Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Spectrum of the discrete bilaplace operator with complex potential

Spektrum diskrétního bilaplaceova operátoru s komplexním potenciálem

Research project

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ZADÁNÍ VÝZKUMNÉHO ÚKOLU

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Pokyny pro vypracování:

1) Pokuste se dokázat/vyvrátit hypotézu o optimálních spektrálních obálkách ze své bakalářské práce.
2) Vyšetřete analytické a topologické vlastnosti nalezených spektrálních obálek diskrétního bilaplaceova operátoru s komplexním potenciálem.
3) Proveďte rešerši důkazů neexistence vložených vlastních hodnot v esenciálním spektru diskrétního Schrödingerova operátoru s jadernou poruchou [1], příp. [2].
4) Diskutujte možnosti analýzy vložených vlastních hodnot v esenciálním spektru diskrétního bilaplaceova operátoru s jadernou poruchou.

Doporučená literatura:

1) L. Golinskii, A remark on the discrete spectrum of non-self-adjoint Jacobi operators, preprint, 2021, arXiv:2101.01974v1.

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3) J. Blank, P. Exner, M. Havlíček, Lineární operátory v kvantové fyzice. Karolinum, Praha, 1993.

4) O. O. Ibrogimov, F. Štampach, Spectral enclosures for non-self-adjoint discrete Schrödinger operators, Integr. Equ. Oper. Theory 91, 2019, 1-15.

5) O. O. Ibrogimov, D. Krejčiřík, A. Laptev: Sharp bounds for eigenvalues of biharmonic operators with complex potentials in low dimensions, Math. Nachr. 294, 2021, 1333-1349.

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Author's declaration:
I declare that this Research project is entirely my own work and I have listed all the used sources in the bibliography.

Prague, August 21, 2023 Tomáš Hrdina

Název práce:

Spektrum diskrétního bilaplaceova operátoru s komplexním potenciálem

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Druh práce: Výzkumný úkol

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Abstrakt: Studujeme spektrum diskrétního bilplaceova operátoru s komplexním potenciálem. Doplňujeme výsledky autorovy bakaláské práce. Dokazujeme hypotézu na odhad Greenova jádra diskrétního bilplaciánu pomocí principu maxima modulu. Tento výsledek nám umožňuje získat optimální spektrální obálky pro porušený bilaplaceův operátor. Hraniční křivky těchto množin jsou analyticky a topologicky studovány a získané výsledky jsou v souladu s numerickými simulacemi. Dále provádíme rešerši důkazu absence vložených vlastních hodnot pro diskrétní Lapalcián a diskutujeme možnost zobecnění tohoto důkazu pro operátor bilplaceův.

Klíčová slova: Diskrétní bilaplacián, diskrétní Laplacián, potenciál, spektrum, spektrální obálky, vlastní čísla

Title:

Spectrum of the discrete bilaplace operator with complex potential

Author: Tomáš Hrdina

Abstract: We study spectrum of the discrete bilaplace operator with a complex potential. We complete the results from authors bachelor degree project. We prove the conjecture on the estimate of the Green kernel of discrete bilaplacian using Maximum modulus principle. It allows us to obtain optimal spectral enclosures for perturbed discrete bilaplace operator. Boundary curves of these enclosures are topologically and analytically analyzed. The analysis is in accordance with results obtained using numerical simulation. We make the recherche of the proof of absence of eigenvalue of the discrete Laplace operator with a complex potential in the interior of its essential spectrum and then discuss the possibility of generalizing this proof for the discrete bilaplace operator.

Key words: Discrete bilaplcian, discrete Laplacian, eigenvalues, potential, spectrum, spectral enclosures

Contents

Introduction

This project is very closely related to my bachelor degree project, where the spectral enclosures for discrete Lapalcian and bilaplacian with a complex potential were derived. Let us now briefly recall some key steps.

We are dealing with a difference operator T defined as

$$Te_n = -e_{n+1} + 2e_n - e_{n-1}, \ \forall n \in \mathbb{Z},$$

where $\{e_n\}_{n\in\mathbb{Z}}$ is standard orthonormal basis of $\ell^2(\mathbb{Z})$. The discrete bilaplacian is a second power of operator T

$$T^2 e_n = e_{n+2} - 4e_{n+1} + 6e_n - 4e_{n-1} + e_{n-2}, \ \forall n \in \mathbb{Z}.$$

Our aim is to analyze spectrum of an operator $T^2 + V$, where

$$Ve_n = v_n e_n, \quad \forall n \in \mathbb{Z}$$

and $v \in \ell^1(\mathbb{Z})$.

Using the Birman-Schwinger principle it was obtained that

$$\sigma(T^2 + V) \subset \left\{ \lambda \in \mathbb{C} \setminus [0, 16] : \left| \frac{2\lambda(\lambda - 16)}{1 - \sqrt{1 - \frac{16}{\lambda}}} \right| \leq \|v\|^2_{\ell^1(\mathbb{Z})} \right\} \cup [0, 16].$$

All details could be find in [3]. These sets are called **spectral enclosures**. The spectrum of the perturbed operator is definitely a subset of the spectral enclosure for a given norm of sequence v. In the Bachelor degree project, a conjecture was considered. For Birman-Schwinger principle, it was necessary to estimate the Green kernel of the operator T^2. There were some simple possibilities which allowed us to obtain spectral enclosures which were not optimal. The conjecture was on an estimate of the Green kernel which allowed us to prove, that the spectral enclosures obtained using this estimate were optimal.

The conjecture on the estimate of the Green kernel of T^2 will be proven in the first chapter. Some basic analytic and topological properties of the enclosures will be shown in the second chapter. Finally, in the third chapter the recherche of the proof of the absence of an eigenvalue of the trace-class perturbed discrete Laplacian in the interior of its essential spectrum will be done. Then, the there will be made a discussion about the generalization of this proof for the discrete bilaplacian T^2.

Chapter 1

Optimal spectral enclosures for the discrete bilaplacian

In this chapter, the conjecture on the estimate of the Green kernel of T^2 will be proven. According to my Bachelor degree project [3], where the conjecture was first introduced, we want to show the following inequality

$$\left|(T^2-\lambda)^{-1}_{m,n}\right| \leq \left|(T^2-\lambda)^{-1}_{0,0}\right|, \ \forall m,n \in \mathbb{Z}. \tag{1.1}$$

The Green kernel was obtain using a transformation of the number λ by a transform

$$\zeta(k) = k^{-2} - 4k^{-1} + 6 - 4k + k^2,$$

which is a bijection from the set $\mathbb{D}_+^\zeta := \{k \in \mathbb{C} : \ |k| < 1, \ \mathrm{Im}(k) > 0\} \cup (-1,0)$ onto $\mathbb{C} \setminus [0,16]$. The formula for the Green kernel is as follows

$$(T^2-\lambda)^{-1}_{m,n} = \frac{k}{2(k-1)^2}\left(\frac{k^{|m-n|}}{k-k^{-1}} - \frac{z_-^{|m-n|}}{z_- - z_-^{-1}}\right), \tag{1.2}$$

where $z_- = z_-(k)$ is a root of the polynomial $p(z) = z^2 + z\left(k-4+k^{-1}\right)+1$. We can divide the inequality (1.1) by the positive number $|k/(2(k-1)^2)|$ and we can also denote $s := |m-n|$ which turns this inequality into a problem with one nonegative integer parameter.

1.1 Proof of the conjecture on optimal enclosures

The aim of this section is to prove following conjecture from my bachelor project. Here, $\mathbb{D}_\pm$ are simply the upper and the lower half of the unit circle respectively.

Conjecture 1.1. Let $k \in \mathbb{D}_+^\zeta$ and let z_- be a root of the polynomial $p(z) = z^2+z\left(k-4+k^{-1}\right)+1$ such that $|z_-| < 1$. Then

$$\forall s \in \mathbb{N}_0 : \quad \left| k^s\left(z_- - z_-^{-1}\right) - z_-^s\left(k-k^{-1}\right)\right| \leq \left|\left(z_- - z_-^{-1}\right) - \left(k-k^{-1}\right)\right|.$$

Remark 1.2. It was proved in [3] that there exist such a root $z_-$of $p(z)$ for any $k \in \mathbb{D}_+^\zeta$.

Lemma 1.3. Let $k \in \mathbb{D}_+$ and let z_- be a root of polynomial $p(z) = z^2+z\left(k-4+k^{-1}\right)+1$ such that $|z_-| < 1$. Then $z_- \in \mathbb{D}_-$.

