak{g}) = \frac{(-\Delta)^{\ell} \mathfrak{g}_n^{(\ell)}}{\mathfrak{g}_n^{(\ell)}} > \frac{n^{\ell-1}}{(n-1)\dots(n-\ell+1)} \frac{(-\Delta)^{\ell} n^{\ell-1/2}}{n^{\ell-1/2}} > \frac{(-\Delta)^{\ell} n^{\ell-1/2}}{n^{\ell-1/2}}.$$

The proof of Theorem 3.14 is complete. □

3.3 More general families of discrete Hardy weights of higher order

The concrete parameter sequence $\mathfrak{g}^{(\ell)}$, defined by the formula (3.19), was considered in Theorem 3.11 because it gives rise to the corresponding optimal weight $\rho^{(\ell)}$ which is of a relatively simple form. However, these are not the only optimal discrete Hardy weights. In this section, we intend to emphasize that the abstract formulation of Theorems 3.6, 3.8, and 3.10 can be used to derive more general discrete Hardy weights of higher order.

3.3.1 q-generalized Hardy weights of higher order

In Theorem 3.10, we restricted ourselves to parametric sequences that satisfy $\mathfrak{g}_n \sim n^{\ell-1/2}$, as $n \to \infty$, see (A4). However, an analogous claim holds even if we let the asymptotic behaviour of $\mathfrak{g}_n$, as $n \to \infty$, depend on a real parameter $s \in (0,1)$.

Proposition 3.20. *Let $\ell \in \mathbb{N}$. Suppose (A1), (A2), (A3), and, in addition, suppose that $\mathfrak{g}$ admits the asymptotic expansion*

$$\mathfrak{g}_n = \sum_{j=0}^{2\ell} \alpha_j n^{\ell-s-j} + \mathcal{O}\left(n^{-\ell-s-1}\right) \text{ for some } \alpha_j \in \mathbb{R} \text{ with } \alpha_0 \neq 0 \text{ and } s \in (0,1), \tag{A4'}$$

as $n \to \infty$. Then the discrete Hardy weight $\rho(\mathfrak{g}) = (-\Delta)^\ell \mathfrak{g}/\mathfrak{g}$ of order ℓ is critical if $s \in [1/2, 1)$, optimal near infinity if $s = 1/2$, and non-attainable if $\mathfrak{g}$ meets the assumption (A5) and $s \in (0, 1/2]$.

Proof. The proof of Proposition 3.20 can be done in a similar manner as the proof of Theorem 3.10 in Section 3.2.4.

a) *Criticality*: Suppose $s \in [1/2, 1)$. The proof of criticality proceeds analogously as the proof of criticality in Section 3.2.4 with the only difference being the resulting estimate

$$\left|R_k^{(\ell)}(\mathfrak{g}) u_n^N\right| \lesssim \frac{1}{n^s \ln N},$$

that can be further estimated from above by $1/(\sqrt{n}\ln N)$, for all $n \geq \ell$ and $N \geq 2$, as in (3.38) nonetheless. The details are left as an exercise for the reader.

The second way of proving the criticality is to restrict ourselves to the case $s \in (1/2, 1)$ since the claim for $s = 1/2$ is a part of Theorem 3.10. Now, similarly as in the proof of criticality in Section 3.2.4, suppose that the sequence $\tilde{\rho} = \{\tilde{\rho}_n\}_{n=\ell}^\infty$ is a discrete Hardy weight of order ℓ satisfying $\tilde{\rho}_n \geq \rho_n(\mathfrak{g})$ for all $n \geq \ell$. Using identity (3.12), with the extension provided by Proposition 3.19, for the weight $\rho(\mathfrak{g})$ together with the discrete Hardy inequality of order ℓ for the weight $\tilde{\rho}$ (which can be straightforwardly extended to all $u \in H^\ell$), we find that

$$0 \leq \sum_{n=\ell}^{\infty} \left(\tilde{\rho}_n - \rho_n(\mathfrak{g})\right) |u_n|^2 \leq \sum_{k=0}^{\ell-1} \mathcal{R}_k^{(\ell)}(\mathfrak{g}; u)$$

for all $u \in \mathcal{D}^\ell$. Invoking Lemma 3.16, we observe that $(-\Delta)^{\ell/2}\mathfrak{g}_n \sim n^{-s}$, hence $\mathfrak{g} \in \mathcal{D}^\ell$. Substituting $u = \mathfrak{g}$ into the above inequality, we conclude that $\tilde{\rho}_n = \rho_n(\mathfrak{g})$ for all $n \geq \ell$, since all the remainders on the right-hand side are annihilated by this particular choice of u.

b) *Optimality near infinity*: If $s = 1/2$ Proposition 3.20 becomes Theorem 3.10, hence the optimality near infinity was already proven in Section 3.2.4.

c) *Non-attainability*: Suppose $s \in (0, 1/2]$. Since $\mathfrak{g}$ satisfies the assumptions (A1) and (A5), we find, similarly as in the proof of non-attainability in Section 3.2.4, that if $u \in H^\ell$ fulfils (3.9) as equality with $\rho = \rho(\mathfrak{g})$, then $u_n = c\mathfrak{g}_n$ for some constant $c \in \mathbb{C}$ and all $n \geq \ell$. Moreover, by assumption (A4') and Lemma 3.16, we have

$$\rho_n(\mathfrak{g})\mathfrak{g}_n^2 \gtrsim \frac{1}{n^{2\ell}} n^{2\ell-2s}$$

for all $n \geq \ell$, which is not a summable sequence for $s \in (0, 1/2]$. Therefore, we conclude that $c = 0$, i.e. $u \equiv 0$, proving the non-attainability. ∎

Proposition 3.20 can be used to derive more Hardy weights of any order. For a parameter $q > 0$, we set

$$\mathfrak{g}_n^{(\ell)}(q) := n^q \prod_{j=1}^{\ell-1}(n-j) \tag{3.45}$$

for all $n \geq 0$ and $\mathfrak{g}_n^{(\ell)}(q) = 0$ if $n < 0$.

Proposition 3.21. *Let $\ell \in \mathbb{N}$ and $q \in (0,1)$. Then the sequence $\rho^{(\ell)}(q)$ given by*

$$\rho_n^{(\ell)}(q) \coloneqq \frac{(-\Delta)^\ell \mathfrak{g}_n^{(\ell)}(q)}{\mathfrak{g}_n^{(\ell)}(q)} \tag{3.46}$$

for all $n \geq \ell$, where $\mathfrak{g}^{(\ell)}(q)$ is defined by (3.45), is a discrete Hardy weight of order ℓ. Furthermore, $\rho^{(\ell)}(q)$ is critical if and only if $q \in (0, 1/2]$, non-attainable if and only if $q \in [1/2, 1)$ and optimal near infinity if and only if $q = 1/2$.

Remark 3.22. If $q = 1$, $\rho^{(\ell)}(1) \equiv 0$ for all $\ell \geq 1$ and therefore it is meaningful to consider only $q \in (0,1)$. In the case $\ell = 1$, weight $\rho^{(1)}(q)$ appeared already in [16]. Clearly, $\mathfrak{g}^{(\ell)}$ defined in (3.19) corresponds to $q = 1/2$, thus $\rho^{(\ell)}(1/2) = \rho^{(\ell)}$ defined by (3.20). Moreover, Proposition 3.21 asserts that the weight $\rho^{(\ell)}(q)$ is optimal if and only if $q = 1/2$.

Proof. For $q \in (0,1)$, claims a) and b) from the proof of Theorem 3.11 in Section 3.2.5 can be verified in an analogous fashion. Consequently, $\mathfrak{g}^{(\ell)}(q)$ meets the assumptions (A1), (A2), and (A3), hence $\rho^{(\ell)}(q)$ is a discrete Hardy weight of order ℓ for all $q \in (0,1)$ by Theorem 3.8. Let us discuss the optimality of $\rho^{(\ell)}(q)$.

a) *Criticality*: Suppose $q \in (0, 1/2]$. Then the assumption (A4') holds for $\mathfrak{g}^{(\ell)}(q)$ with $s = 1 - q \in [1/2, 1)$, and therefore the weight $\rho^{(\ell)}(q)$ is critical by Proposition 3.20.