Proof. It holds

$$z^2 + z\left(k - 4 + k^{-1}\right) + k = (z - z_-)\left(z - z_-^{-1}\right).$$

Hence

$$\begin{aligned} &-(z_- + z_-^{-1}) = k - 4 + k^{-1}, \\ \Longrightarrow \quad &-\operatorname{Im}\left(z_- + z_-^{-1}\right) = \operatorname{Im}\left(k + k^{-1}\right). \end{aligned}$$

Now put $z_- := re^{\mathrm{i}\varphi}, r \in (0,1), \varphi \in (-\pi, \pi]$. Hence

$$\left(r - r^{-1}\right)\sin(\varphi) = -\operatorname{Im}\left(k + k^{-1}\right) > 0 \Longleftarrow k \in \mathbb{D}_+.$$

Thus

$$\underbrace{\left(r - r^{-1}\right)}_{<0, \forall r \in (0,1)} \underbrace{\sin(\varphi)}_{\text{must be}<0} > 0,$$

therefore φ must be in $(-\pi, 0)$, which was our aim to show. Let us look closer on proposition $\operatorname{Im}\left(k + k^{-1}\right) < 0$. Put $k := se^{\mathrm{i}\psi} \in \mathbb{D}_+$. It follows that $s \in (0,1)$ and $\psi \in (0, \pi)$ and then

$$k + k^{-1} = \left(s + s^{-1}\right)\cos(\psi) + \mathrm{i}\left(s - s^{-1}\right)\sin(\psi).$$

Indeed $\operatorname{Im}\left(k + k^{-1}\right) < 0$.

□

Lemma 1.3 implies that it is sufficient to prove the following theorem , which is actually more general.

Theorem 1.4. Let $s \in \mathbb{N}_0$, then

$$\forall u \in \mathbb{D}_+, \forall v \in \mathbb{D}_- : \quad \mid u^s\left(v - v^{-1}\right) - v^s\left(u - u^{-1}\right) \mid \leq \mid \left(v - v^{-1}\right) - \left(u - u^{-1}\right) \mid .$$

To prove the Theorem 1.4 we will use Maximum modulus principle (MMP).

Theorem 1.5. (Maximum Modulus Principle).
Let Ω be a connected open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ holomorphic on Ω and $f \neq$ const. Then $|f|$ cannot exhibit a strict local maximum in Ω.

Corollary 1.6. Let Ω be a bounded connected open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ holomorphic on Ω and continuous on $\bar{\Omega}$. Then

$$\max_{\bar{\Omega}} |f| = \max_{\partial\Omega} |f|$$

Remark 1.7. Consider $s \geq 0$ and the function

$$f_s(u,v) := \frac{u^s\left(v - v^{-1}\right) - v^s\left(u - u^{-1}\right)}{\left(v - v^{-1}\right) - \left(u - u^{-1}\right)} = \frac{u^{s+1}\left(v^2 - 1\right) - v^{s+1}\left(u^2 - 1\right)}{(1 + uv)(v - u)}.$$

Theorem 1.4 is equivalent to

$$|f_s(u,v)| \leq 1, \forall u \in \mathbb{D}_+, \forall v \in \mathbb{D}_-. \tag{1.3}$$

We choose a fixed $v \in \mathbb{D}_-$ arbitrarily and look closer to the function $f_s(\cdot, v)$, which is analytic on $\mathbb{D}_+$ and continuous on $\overline{\mathbb{D}}_+$. Indeed, it holds that the limit

$$\lim_{u \to v} f_s(u,v)$$

exists and is finite. And the second factor in the denominator of f_s could generate singularity if $1+uv=0$, but in this case $u=-1/v$, thus $u \notin \overline{\mathbb{D}_+}$, since $v \in \mathbb{D}_-$ and $|u| = |1/v| > 1$. We can reformulate (1.3) using MMP as follows

$$|f_s(u,v)| \le 1, \forall u \in \partial\mathbb{D}_+, \forall v \in \mathbb{D}_-. \tag{1.4}$$

Now, we take fixed $u \in \partial\mathbb{D}_+$ and we will analyze the function $f_s(u,\cdot)$. Function f is analytic on $\mathbb{D}_-$. indeed, only term in the definition of function f_s which could generate a singularity is $(1+uv)$. It follows that $v = -1/u$ and $u \in \partial\mathbb{D}_+$. We look closer on following situations:

$$\begin{aligned}
&u = e^{\mathrm{i}\phi}, \phi \in (0,\pi): \quad v = -\frac{1}{u} = e^{\mathrm{i}\phi} \notin \mathbb{D}_-, \\
&u \in (-1,1): \quad v = -\frac{1}{u} \notin \mathbb{D}_- \text{since } \left|\frac{1}{u}\right| > 1, \\
&u = \pm 1: \quad v = -\frac{1}{u} = \mp 1.
\end{aligned}$$

Functions $f_s(\pm 1, v)$ do not have singularities at ∓ 1. Thus we can simplify (1.4), using MPP again and we get

$$|f_s(u,v)| \le 1, \forall u \in \partial\mathbb{D}_+, \forall v \in \partial\mathbb{D}_-. \tag{1.5}$$

Based on the Remark 1.7, it is sufficient to prove following theorem.

Theorem 1.8. Let $s \in \mathbb{N}_0$, then

$$\forall u \in \partial\mathbb{D}_+, \forall v \in \partial\mathbb{D}_- : \quad \left| u^s \left(v - v^{-1}\right) - v^s \left(u - u^{-1}\right) \right| \le \left| \left(v - v^{-1}\right) - \left(u - u^{-1}\right) \right|. \tag{1.6}$$

Proof. The Theorem 1.8 will be proven using following strategy. We divide the boundaries of the upper and lower half of the unit circle into parts and then we will prove the theorem considering all the positions of u and v step by step.

In following, we assume

$$u = r_u e^{\mathrm{i}\phi_u}, \quad v = r_v e^{\mathrm{i}\phi_v}.$$

1. We consider the case, when both u and v are real. Without loss of generality, let us assume that $0 < r_u < r_v < 1$. If $r_u > r_v$ we can use the symmetry of u, v in this problem. If $0 > r_u > r_v > -1$ we can get the same problem as if the numbers are positive thanks to the absolute value in (1.6).

   We have $u = r_u, v = r_v$, thus the inequality is in the form

   $$\begin{aligned}
   \left|r_u^s \left(r_v - r_v^{-1}\right) - r_v^s \left(r_u - r_u^{-1}\right)\right| &\le \left|\left(r_v - r_v^{-1}\right) - \left(r_u - r_u^{-1}\right)\right|, \\
   \left|r_u^{s+1}\left(1 - r_v^2\right) - r_v^{s+1}\left(1 - r_u^2\right)\right| &\le \left|r_u\left(1 - r_v^2\right) - r_v\left(1 - r_u^2\right)\right|.
   \end{aligned}$$

   Let us denote RHS $:= r_v\left(1 - r_u^2\right) - r_u\left(1 - r_v^2\right)$ and LHS $:= r_v^{s+1}\left(1 - r_u^2\right) - r_u^{s+1}\left(1 - r_v^2\right)$. Since RHS ≥ 0 and LHS ≥ 0, $\forall s \in \mathbb{N}_0$, the problem is as follows

   $$\begin{aligned}
   0 \le \text{ RHS } - \text{ LHS } &= r_v\left(1 - r_u^2\right)\left(1 - r_v^s\right) - r_u\left(1 - r_v^2\right)\left(1 - r_u^s\right), \\
   0 &\le \frac{r_v\left(1 - r_v^s\right)}{1 - r_v^2} - \frac{r_u\left(1 - r_u^s\right)}{1 - r_u^2}.
   \end{aligned}$$

Consider the real function

$$\xi(x) = \frac{x\left(1 - x^s\right)}{1 - x^2},$$

which is increasing on interval $(0, 1)$. Hence we get $0 \leq \xi(y) - \xi(x)$ for any $x, y \in (0, 1), y > x$. If we put $x := r_u$ and $y := r_v$ the lemma is proved. It also follows from this proof that the inequality holds $\forall r_u, r_v \in (-1, 1)$. Indeed for odd s we can use directly previous part of this proof. For even s is function ξ odd and we can also prove it the same way. It remains to prove that the function ξ is increasing.

Indeed, for $s = 0$ it is clear, for $s \geq 1$ it holds

$$\xi(x) = \frac{x}{1 + x} \sum_{j=0}^{s-1} x^j$$

and thus

$$\begin{aligned}
\xi'(x) &= \frac{1}{(1 + x)^2} \left((1 + x) \sum_{j=0}^{s-1} (j + 1) x^j - \sum_{j=0}^{s-1} x^{j+1} \right) = \\
&= \frac{1}{(1 + x)^2} \left(\sum_{j=0}^{s-1} j x^j + \sum_{j=0}^{s-1} j x^{j+1} + \sum_{j=0}^{s-1} x^j \right) \geq 0,
\end{aligned}$$

for $x \in (0, 1)$. Which was to be proved.