On the other hand, if $q \in (1/2, 1)$, $\rho^{(\ell)}(q)$ is not critical because, in this case,

$$\rho_n^{(\ell)}(q) < \rho_n^{(\ell)}(1/2) \text{ for all } n \geq \ell. \tag{3.47}$$

Indeed, in view of the definition (3.46) and the formula (3.41), statement (3.47) can be equivalently written as

$$\sum_{j=1}^{\ell} s(\ell, j)(-\Delta)^\ell n^{j-1+q} < \sum_{j=1}^{\ell} s(\ell, j) n^{q-1/2} (-\Delta)^\ell n^{j-1/2}$$

for all $n \geq \ell$. Recalling that $(-1)^{\ell+j} s(\ell, j) > 0$ for all $\ell \in \mathbb{N}$ and $j \in \{1, \dots, \ell\}$, it suffices to show that

$$(-1)^{\ell+j} (-\Delta)^\ell n^{j-1+q} < (-1)^{\ell+j} n^{q-1/2} (-\Delta)^\ell n^{j-1/2}$$

for all $n \geq \ell$ and $j \in \{1, \dots, \ell\}$. With the aid of the expansion (3.43) and the fact that $X_m^{(\ell)} = 0$ whenever m is odd, the above expression can be written as

$$(-1)^{\ell+j} n^{j-1+q} \sum_{m=\ell}^{\infty} \binom{j-1+q}{2m} \frac{X_{2m}^{(\ell)}}{n^{2m}} < (-1)^{\ell+j} n^{q-1/2} n^{j-1/2} \sum_{m=\ell}^{\infty} \binom{j-1/2}{2m} \frac{X_{2m}^{(\ell)}}{n^{2m}}.$$

Moreover, $(-1)^\ell X_{2m}^{(\ell)} > 0$ for all $m \geq \ell$ by (3.25), hence the inequality (3.47) follows from the fact that

$$(-1)^j \binom{j-1+q}{2m} < (-1)^j \binom{j-1/2}{2m}$$

for all $m \geq \ell$ and $j \in \{1, \dots, \ell\}$. This can be verified, realizing that both expressions are positive, see (3.44), and that $(q+k)(1-q+k) < (k+1/2)^2$, which holds true for all $q \in (1/2, 1)$ and $k \in \mathbb{N}_0$.

b) *Non-attainability*: Suppose $q \in [1/2, 1)$. Then the assumption (A4') holds for $\mathfrak{g}^{(\ell)}(q)$ with $s = 1 - q \in (0, 1/2]$, and therefore the weight $\rho^{(\ell)}(q)$ is non-attainable by Proposition 3.20.

Conversely, for $q \in (0, 1/2)$, the Hardy inequality of order ℓ (3.9), with the weight $\rho^{(\ell)}(q)$, holds as equality for $u = \mathfrak{g}^{(\ell)}(q)$, while both sides are finite. This stems from the fact that

$$(-\Delta)^{\ell/2}\mathfrak{g}_n^{(\ell)}(q) \sim \frac{1}{n^{1-q}}, \quad n \to \infty,$$

hence $\mathfrak{g}^{(\ell)}(q) \in \mathcal{D}^\ell$. Proposition 3.19 then implies, that the identity (3.12) holds for $u = \mathfrak{g}^{(\ell)}(q)$, while all the remainders on the right-hand side vanish. Moreover, with the aid of (3.43) and (3.25), we find that

$$\rho_n^{(\ell)}(q) = \binom{\ell - 1 + q}{2\ell}\frac{X_{2\ell}^{(\ell)}}{n^{2\ell}} + \mathcal{O}\left(\frac{1}{n^{2\ell+1}}\right) = \frac{(q)_\ell(1-q)_\ell}{n^{2\ell}} + \mathcal{O}\left(\frac{1}{n^{2\ell+1}}\right), \tag{3.48}$$

as $n \to \infty$, hence

$$\rho_n^{(\ell)}(q)(\mathfrak{g}_n^{(\ell)}(q))^2 \sim \frac{1}{n^{2-2q}}$$

for all $n \geq \ell$, from which we infer that $\sqrt{\rho^{(\ell)}(q)}\mathfrak{g}^{(\ell)}(q) \in \mathcal{H}^\ell$ for all $q \in (0, 1/2)$.