2. Now consider u and v such that $r_u = r_v = 1$. Let us denote

$$\mathrm{g}(s) := \left| u^s \left(v - v^{-1}\right) - v^s \left(u - u^{-1}\right)\right|^2$$

We have $u = e^{\mathrm{i}\phi_u}$ and $v = e^{\mathrm{i}\phi_v}$, $\phi_u \in (0, \pi)$, $\phi_v \in (-\pi, 0)$ and using the definition of absolute value of the complex number and some trigonometric identities we obtain

$$\begin{aligned}
\mathrm{g}(s) &= \left| e^{\mathrm{i}s\phi_u} 2\mathrm{i} \sin \phi_v - e^{\mathrm{i}s\phi_v} 2\mathrm{i} \sin \phi_u \right|^2 = \\
&= 4 \left(\left(\sin\left(s\phi_u\right) \sin\left(\phi_v\right) - \sin\left(s\phi_v\right) \sin\left(\phi_u\right)\right)^2 - \left(\cos\left(s\phi_u\right) \sin\left(\phi_v\right) - \cos\left(s\phi_v\right) \sin\left(\phi_u\right)\right)^2 \right) = \\
&= 4 \left(\sin^2\left(\phi_u\right) + \sin^2\left(\phi_v\right) - 2 \sin\left(\phi_u\right) \sin\left(\phi_v\right) \underbrace{\left(\cos\left(s\phi_u\right) \cos\left(s\phi_v\right) + \sin\left(s\phi_u\right) \sin\left(s\phi_v\right)\right)}_{\cos(s(\phi_u - \phi_v))} \right).
\end{aligned}$$

Now it is not hard to verify that $\mathrm{g}(s) \leq \mathrm{g}(0)$. Let us analyze

$$\mathrm{g}(0) - \mathrm{g}(s) = -8 \underbrace{\sin\left(\phi_u\right)}_{\geq 0} \underbrace{\sin\left(\phi_v\right)}_{\leq 0} \underbrace{\left(1 - \cos\left(s\left(\phi_u - \phi_v\right)\right)\right)}_{2\sin^2\left(\frac{s}{2}(\phi_u - \phi_v)\right)} \geq 0.$$

Thus the inequality (1.6) holds for this range of u and v.

3. Now consider number v to be real and positive, u to be complex. It is $r_u = 1, r_v \in [0,1]$ and $\phi_u \in (0,\pi), \phi_v = 0$. Denote

$$\mathrm{h}(s) := \left| u^s \left(v - v^{-1}\right) - v^s \left(u - u^{-1}\right)\right|^2$$

As in the previous parts, we have $u = e^{\mathrm{i}\phi_u}$ and $v = r_v$ and we obtain

$$\begin{aligned}\mathrm{h}(s) &= \left|e^{\mathrm{i}s\phi_u}\left(r_v - r_v^{-1}\right) - r_v^s 2\mathrm{i}\sin\phi_u\right|^2 = \\ &= \left(r_v - r_v^{-1}\right)\cos^2\left(s\phi_u\right) + \left(\left(r_v - r_v^{-1}\right)\sin\left(s\phi_u\right) - 2r_v^s\sin\left(\phi_u\right)\right)^2 \\ &= \left(r_v - r_v^{-1}\right)^2 - 4r_v^s\left(r_v - r_v^{-1}\right)\sin\left(s\phi_u\right)\sin\left(\phi_u\right) + 4r_v^{2s}\sin^2\left(\phi_u\right).\end{aligned}$$

In fact, the difference $\mathrm{h}(0) - \mathrm{h}(s)$ is non-negative,

$$\mathrm{h}(0) - \mathrm{h}(s) = 4\sin^2\left(\phi_u\right)\left(1 - r_v^{2s}\right) + 4r_v^s\left(r_v - r_v^{-1}\right)\sin\left(s\phi_u\right)\sin\left(\phi_u\right) \overset{?}{\geq} 0$$

Dividing both sides by positive terms $4\sin^2\left(\phi_u\right), r_v^{-1} - r_v, r_v^s$, we get

$$0 \leq \frac{r_v^{-s} - r_v^s}{r_v^{-1} - r_v} - \frac{\sin\left(s\phi_u\right)}{\sin\left(\phi_u\right)}. \tag{1.7}$$

Since the function

$$\eta(x) = \frac{x^{-s} - x^s}{x^{-1} - x}$$

is decreasing on $(0,1), \lim_{x\to 1^-}\eta(x) = s$ (see the end of the proof) and

$$\max_{\phi_u\in(0,\pi)}\left(\frac{\sin\left(s\phi_u\right)}{\sin\left(\phi_u\right)}\right) = s$$

(see the end of the proof), the inequality (1.7) holds. We can easily prove that the inequality (1.6) holds for $u \in \mathbb{D}_+, v \in \mathbb{D}_-$ such that $r_v = 1, r_u \in [0,1]$ and $\phi_v \in (-\pi, 0), \phi_u = 0$. It is enough to use the symmetry of u and v in the problem and the fact that

$$\max_{\phi\in(0,\pi)}\left(\frac{\sin\left(s\phi\right)}{\sin\left(\phi\right)}\right) = s.$$

4. Now consider similar case, but $r_u = 1, r_v \in [-1, 0]$ and $\phi_u \in (0,\pi), \phi_v = 0$. Denote

$$\mathrm{h}(s) := \left| u^s \left(v - v^{-1}\right) - v^s \left(u - u^{-1}\right)\right|^2.$$

Using exactly the same method as in the previous case we get

$$\mathrm{h}(0) - \mathrm{h}(s) = 4\sin^2\left(\phi_u\right)\left(1 - r_v^{2s}\right) + 4r_v^s\left(r_v - r_v^{-1}\right)\sin\left(s\phi_u\right)\sin\left(\phi_u\right) \geq 0. \tag{1.8}$$

Now we have to separately discuss two situations. Firstly, consider s is odd. In this case, we divide both sides of the inequality (1.8) by the same terms as in the proof of previous part, i.e. $4\sin^2\left(\phi_u\right) > 0, r_v^{-1} - r_v < 0, r_v^s < 0$. Now we have the same inequality for different range of parameters

$$0 \leq \frac{r_v^{-s} - r_v^s}{r_v^{-1} - r_v} - \frac{\sin\left(s\phi_u\right)}{\sin\left(\phi_u\right)}.$$

Since both functions are even in this case, the inequality follows directly from the proof of previous case.

To finish the proof, we assume s even. Again, we divide both sides of the inequality (1.8) by $4\sin^2(\phi_u) > 0$, $r_v^{-1} - r_v < 0, r_v^s < 0$ and obtain

$$0 \geq \frac{r_v^{-s} - r_v^s}{r_v^{-1} - r_v} + \left(-\frac{\sin(s\phi_u)}{\sin(\phi_u)}\right)$$

Now $\eta(x) = \frac{x^{-s}-x^s}{x^{-1}-x}$ is odd function and $\frac{\sin(s\phi_u)}{\sin(\phi_u)}$ is odd. It is easy to see that the inequality holds.

One can easily prove (1.6) for $u \in \mathbb{D}_+, v \in \mathbb{D}_-$ such that $r_v = 1, r_u \in [-1, 0]$ and $\phi_v \in (-\pi, 0), \phi_u = 0$. It follows directly from symmetry of u and v in (1.6) and properties of the functions η and $\sin(sx)/\sin(x)$.

Now it remains to prove that the function η is decreasing. Moreover, η is odd function for even s and even function for odd s. Indeed, the oddness/evenness follows directly from the definition and the monotonicity is easy for $s \in \{0, 1\}$, for $s \geq 2$

$$\eta(x) = \frac{x^{-s} - x^s}{x^{-1} - x} = x^{-s+1}\frac{x^{2s} - 1}{x^2 - 1} = x^{-s+1}\sum_{k=0}^{s-1} x^{2k} = \sum_{k=0}^{s-1} x^{2k-s+1}.$$

Hence

$$\lim_{x\to 1^-} \eta(x) = s \quad \& \quad \lim_{x\to -1^+} \eta(x) = \begin{cases} s: & s \quad \text{odd} \\ -s: & s \quad \text{even} \end{cases}$$

and also

$$\eta'(x) = \sum_{k=0}^{s-1} (2k - s + 1)x^{2k-s}.$$

Now we look closer to the sum. There are s terms in the sum for every even $s \geq 2$ and $s - 1$ terms for odd $s > 2$. There are also $\lfloor s/2 \rfloor$ positive terms and $\lfloor s/2 \rfloor$ negative terms. It holds that for any $x \in (0, 1)$

$$\forall s \geq 2, \forall l \in \{0, 1, \ldots, \lfloor s/2 \rfloor - 1\} : \left|(2l - s + 1)x^{2l-s}\right| \geq \left|(2(s - 1 - l) - s + 1)x^{2(s-1-l)-s}\right|.$$

Indeed, using standard algebraic manipulations we get

$$\begin{aligned} \left|(2l - s + 1)x^{2l-s}\right| &\geq \left|(2(s - 1 - l) - s + 1)x^{2(s-1-l)-s}\right|, \\ \left|(2l - s + 1)x^{2l-s}\right| | &\geq \left|-(2l - s + 1)x^{-2l+s-2}\right|, \\ x^{(2l-s)-(-2l+s-2)} &= x^{4l-2s+2} \geq 1. \end{aligned}$$

It holds, because $4l - 2s + 2 \leq 0, \forall s \geq 2, \forall l \in \{0, 1, \ldots, \lfloor s/2 \rfloor - 1\}$ and x is from $(0, 1)$.

Last proposition to show to finish the proof is that for every $s \in \mathbb{N}_0$ is

$$\max_{\phi \in (0,\pi)} \frac{\sin(s\phi)}{\sin(\phi)} = s.$$

It is true, because the function

$$\frac{\sin(s\phi)}{\sin(\phi)} = U_{n-1}(\cos\phi),$$

where $U_n(x)$ is the second kind Chebyshev polynomial which has extreme values at ± 1. The proof can be found in the first chapter in [6]. It is clear that the value of the function at 0 is s.

$\square$

Chapter 2

The analysis of boundary curves of the spectral enclosures

In this chapter, we will do the analysis of the boundary curves of the spectral enclosures. Now, according to the proven conjecture, we know that these are the optimal conjectures. By the optimality it is now meant, that to any point z of the boundary it can be found a potential V such that z is the eigenvalue of $T^2 + V$. Our goal is to justify analytically the shape of the curves which was, until now, know just as a result of some numerical computations.

Let us start with few general propositions from complex analysis. We will be dealing with a function f which is analytic on some open set $U \subset \mathbb{C}$. By an **level set** of f we understand

$$\Gamma_c := \{z \in U : \ |f(z)| = c\}, \quad \text{for any } c > 0. \tag{2.1}$$

A level set Γ_c does have to be simple in general. A curve being simple now means that it can be described by a graph of the 1-variable function locally. The function is $y = y(x)$ or $x = x(y)$, where $y-$axis is considered to be Im-axis and $x-$axis the Re-axis in the complex plain. Shortly said, the curve does not have any intersections with itself. But we know some sufficient condition for simplicity.

Lemma 2.1. Let function f be analytic on an open $U \subset \mathbb{C}$ and $c > 0$. And let $f \neq 0$ and $f' \neq 0$. Then Γ_c is either $\emptyset$ or it is simple.

Proof. Consider a function $F(x, y) := |f(x + \mathrm{i}y)|$ where $z = x + \mathrm{i}y$. Then by Cauchy-Riemann conditions either $\partial_y F \neq 0$ or $\partial_x F \neq 0$. The proposition follows directly from the the Implicit function theorem applied to the function F. □

Theorem 2.2. (Open Mapping Theorem) Let f be analytic on some open set $U \subset \mathbb{C}$ and $f \neq \text{const}$ on U. Than the image of any open subset of U is an open subset in $\mathbb{C}$.

Lemma 2.3. Let f be analytic function on a disc $D \subset \mathbb{C}$, $f \neq 0$ on D, and $c > 0$, we have $\Gamma_c = \{z \in D : \ |f(z)| = c\}$. Then

$$\text{if } \Gamma_c \neq 0 \ \& \ D \setminus \Gamma_c \text{ is connected, then } f \text{ is constant.}$$

Proof. For a contradiction, let us assume that f is not constant on D and the conditions hold. Let us denote $h := \log |f(x + \mathrm{i}y)|$ where $z = x + \mathrm{i}y$. Then, according to Cauchy-Riemann conditions,

$$\partial_x^2 h + \partial_y^2 h = 0, \tag{2.2}$$

i.e. h is a harmonic function. It means h is continuous on D and moreover $h \neq \tilde{c}$ on $D \setminus \Gamma_c$, where $\tilde{c} := \log(c)$. It follows that $h > \tilde{c}$ or $h < \tilde{c}$ on $D \setminus \Gamma_c$.

Let us assume, without loss of generality, that $h < \tilde{c}$ on $D \setminus \Gamma_c$. Then h exhibits its maximum $\tilde{c}$ on Γ_c and according to the MMP for harmonic functions, $h = \tilde{c}$ on D, i.e. $|f| = c$ on D. By the Open Mapping Theorem the image $f(D)$ is open in $\mathbb{C}$. Since $|f| = c$ on D, the set $f(D) \subset \{w \in \mathbb{C} : \ |w| = c\}$. It follows that f is constant on D. This is the contradiction. $\square$

Remark 2.4. It follows from the Lemma 2.3, that the level set of non-constant analytic function f cannot "end" in the middle of the disc D. It must start at the boundary of D and continue through the interior of D to another point of the boundary or it must be closed in D.

Lemma 2.5. Let f be analytic on a bounded nonempty connected set $\Omega \subset \mathbb{C}$. Let $f \in C(\overline{\Omega})$. If $f = c$ on $\partial\Omega$, then $f = const.$ on Ω.

Proof. It directly follows from the maximum modulus principle. $\square$

Remark 2.6. If a level set Γ_c of some analytic function had a loop, the interior of this loop is a bounded nonempty connected set. Then according to Lemma 2.5, the function (if it fulfills the assumptions) must be constant on the interior of this loop and since the function is continuous, it must be equal to c too. It follows that non-constant analytic function cannot have level sets with loops.

2.1 Shape of boundary curves and intersections with the real axis

Boundary curves of the enclosures are given by the equation

$$\left| \frac{2\lambda(\lambda - 16)}{1 - \sqrt{1 - \frac{16}{\lambda}}} \right| = \|v\|^2_{\ell^1(\mathbb{Z})} \, . \tag{2.3}$$

In this section, let us denote $Q := \|v\|^2_{\ell^1(\mathbb{Z})}$. Since v is the V potential-generating sequence, we consider $v \neq 0$ and thus $Q > 0$.

In keeping with the general result in the beginning of this chapter we now consider a concrete function

$$f(\lambda) := \frac{2\lambda(\lambda - 16)}{1 - \sqrt{1 - \frac{16}{\lambda}}}. \tag{2.4}$$

Boundaries of spectral enclosures are then

$$\{\lambda \in \mathbb{C} : \ |f(\lambda)| = Q\} = \Gamma_Q. \tag{2.5}$$

To apply Lemmas 2.1, 2.3, 2.5, we need the f to be analytic. Function (2.4) is definitely analytic and continuous on $\mathbb{C} \setminus [0, 16]$.

2.1.1 Transformation of the boundary curve

To make the analysis more convenient, we introduce following transformation. Let us put

$$\lambda = \frac{16}{\sin^2(z)}, \quad \text{for } z \in (0, \pi/2) \times \mathrm{i}\mathbb{R} \cup i(0, +\infty). \tag{2.6}$$

The bijectivity of this transformation should be verified. Let us consider $S := (0, \pi/2) \times \mathrm{i}\mathbb{R}$ and show the bijectivity for it onto $\mathbb{C} \setminus [0, 16]$ first, adding the upper half of the Im axis is just an easy modification.