c) *Optimality near infinity*: Theorem 3.11 asserts optimality near infinity of $\rho^{(\ell)}(1/2)$. The non-optimality near infinity of $\rho^{(\ell)}(q)$ for $q \neq 1/2$ is a consequence of the fact that the constant by the leading term in (3.48) is smaller than for $q = 1/2$, i.e.

$$(q)_\ell(1-q)_\ell < \left(\frac{1}{2}\right)_\ell^2$$

for $q \neq 1/2$. This can be seen from the definition of the Pochhammer symbol and from the inequality $(q+k)(1-q+k) < (k+1/2)^2$ once again. Consequently, for $q \neq 1/2$ fixed, we find $\varepsilon > 0$ sufficiently small, such that

$$(1+\varepsilon)\rho_n^{(\ell)}(q) \leq \rho_n^{(\ell)}$$

for all n sufficiently large. Then, for all $M \in \mathbb{N}$ sufficiently large and any $u \in \mathcal{H}_0^M$, we have

$$\sum_{n=\lceil \ell/2 \rceil}^{\infty} \left|(-\Delta)^{\ell/2}u_n\right|^2 \geq \sum_{n=M}^{\infty} \rho_n^{(\ell)} |u_n|^2 \geq (1+\varepsilon) \sum_{n=M}^{\infty} \rho_n^{(\ell)}(q) |u_n|^2,$$

contradicting (3.10). □

3.3.2 Countable sets of Hardy weights of higher order

Recall that the concrete parameter sequence $\mathfrak{g}^{(\ell)}$ defined by (3.19) meets all the assumptions of Theorems 3.6, 3.8, and 3.10 for all $\ell \in \mathbb{N}$. In fact, any family of parameter sequences with this property gives rise to a countably infinite set of new optimal discrete Hardy weights of order ℓ. In the next statement, $\mathfrak{g}^{[\ell]}$ stands for any family of parameter sequences with explicitly designated dependence on the integer ℓ.

Proposition 3.23. *Suppose that the sequences $\mathfrak{g}^{[\ell]}$ fulfill the assumptions (A1), (A2), and (A3) for all $\ell \in \mathbb{N}$. Then for any $k \in \mathbb{N}_0$, the sequence $\rho^{[\ell,k]}$, given by*

$$\rho_n^{[\ell,k]} := \frac{(-\Delta)^\ell \operatorname{div}^k \mathfrak{g}_n^{[\ell+k]}}{\operatorname{div}^k \mathfrak{g}_n^{[\ell+k]}}$$

for all $n \geq \ell$, is a discrete Hardy weight of order ℓ. If, in addition, $\mathfrak{g}^{[\ell]}$ satisfy the assumption (A4') for fixed $s \in (0,1)$ and all $\ell \in \mathbb{N}$, then the weight $\rho^{[\ell,k]}$ is critical if $s \in [1/2, 1)$, optimal near infinity if $s = 1/2$, and non-attainable if $\mathfrak{g}^{[\ell]}$ meet the assumption (A5) and $s \in (0, 1/2]$.

Proof. We will prove both claims of Proposition 3.23 separately.

a) First, we show that if the sequences $\mathfrak{g}^{[\ell+1]}$ fulfill all the assumptions (A1), (A2), and (A3) with $\ell+1$, then the sequences $\mathrm{div}\mathfrak{g}^{[\ell+1]}$ also meet these assumptions, but with ℓ. By induction, we find that, for all $k \in \mathbb{N}_0$, the sequences $\mathrm{div}^k\mathfrak{g}^{[\ell+k]}$ also satisfy the assumptions (A1), (A2), and (A3), hence Theorem 3.8 implies that $\rho^{[\ell,k]}$ are discrete Hardy weights of order ℓ for all $k \in \mathbb{N}_0$.

Suppose $\ell \in \mathbb{N}$. Evidently, the assumption (A1) for $\mathfrak{g}^{[\ell+1]}$ (with $\ell+1$) implies that $\mathrm{div}\mathfrak{g}^{[\ell+1]}$ also satisfies the assumption (A1) (with ℓ). Analogously, inequalities of assumption (A2) for $\mathfrak{g}^{[\ell+1]}$ include the respective inequalities of (A2) for $\mathrm{div}\mathfrak{g}^{[\ell+1]}$.