Let us start just with sine function transform. Complex sin is injective on S. Indeed, we choose $z \in S$, it means $z = x + \mathrm{i}y$ where $x \in (0, \pi/2)$ and $y \in \mathbb{R}$. Then

$$\sin(x + \mathrm{i}y) = \underbrace{\sin(x)\cosh(y)}_{=:\alpha>0} + \mathrm{i}\underbrace{\cos(x)\sinh(y)}_{=:\beta},$$

it follows that

$$(\alpha\cos(x))^2 - (\beta\sin(x))^2 = \sin^2(x)\cos^2(x).$$

Now denote $t := \sin^2(x) \in (0, 1)$ and we obtain

$$\alpha^2(1-t) - \beta^2 t - (1-t)t = 0,$$

it can be rewritten as

$$\underbrace{t^2 - (\alpha^2 + \beta^2 + 1)t + \alpha^2}_{=:p(t)} = 0. \tag{2.7}$$

It holds that $p(0) = \alpha^2 > 0$ and $p(1) = -\beta^2 < 0$. It follows that there exist

$$t_0 \in (0, 1), \; p(t_0) = 0.$$

Solutions of the quadratic equation (2.7) are of following form

$$t_\pm = \frac{1}{2}\left((\alpha^2 + \beta^2 + 1) \pm \sqrt{(\alpha^2 + \beta^2 + 1)^2 - 4\alpha^2}\right). \tag{2.8}$$

It is clear that they are both positive and the root t_0 must be $t_0 = t_-$. For such a t_0 we have

$$\sin(x) = \sqrt{t_0} \quad \Longrightarrow \quad x = \arcsin(\sqrt{t_0}).$$

Next, with this result we have

$$\alpha = \sqrt{t_0}\cosh(y),$$
$$\beta = \sqrt{1 - t_0}\sinh(y),$$

it follows that $y = \operatorname{argsinh}(\frac{\beta}{\sqrt{1-t_0}})$. From the first expression for α we have

$$\frac{\alpha}{\sqrt{t_0}} \geq 1,$$

and thus we must consider this restriction. But the inequality holds using the equivalent formulation $\alpha^2 \geq t_0$ and (2.8) for $t_- = t_0$.

Based on this, we know where the exact form of S came from. We have the expression for reverse transform now. As it was said sin is injective on this set.

The image $\sin(S)$ is then as follows

$$\sin(S) = \{w \in \mathbb{C} : \mathrm{Re}(w) > 0\ \} \setminus [1, +\infty]. \tag{2.9}$$

If we put $S' := S \cup \mathrm{i}[0, +\infty]$, we obtain

$$\sin(S') = \{w \in \mathbb{C} : \mathrm{Re}(w) > 0\ \} \setminus [1, +\infty] \cup \mathrm{i}[0, +\infty]. \tag{2.10}$$

Then it is easy to see that

$$\sin^2(S') = \mathbb{C} \setminus [1, +\infty] \setminus \{0\}, \tag{2.11}$$

If we do inverted value for any number in $\sin^2(S')$ we get

$$\frac{1}{\sin^2(S')} = \mathbb{C} \setminus [0, 1]. \tag{2.12}$$

Multiplying by 16 it becomes

$$\frac{16}{\sin^2(S')} = \mathbb{C} \setminus [0, 16]. \tag{2.13}$$

The injectivity is preserved during these transformations.

2.1.2 Intersections with the axes

Using the transformation $\lambda = \frac{16}{\sin^2(z)}$, we can rewrite formula

$$\left| \frac{2\lambda(\lambda - 16)}{1 - \sqrt{1 - \frac{16}{\lambda}}} \right| = Q$$

as follows. First we have

$$\begin{aligned} 1 - \sqrt{1 - \frac{16}{\lambda}} &= 1 - \cos(z) = 2\sin^2(z/2), \\ \lambda(\lambda - 16) &= \frac{256\cos^2(z)}{\sin^4(z)}. \end{aligned}$$

The equation then takes the following form

$$256 \left| \frac{\cos(z)}{\sin^2(z)\sin(z/2)} \right|^2 = Q. \tag{2.14}$$

In what follows, we will use the well known formulas

$$\sin(x + \mathrm{i}y) = \sin(x)\cosh(y) + \mathrm{i}\cos(x)\sinh(y), \tag{2.15}$$
$$\cos(x + \mathrm{i}y) = \cos(x)\cosh(y) - \mathrm{i}\sin(x)\sinh(y). \tag{2.16}$$

1. **Intersections with the interval [0,16]**.
If we want λ to be in [0,16] then $z \in \pi/2 + \mathrm{i}[0, +\infty]$. Thus we consider $z = \pi/2 + \mathrm{i}y$. We substitute into (2.14). We also denote a constant on the RHS of the equation by c and any time we divide the equation by a positive constant, the value of c changes, but we will not change the symbol. We obtain

$$\frac{2\sinh^2(y)}{\cosh^4(y)|1 + \mathrm{i}\sinh(y)|} = c.$$

We used (2.15),(2.16) and $|2\sin^2(z/2)| = |1 - \cos(z)|$, then

$$2\sinh^2(y) = c\cosh^4(y) \underbrace{|1 + \mathrm{i}\sinh(y)|}_{=\sqrt{1+\sinh^2(y)}=\cosh(y)}.$$

Now divide both sides by 2 and then add 1 to both sides

$$\cosh^2(y) = c\cosh^5(y) + 1.$$

If we put $t := \cosh^2(y)$ we obtain

$$\underbrace{ct^5 - t^2 + 1}_{=:g(t)} = 0. \tag{2.17}$$

It is clear, that if there exists $t^* > 1$ such that $g(t^*) = 0$ than there is an intersection of the boundary curve in the $\lambda-$plane with the interval $[0, 16]$. Number of the roots is equivalent to the number of intersections. The restriction $t > 1$ is due to the substitution.

Let us analyze the polynomial. The derivative

$$g'(t) = 5ct^4 - 2t = 0 \iff t = \sqrt[3]{2c/5} =: t_c.$$

We find, such a c, that the value

$$\begin{aligned} g(t_c) = g(\sqrt[3]{2c/5}) &= 0, \\ (2/5)^{5/3}\frac{1}{c^{2/3}} - (2/5)^{2/3}\frac{1}{c^{2/3}} + 1 &= 0, \\ \frac{1}{c^{2/3}} &= \frac{1}{(2/5)^{5/3} - (2/5)^{2/3}}, \\ c = (2/5)(3/5)^{3/2} &= c_0 \cong 0,186. \end{aligned}$$

Thus

$$t_{c_0} = (2/5)^{1/3}(5/2)^{1/3}(5/3)^{1/2} = \sqrt{\frac{5}{3}}. \tag{2.18}$$

It is easy to see that $\lim_{t\to+\infty} g(t) = +\infty$ and $g(1) = d > 0$. Then we can make a following conclusion. Note, that the size of constant c can be chosen arbitrarily in $(0, +\infty)$ because it is dependent only on the $\ell^1(\mathbb{Z})-$ norm of the sequence v and now it is just multiplied by some positive constants.

- If $c = c_0$ then $g(t_c) = 0$ and there is only one root of $g(t)$ which is a local minimum at the same time. Since $t_c > 1$, the root fulfills the restriction and there is only one intersection of the boundary curve with the interval $[0, 16]$.
- If $c \in (0, c_0)$ Then $t_c > t_{c_0}$ and $g(t_c) < 0$. Thus there are two roots of $g(t)$ bigger than 1 and there are also two intersections of the boundary curve with the interval $[0, 16]$.
- If $(c > c_0)$, then $t_c < 0$ and both roots are smaller than 1. It means that there is no intersection of the boundary curve with the interval $[0, 16]$.

As can bee seen from the figures 2.2, the results correspond to the plots made by a CAS.

2. **Intersections with the interval** $(16, +\infty)$
In this case, we consider $z \in (0, \pi/2)$, i.e. $z = x \in (0, \pi/2)$. Again, we substitute into (2.14) and obtain

$$\frac{\cos^2(x)}{(1-\cos^2(x))^2|\sin^2(x/2)|} = c,$$
$$2\cos^2(x)(1-\cos^2(z))^2|\sin^2(x/2)| = c(1-\cos^2(x))^2(1-\cos(x)).$$

If we put $t := \cos(x) \in (0, 1)$, mind that $x \in (0, \pi/2)$, we obtain

$$c(1-t^2)^2(1-t) = 2t^2,$$
$$c(1-2t^2+t^4)(1-t) = 2t^2,$$
$$c(1-2t^2+t^4-t+2t^3-t^5) - 2t^2 = 0,$$
$$\underbrace{-c+2(c+1)t^2 - ct^4 + ct - 2ct^3 + ct^5}_{=:p(t)} = 0.$$

It is easy to see that $p(1) = 2$ and $g(0) = -c < 0$. Since p is continuous, there is a root $t_0 \in (0, 1)$, by the analysis of derivatives of p (the first derivative is positive in $(0, 1)$, it can be verified the monotonicity of p on [0,1], thus there is a single root. It means that there is only one intersection of the boundary curve with the interval $(16, +\infty)$.