Moreover, the assumption (A2) for $\mathfrak{g}^{[\ell+1]}$ with $k=1$, yields inequalities

$$(-\Delta)^\ell \mathrm{div}\mathfrak{g}_n^{[\ell+1]} \geq 0 \text{ for all } n \geq \ell+1. \tag{3.49}$$

In order to deduce it also for $n=\ell$, and hence to check non-strict inequalities in the assumption (A3) for $\mathrm{div}\mathfrak{g}^{[\ell+1]}$, we need to apply (A3) to $\mathfrak{g}^{[\ell+1]}$, which can be written as the inequality

$$-\nabla(-\Delta)^\ell \mathrm{div}\mathfrak{g}_n^{[\ell+1]} > 0$$

for all $n \geq \ell+1$. It follows that, for all $n \geq \ell+1$, we have

$$(-\Delta)^\ell \mathrm{div}\mathfrak{g}_{n-1}^{[\ell+1]} > (-\Delta)^\ell \mathrm{div}\mathfrak{g}_n^{[\ell+1]}, \tag{3.50}$$

which, together with (3.49), implies that the inequality (3.49) holds also with $n=\ell$. To complete part a) of this proof, we must show that the inequalities in (3.49) are strict for all $n \geq \ell$, and therefore that $\mathrm{div}\mathfrak{g}^{[\ell+1]}$ meets the assumption (A3). Indeed, from (3.50), we infer that if there exists $n_0 \geq \ell$, such that $(-\Delta)^\ell \mathrm{div}\mathfrak{g}_{n_0}^{[\ell+1]} = 0$, then for all $n > n_0$, we have $(-\Delta)^\ell \mathrm{div}\mathfrak{g}_n^{[\ell+1]} < 0$ contradicting (3.49).

b) It is easy to see that, if $\mathfrak{g}^{[\ell+1]}$ admits the expansion (A4') with $\ell+1$, $\alpha_0 \neq 0$, and $s \in (0,1)$, then $\mathrm{div}\mathfrak{g}^{[\ell+1]}$ fulfils (A4') with the same s and α_0 replaced by $(\ell+1-s)\alpha_0 \neq 0$. In order to finish the proof of the second part of Proposition 3.23, it suffices to show that if $\mathfrak{g}^{[\ell+1]}$ meets also the assumption (A5), then the sequence $\mathrm{div}\mathfrak{g}^{[\ell+1]}$ satisfies (A5) too. The rest readily follows by induction and Proposition 3.20.

Indeed, if $\ell=1$, then the assumption (A5) (with $\ell=1$) is trivially true for $\mathrm{div}\mathfrak{g}^{[\ell+1]}$. Furthermore, for $\ell \geq 2$, the assumption (A2) applied to $\mathfrak{g}^{[\ell+1]}$ with $k=2$ yields

$$(-\Delta)^{\ell-1}\mathrm{div}^2\mathfrak{g}_n^{[\ell+1]} \geq 0 \tag{3.51}$$

for all $n \geq \ell$. In order to show that inequalities (3.51) are actually strict, we proceed analogously as in part a) of this proof. Suppose that there exists $n_0 \geq \ell$ such that $(-\Delta)^{\ell-1}\mathrm{div}^2\mathfrak{g}_{n_0}^{[\ell+1]} = 0$. The assumption (A5) imposed on $\mathfrak{g}^{[\ell+1]}$ tells us that

$$(-\Delta)^{\ell-1}\mathrm{div}^2\mathfrak{g}_n^{[\ell+1]} < (-\Delta)^{\ell-1}\mathrm{div}^2\mathfrak{g}_{n-1}^{[\ell+1]}$$

for all $n \geq \ell+1$, hence $(-\Delta)^{\ell-1}\mathrm{div}^2\mathfrak{g}_n^{[\ell+1]} < 0$ for all $n > n_0$, contradicting (3.51). □

Proposition 3.23 is evidently applicable for the parameter-dependent sequences (3.45). The following extension of Proposition 3.21 yields a countable set of Hardy weights of any order for any $q \in (0,1)$. Since the proof of *attainability* and *non-optimality near infinity* in Proposition 3.21 relies on the asymptotic behaviour of $\mathfrak{g}^{(\ell)}(q)$ and $\rho^{(\ell)}(q)$, defined by (3.45) and (3.46) respectively, these properties can be verified in an analogous fashion, and therefore the proof will be omitted.

Proposition 3.24. *Let $\ell \in \mathbb{N}$. For any $k \in \mathbb{N}_0$ and $q \in (0,1)$, the sequence $\rho^{(\ell,k)}(q)$, given by*

$$\rho_n^{(\ell,k)}(q) := \frac{(-\Delta)^\ell \mathrm{div}^k \mathfrak{g}_n^{(\ell+k)}(q)}{\mathrm{div}^k \mathfrak{g}_n^{(\ell+k)}(q)}$$

for all $n \geq \ell$, where $\mathfrak{g}^{(m)}(q)$ is defined by (3.45) for any $m \in \mathbb{N}$, is a discrete Hardy weight of order ℓ. Furthermore, $\rho^{(\ell,k)}(q)$ is critical if $q \in (0, 1/2]$, non-attainable if and only if $q \in [1/2, 1)$ and optimal near infinity if and only if $q = 1/2$.

Remark 3.25. Evidently, Proposition 3.24 asserts Proposition 3.21, for any $\ell \in \mathbb{N}$, since $\rho^{(\ell,0)}(q) = \rho^{(\ell)}(q)$. Furthermore, for $q = 1/2$, we obtain a countable set of explicit optimal discrete Hardy weights of any order. For example, for $\ell = 2$ and $k = 1$, we have the optimal discrete Rellich weight

$$\rho_n^{(2,1)}(1/2) = \frac{(-\Delta)^2 \operatorname{div}\mathfrak{g}_n^{(3)}}{\operatorname{div}\mathfrak{g}_n^{(3)}} = \frac{9}{16n^4} + \frac{63}{40n^5} + \frac{9357}{3200n^6} + \mathcal{O}\left(\frac{1}{n^7}\right).$$

Notice that the coefficient $63/40$ by the second term improves upon the analogous term $3/2$ in the expansion of $\rho^{(2)}$, see claim (iii) of Remark 3.15. On the other hand, for the weight $\rho^{(2,1)}(1/2)$ it does not hold, that all the terms in the expansion are non-negative (unlike for $\rho^{(2)}$).

3.3.3 Multi-parameter families of optimal discrete Hardy weights of higher order

For $\ell \geq 2$, more optimal weights generalizing (3.20) in $(\ell-1)$-parameters can be found. The basic idea for their detection is reminiscent of the one developed in [10], where the authors related Hardy weights ($\ell = 1$) to positive harmonic functions. For $\ell \geq 2$, we seek poly-harmonic functions, i.e. solutions of the equation

$$(-\Delta)^\ell \mathfrak{h}_n = 0$$

for all $n \geq \ell$, satisfying the boundary condition $\mathfrak{h}_0 = \cdots = \mathfrak{h}_{\ell-1} = 0$, and then we take $\mathfrak{g} := \sqrt{\mathfrak{h}}$, provided that $\mathfrak{h} \geq 0$, as a candidate for the parameter sequence. Up to a multiplicative constant, a general solution $\mathfrak{h}$ of this problem can be expressed as

$$\mathfrak{h}_n = \prod_{j=0}^{\ell-1}(n-j) \prod_{k=1}^{\ell-1}(n-\alpha_k), \tag{3.52}$$

where $\alpha_1, \dots, \alpha_{\ell-1} \in \mathbb{R}$ are parameters.

Notice, that if $\alpha_k = k$, then $\sqrt{\mathfrak{h}}$ coincides with the sequence $\mathfrak{g}^{(\ell)}$ given by (3.19). However, for general $\alpha_1, \dots, \alpha_{\ell-1}$, we find it difficult to formulate further restrictions, directly in terms of the parameters, so that the sequence $\mathfrak{g} = \sqrt{\mathfrak{h}}$ would satisfy the assumptions (A1), (A2), and (A3). Nevertheless, the verification of claims a) and b) from Section 3.2.5 for the sequence $\mathfrak{g}^{(\ell)}$ and perturbation arguments imply that the set of admissible values of $\alpha_1, \dots, \alpha_{\ell-1}$ contains other solutions than the one corresponding to the particular parameter sequence $\mathfrak{g}^{(\ell)}$.