3. **Intersections with the interval** $(-\infty, 0)$.
Finally for $\lambda \in (-\infty, 0)$ we consider $z \in \mathrm{i}(0, +\infty)$, i.e. $z = \mathrm{i}y$, $y > 0$. We substitute into (2.14) and using the same formulas as before, we obtain

$$\frac{2\cosh^2(y)}{\sinh^4(y)|1-\cosh(y)|} = c,$$
$$c(1-\cosh^2(y))^2(\cosh(y)-1) = 2\cosh^2(y).$$

Let us now make the substitution $t := \cosh(y) > 1$. We reduce the problem to finding roots of some polynomial again. We have

$$c(t^2-1)^2(t-1) = 2t^2,$$
$$\underbrace{ct^5 - ct^4 - 2ct^3 + 2(c-1)t^2 + ct - c}_{=:p(t)} = 0.$$

Now, it is easy to see that $p(1) = -2 < 0$ and $\lim_{t\to+\infty} p(t) = +\infty$. Since p is continuous, there is a root of p in $(1, +\infty)$. According to analysis of derivatives, there is only one root. It means, that there is only one intersection with the interval of the boundary curve with the interval $(-\infty, 0)$.

This brings us to the next part where we will discuss some properties of boundary curves with relation to Lemma 2.1. We are dealing with function (2.4) after transform $\lambda := \frac{1}{\sin^2(z)}$, where $z \in S'$.Thus f is up to real constant (consider the constant to be 1, without loss of generality) as follows

$$
\begin{aligned}
f(z) &= \frac{\cos(z)}{\sin^2(z)\sin(z/2)},\\
f'(z) &= -\sin^3(z)\sin(z/2) - 2\sin(z)\underbrace{\cos^2(z)}_{=1-\sin^2(z)}\sin(z/2) - \frac{1}{2}\sin^2(z)\cos(z)\cos(z/2)) =\\
&= \sin^3(z)\sin(z/2) - \sin(z)\sin(z/2) - \frac{1}{2}\sin^2(z)\cos(z)\cos(z/2))\\
&= \underbrace{\sin(z)\cos(z)}_{\neq 0,\ z\in S'}\left(\sin(z/2)\cos(z) + \frac{1}{2}\sin(z)\cos(z/2)\right).
\end{aligned}
$$

If we want $f'(z) = 0$ then it has to hold

$$
\begin{aligned}
\sin(z/2)\cos(z) + \frac{1}{2}\sin(z)\cos(z/2) &= 0,\\
\sin(z/2)\left(\cos^2(z/2) - \sin^2(z/2)\right) + sin(z/2)\cos^2(z/2) &= 0,\\
\underbrace{\sin(z/2)}_{\neq 0,\ z\in S'}\left(3\cos^2(z/2) - 1\right) &= 0,\\
\cos(z) &= \frac{1}{\sqrt{3}}.
\end{aligned}
$$

Solutions of this equation are not in S'. Thus $f'(z) \neq 0$ on S' and the Lemma 2.1 can be applied. The level curves of the function s are simple curves (in the sense of Lemma 2.1).

Moreover, assumptions of Lemma 2.3 and Lemma 2.5 are fulfilled too. Using Remarks 2.4 and 2.6, we can see that the boundary curves have no loops and they simply cannot end somewhere in the interior of S' and since $f(z)$ is analytic on whole S', the level set cannot be closed in S'. The situation look different when we are in $\lambda-$plane, because , as we can see from the pictures, the level sets are kind closed of ovals. But mind that the oval is surrounding the interval $[0, 16]$ where is the function $f(\lambda)$ given by (2.4) not analytic.

Let us now briefly discuss the shape of the boundary curves. The constants Q in (2.14) and c from calculations can be chosen arbitrarily positive and it holds $c = Q \cdot \text{const}$. In our analysis, we will restrict our $z \in S$ and $\lambda \in \mathbb{C} \setminus [0, 16]$ to the ones with non-negative imaginary part. It is because of the curves are symmetric with respect to the real axis.

Starting with the range of constant Q in (2.14) such that there are just two intersections of the non-transformed (i.e. in $\lambda-$plane) boundary curve with real axis. Consider now the $z-$plane with the non-negative imaginary part restriction. Than according to the known position of the intersections, one on $\mathrm{i}(0, +\infty)$ and the second in $(0, \pi/2)$ and also to the "no loop" and "no-ending" argument (Lemma 2.3 and Lemma 2.5), the level sets look topologically as in the Figure

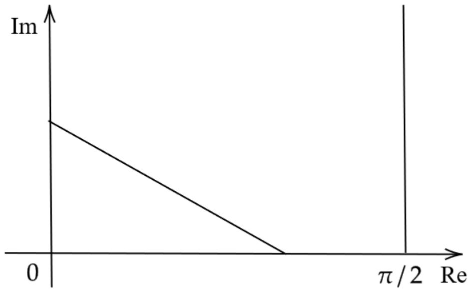

Figure 2.1: Level sets in $z-$plane, the case of 2 intersections.

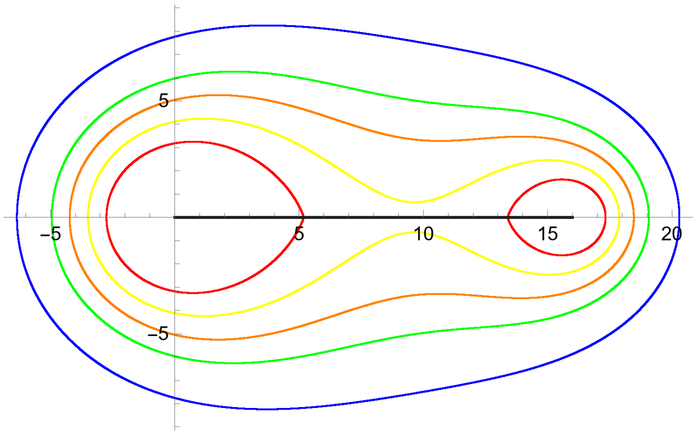

Figure 2.2: Level sets in $\lambda-$plane, all cases.

2.1. And the situation in $\lambda-$plane is then in accordance with the Figure 2.2.

There are more options when it comes to three intersections of the non-transformed boundary curve with real axis. Again according to same we have 4 possibilities They can be seen in the Figure 2.3. The configuration B is the one from numerical simulations. Configuration A can not be possible, since it is in contradiction with "no-loop" argument. And configuration C and D might be excluded using implicit function theorem. Indeed, for $z = x + \mathrm{i}y$ we define

$$F(x, y) := |f(z)|^2 = \left| \frac{\cos(z)}{\sin^2(z)\sin(z/2)} \right|^2. \tag{2.19}$$

Using the standard formulas (2.16),(2.15) we obtain

$$F(x, y) = \frac{\cos(2x) + \cosh(2y)}{(\cosh(y) - \cos(x))^3(\cosh(y) + \cos(x))^3}. \tag{2.20}$$

Then consider $y > 0$, because we now want to reject configuration C, we have

$$\partial_y F(0, y) = -\frac{1}{128}\left(\underbrace{2 + 17\cosh(y) + 2\cosh(2y) + 3\cosh(3y)}_{>0}\right)\underbrace{\frac{1}{\sinh^7(y/2)\cosh^5(y/2)}}_{\neq 0}. \quad (2.21)$$

Thus $\partial_y F(0, y) \neq 0$ for any $y > 0$ and the level curve, which is the solution of an equation

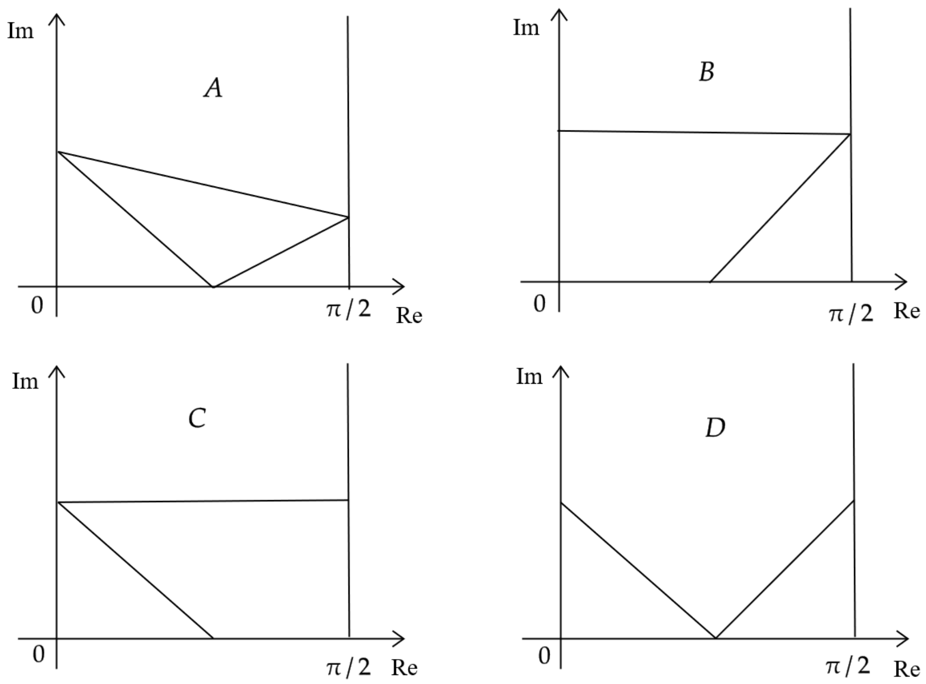

Figure 2.3: Level sets in $z-$plane, the cases of 3 intersections.

$F(x, y) = c$ should be a graph of some function $y = y(x)$ on the neighborhood of the intersection. Mind that we do not need to know exactly its position, since we know that it is on $\mathrm{i}(0, +\infty)$. To conclude, the curve must be graph of the function $y = y(x)$, but the configuration C is definitely not the case.
In the same way, we will exclude the configuration D. We choose $x \in (0, \pi/2)$, then it holds

$$\partial_y F(0, y) = -\frac{1}{128}\left(\underbrace{2 + 17\cos(x) + 2\cos(2x) + 3\cos(3x)}_{=:p(\cos(x)}\right)\underbrace{\frac{1}{\sin^7(y/2)\cos^5(y/2)}}_{\neq 0}. \quad (2.22)$$

To show $\partial_x F(x, 0) \neq 0$ it is sufficient to show

$$\begin{aligned} &p(\cos(x)) \neq 0, \quad \text{for } x \in (0, \pi/2), \\ &2 + 17\cos(x) + 2\cos(2x) + 3\cos(3x) \neq 0. \end{aligned}$$

We will show that

$$2 + 17\cos(x) + 2\cos(2x) + 3\cos(3x) > 0.$$

Indeed, using formulas for $\cos(2x)$ and $\cos(2x + x)$ we obtain

$$\begin{aligned}2 + 17\cos(x) + 2\cos^2(x) - 2\sin^2(x) + 3\cos(2x)\cos(x) - 3\sin(2x)\sin(x) > 0,\\ 2 + 17\cos(x) + 2\cos^2(x) - 2\sin^2(x) + 3\cos^3(x) - 3\sin^2(x)\cos(x) - 6\sin^2(x)\cos(x) > 0,\\ 2 + 17\cos(x) + 2\cos^2(x) + 3\cos^3(x) > 2\sin^2(x) + 9\sin^2(x)\cos(x).\end{aligned}$$

As usual, we put $t := cos(x) \in (0, 1)$ and get

$$12t^3 + 4t^2 + 8t > 0.$$

Which holds, since for $t \in (0, 1)$ is it just a sum of positive numbers. To conclude, it means, that in the neighborhood of the intersection with the real axis in $z-$plane must the curve must be a graph of function $x = x(y)$. It is definitely not. Thus the only possible configuration is B.

Having the case of 4 intersections, there are again more possible configurations. The one which is observed in numerical simulations is in the Figure 2.4. According to results in previous parts, we can immediately exclude most of them. We know, that there is exactly one intersection in $(0, \pi/2)$ and one in $\mathrm{i}(0, +\infty)$. We also know that there have to be 2 intersections in $\pi/2+\mathrm{i}(0, +\infty)$. According to the Implicit Function Theorem, it was shown in the case of three intersections, that in the neighborhood of $(0, \pi/2)$ the level set must be a graph of the function $x = x(y)$ and in the neighborhood of $\mathrm{i}(0, +\infty)$ it must be a graph of function $y = y(x)$ where we put $x + \mathrm{i}y = z$. It follows that there must be exactly one line from the real axis to $\pi/2+\mathrm{i}(0, +\infty)$ and one line from the imaginary axis to $\pi/2 + \mathrm{i}(0, +\infty)$. If these two lines had a common intersection, it would be a contradiction with Lemma 2.1.

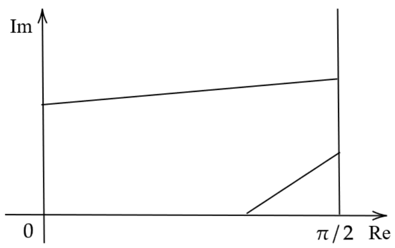

Figure 2.4: Level sets in $z-$plane, the case of 4 intersections.

Chapter 3

Absence of the eigenvalues in the interior of essential spectrum of unperturbed discrete (bi)-Laplace operator

In this chapter, we will go through a proof of absence of the eigenvalues of perturbed discrete Laplacian Δ in the interior of the essential spectrum, which is a well known result for discrete Schrödinger operators. As a result of the perturbation by a compact potential V, the spectrum of the Schrödinger operator $\Delta + V$ consists of $\sigma_{\text{ess}}(\Delta + V) = \sigma_{\text{ess}}(\Delta) = [-2, 2]$ and possible eigenvalues outside the essential spectrum. It will be proven here, that there cannot be eigenvalues in the interior of essential spectrum which is the interval $(-2, 2)$.

There is more than one possibility how to approach this result, we will choose a direct one, which was proved in [1]. In this project, the proof will be done exactly in the same way.The proposition for Laplacian and complex potential then follows directly from this result. We will discuss possibilities of generalization of this proof for the bilaplace operator T^2. Let us start with some basic definitions. We consider operators on the Hilbert space $\ell^2(\mathbb{Z})$.

Definition 3.1. Let $a, b, c \in \ell^2(\mathbb{Z})$. By a general Jacobi operator is understood an operator $J = J(a, b, c)$ which is defined

$$Je_n = c_n e_{n-1} + b_{n+1} e_n + c_{n+1} e_{n+1}, \quad \forall n \in \mathbb{Z}. \tag{3.1}$$

Remark 3.2. Matrix of the operator J is called Jacobi matrix and it is of following form

$$J = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & a_{-1} & b_0 & c_0 & & & \\ & & a_0 & \boxed{b_1} & c_1 & & \\ & & & a_1 & b_2 & c_2 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Definition 3.3. We define the operator $J_0 := J(\{1\}, 0, \{1\})$.

Spectrum of the discrete Laplacian J_0 is well known. It is $\sigma(J_0) = \sigma_{\text{ess}}(J_0) = [-2,2] =$. It was shown for e.g. in [3].

Definition 3.4. Perturbation $V = J - J_0$ is called trace class if it holds

$$\sum_{n\in\mathbb{Z}} (|1-a_n| + |b_n| + |1-c_n|) < +\infty. \tag{3.2}$$

Remark 3.5. It follows directly from the necessary condition for convergent series that

$$\lim_{n\to\pm\infty} a_n = \lim_{n\to\pm\infty} c_n = 1,$$
$$\lim_{n\to\pm\infty} b_n = 0.$$

Now we will study the equation for eigenvalues of J

$$Ju = \lambda u, \tag{3.3}$$

which is equivalent to

$$a_{i-1}u_{i-1} + b_i u_k + c_i u_{i+1} = (k + \frac{1}{k})u_i, \quad i \in \mathbb{Z}, \tag{3.4}$$

where the Joukowski transform $\lambda = \lambda(k) = k + 1/k$ was used. We would like to study Jost solutions of this equation.

Definition 3.6. Solutions $u^{\pm}$ of (3.4) are called **Jost solutions** at $\pm\infty$ if

$$\lim_{n\to\pm\infty} k^{\mp n} u_n^{\pm} = 1, \quad \forall k \in \mathbb{D} \setminus \{0\}.$$

Our aim is to generalize following part for the dicsrete bilaplacian with a complex potential, thus we do not need to consider general trace class perturbation but only the diagonal one. Since now, we put $a_n = c_n = 1, \ \forall n \in \mathbb{Z}$.

Remark 3.7. In the original proof general sequences a and c are considered. In that case, it is necessary to reformulate the equation (3.4) in the way that there is coefficient 1 in front of the element u_{i+1} or u_{i-1} for iteration to plus or minus infinity respectively.

Remark 3.8. It is a well know fact, that the Green kernel of the discrete Laplacian is

$$(J_0 - \lambda(k))^{-1}_{m,n} = \frac{k^{|m-n|}}{k - k^{-1}}, \tag{3.5}$$

where $\lambda = \lambda(k) = k + k^{-1}$ is the Joukowski transform. Proof can be found in [3].