As far as the optimality is concerned, notice that the assumption (A4) always holds for $\mathfrak{g} = \sqrt{\mathfrak{h}}$, with $\mathfrak{h}$ given by (3.52), by the generalized binomial theorem. Therefore, the resulting candidate weight $\rho(\mathfrak{g}) = (-\Delta)^\ell \mathfrak{g}/\mathfrak{g}$ is critical and optimal near infinity. The non-attainability of $\rho(\mathfrak{g})$ is again a question of additional restrictions of the parameters $\alpha_1, \dots, \alpha_{\ell-1}$ guaranteeing the assumption (A5).

We illustrate the situation in the still relatively simple case $\ell = 2$ when our candidate is

$$\mathfrak{g}_n(\alpha) := \sqrt{n(n-1)(n-\alpha)}.$$

Assumption (A1) requires $\mathfrak{g}_{n+1}(\alpha) > \mathfrak{g}_n(\alpha) > 0$ for all $n \geq 2$. The positivity of $\mathfrak{g}_n(\alpha)$ for all $n \geq 2$ induces the restriction $\alpha < 2$ which is also sufficient for the monotonicity $\mathfrak{g}_{n+1}(\alpha) > \mathfrak{g}_n(\alpha)$ for all $n \geq 2$. Assumptions (A5) (hence also (A2)) and (A3) amount to inequalities $0 < (-\Delta)\operatorname{div}\mathfrak{g}_n(\alpha) < (-\Delta)\operatorname{div}\mathfrak{g}_{n-1}(\alpha)$ for all $n \geq 2$, from which only the second inequality introduces new restrictions on α since

$$(-\Delta)\operatorname{div}\mathfrak{g}_n(\alpha) = \frac{3}{8n^{3/2}} + \mathcal{O}\left(\frac{1}{n^{5/2}}\right), \qquad \text{as } n \to \infty.$$

Thus, the final range for $\alpha < 2$ is determined by the requirement $(-\Delta)^2\mathfrak{g}_n(\alpha) > 0$ for all $n \geq 2$. However, it seems difficult to find the solution analytically. Nevertheless, numerically we get the approximate range $0.847 < \alpha < 1.307$ (a suitable CAS such as Wolfram Mathematica is capable of expressing the lower and upper bounds in radicals). Thus, we conclude that for any α approximately within this range, the weight

$$\rho^{(2)}(\alpha) \coloneqq \frac{(-\Delta)^2\mathfrak{g}(\alpha)}{\mathfrak{g}(\alpha)}$$

is an optimal Rellich weight by Theorem 3.10.

Conclusion

In this thesis, we first studied the classical discrete Hardy inequality (1) in Chapter 1, summarizing its existing proofs and the historical circumstances leading to its discovery. In Chapter 2, we looked into the recent development of the improved optimal discrete Hardy inequality (2.1). Finally, in Chapter 3, we tackled the contemporary problem of optimal discrete Hardy inequalities of higher order. As our main result, we discovered the optimal Hardy weight (3.20) of an arbitrary order $\ell \in \mathbb{N}$. For $\ell = 1$, we rediscovered the optimal Hardy weight (2.2) of Keller–Pinchover–Pogorzelski. For $\ell = 2$, we improved upon the best known Rellich weights due to Gerhat–Krejčiřík–Štampach (3.4) and Huang–Ye (3.6). For $\ell \geq 3$, we proved the conjecture (3.5) by Gerhat–Krejčiřík–Štampach and improved the classical discrete weights (3.2) due to Huang–Ye to optimal weights. Moreover, by means of Theorems 3.6, 3.8, and 3.10, we provided a way of detecting and constructing additional optimal weights. Some examples were given in Section 3.3.

Nevertheless, it is also important to mention some related questions and possible open problems. For example, as we mentioned in Remark 3.7, there are numerous unanswered questions concerning weighted discrete Hardy-type inequalities such as the Knopp inequality, which was studied in [19], or the Hardy inequality with shifting weight investigated in [13]. As of our knowledge, optimal versions of these inequalities remain an unsolved mystery, for which identity (3.14) might be a potent tool. Lastly, optimal Hardy weights (of any order) in a general ℓ^p setting are yet to be found, see [12] and [13] for some improvements in this regard, however it seems that the approach from Chapter 3 might not be viable.

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