Now, let us define few more things. First, according to the formula (3.5) we define non-symmetric Green kernels

$$G_r(n,m) := \begin{cases} \frac{k^{m-n} - k^{n-m}}{k-k^{-1}}, & m \geq n, \\ 0, & m \leq n, \end{cases}, \tag{3.6}$$

$$G_l(n,m) := \begin{cases} 0, & m \geq n, \\ \frac{k^{n-m} - k^{m-n}}{k-k^{-1}}, & m \leq n. \end{cases} \tag{3.7}$$

Using the definitions in (3.6) and (3.7), one can easily see that

$$G_{r,l}(n,m-1)+G_{r,l}(n,m+1)-(k+\frac{1}{k})G_{r,l}(n,m)=\delta_{m,n}, \tag{3.8}$$

$$G_{r,l}(n-1,m)+G_{r,l}(n+1,m)-(k+\frac{1}{k})G_{r,l}(n,m)=\delta_{m,n}. \tag{3.9}$$

Following two theorems introduce an iterative way to find the Jost solutions of (3.4).

Theorem 3.9. The Jost solution $v^+=\{v_n^+\}_{n\in\mathbb{Z}}$ of the difference equation (3.4) at $+\infty$ satisfies the discrete Volterra-type equation

$$v_n^+=k^n+\sum_{m=n+1}^{\infty}-b_mG_r(n,m)v_m^+, \quad n\in\mathbb{Z}, \quad k\in\mathbb{D}\setminus\{0\}. \tag{3.10}$$

On the other hand, every solution $v=\{v_n\}_{n\in\mathbb{Z}}$ of the Volterra-type equation (3.10) solves the equation (3.4).

Proof. We will show the first part of the statement. Let the $\{v_m^+\}$ be the Jost solution at $+\infty$ and multiply the equation (3.8) by v_m^+ and the eigenvalue equation by $G_r(n,m)$ and then subtract them. The equations are

$$G_r(n,m-1)v_m^+ + G_r(n,m+1)v_m^+ - (k+\frac{1}{k})G_r(n,m)v_m^+ = \delta_{m,n}v_m^+,$$
$$v_{m-1}^+G_r(n,m)+b_mv_m^+G_r(n,m)+v_{m+1}^+G_r(n,m)=(k+\frac{1}{k})v_m^+G_r(n,m).$$

And after subtracting we get

$$\left[G_r(n,m+1)v_m^+ - G_r(n,m)v_{m-1}^+\right]+\left[G_r(n,m-1)-b_mG_r(n,m)\right]v_m^+ - v_{m+1}^+G_r(n,m)=\delta_{m,n}v_m^+.$$

From the definition of the kernel (3.6) it is clear that $G_r(n,n)=0,\ G_r(n,n+1)=1,\ \forall n\in\mathbb{Z}$. Using this, one can sum the last relation from $n+1$ to $N>n$ over m. The first term in the equation is a telescopic sum, only first and last term remain after summation. Indeed,

$$G_r(n,N+1)v_N^+ - \underbrace{G_r(n,n+1)}_{=1}v_n^+ + \sum_{m=n+1}^{N}\left[G_r(n,m-1)-b_mG_r(n,m)\right]v_m^+$$
$$-\underbrace{\sum_{m=n}^{N}v_{m+1}^+G_r(n,m)}_{v_{n+1}G_r(n,n)=0 \text{ was added}} = 0.$$

Now we can shift index in the last sum and then subtract it with the part of the second sum. Hence

$$v_n^+=G_r(n,N+1)v_N^+ - G_r(n,N)v_{N+1}^+ + \sum_{m=n+1}^{N}-b_mG_r(n,m)v_m^+. \tag{3.11}$$

To finish this part of the proof, we just do the limit $N\to+\infty$. It holds that

$$\lim_{N\to+\infty}G_r(n,n+1)v_N^+ - G_r(n,N)v_{N+1}^+ = k^n.$$

Indeed, we use the definitions of the kernels and the fact from Definition 3.6,

$$G_r(n,n+1)v_N^+ - G_r(n,N)v_{N+1}^+ = -\frac{k^{n-(N+1)} - k^{N+1-n}}{k-k^{-1}}v_N^+ + \frac{k^{n-N} - k^{N-n}}{k-k^{-1}}v_{N+1}^+ =$$

$$\underbrace{-k^{-N}v_N^+}_{\to -1}\left(\frac{k^{n-1}}{k-k^{-1}}\right) + \underbrace{k^{-N}v_N^+}_{\to 1}\underbrace{\left(\frac{k^{2N-n+1}}{k-k^{-1}}\right)}_{\to 0} + \underbrace{k^{-N+1}v_{N+1}^+}_{\to 1}\left(\frac{k^{n+1}}{k-k^{-1}}\right)$$

$$-\underbrace{k^{-(N+1)}v_{N+1}^+}_{\to 1}\underbrace{\left(\frac{k^{2N-n+1}}{k-k^{-1}}\right)}_{\to 0} \underset{N\to+\infty}{\longrightarrow} k^n\frac{k-k^{-1}}{k-k^{-1}} = k^n.$$

.

To prove the second part, we consider a solution $\{v_n\}_{n\in\mathbb{Z}}$ of the (3.10). One has

$$v_{n-1} + v_{n+1} = k^n(k+1/k) - b_n\underbrace{G_r(n-1,n)v_n}_{=-b_nv_n} - b_{n+1}G_r(n-1,n+1)v_{n+1}$$

$$+\sum_{m=n+2}^{+\infty} -b_m[G_r(n-1,m) + G_r(n+1,m)]v_m,$$

$$G_r(n-1,m) + G_r(n+1,m) = \frac{k^{m-n+1} - k^{n-m-1} + k^{m-n-1} - k^{n-m+1}}{k+k^{-1}} = (k+k^{-1})G_r(m,n),$$

$$G_r(n-1,n+1) = k+k^{-1}, \text{ and thus}$$

$$v_{n-1} + b_nv_n + v_{n+1} = (k+k^{-1})\left(\underbrace{k^n + \sum_{m=n+1}^{\infty} -b_mG_r(n,m)v_m}_{=v_n}\right).$$

Which finishes the proof. □

Theorem 3.10. The Jost solution $v^- = \{v_n^-\}_{n\in\mathbb{Z}}$ of the difference equation (3.4) at $-\infty$ satisfies the discrete Volterra-type equation

$$v_n^- = k^{-n} + \sum_{m=-\infty}^{n-1} -b_mG_l(n,m)v_m^-, \quad n\in\mathbb{Z}, \quad k\in\mathbb{D}\setminus\{0\}. \tag{3.12}$$

On the other hand, every solution $v = \{v_n\}_{n\in\mathbb{Z}}$ of the Volterra-type equation (3.12) solves the equation (3.4).

Proof. We proceed in the exactly same way as in the previous proof but we use the equations (3.8), (3.4) for G_l instead of G_r. □

We obtained two linearly independent solutions of the second order difference equation, thus we have a fundamental system. We will analyze their properties more in detail.

Now, put

$$f_m^r := v_m^+k^{-m} - 1, \quad \tilde{G}_r(n,m) = G_r(n,m)k^{m-n}, \tag{3.13}$$

$$f_m^l := v_m^-k^m - 1, \quad \tilde{G}_l(n,m) = G_l(n,m)k^{n-m}. \tag{3.14}$$

It holds that $\tilde{G}_r(n,m)$ are polynomials in k. Indeed, for $m > n$ it follows from

$$\tilde{G}_r(n,m) = \frac{k^{m-n} - k^{n-m}}{k - k^{-1}} k^{m-n} = k\frac{k^{2(m-n)} - 1}{k^2 - 1} = k\frac{(k^{m-n} - 1)(k^{m-n} + 1)}{(k-1)(k+1)}. \tag{3.15}$$

We can reduce the factors $(k-1)$ and $(k+1)$ in the fraction. For the $\tilde{G}_l$ in the same way.

Equations (3.10) and (3.12) are now of the form

$$f_n^r(k) = \sum_{m=n+1}^{+\infty} -b_m \tilde{G}_r(n,m) + \sum_{m=n+1}^{+\infty} -b_m \tilde{G}_r(n,m) f_m^r \tag{3.16}$$

and

$$f_n^l(k) = \sum_{m=-\infty}^{n-1} -b_m \tilde{G}_l(n,m) + \sum_{m=-\infty}^{n-1} -b_m \tilde{G}_l(n,m) f_m^l, \tag{3.17}$$

it follows directly from the definitions of $f^{r,l}$ and $G_{r,l}$. We cane easily estimate

$$|-b_m G_{r,l}(m,n)| \leq |b_m| \left|\frac{k}{k^2 - 1}\right|. \tag{3.18}$$

It follows from (3.15) and the fact that $|k^{2(|m|)}| \leq 1$. And since b is a summable sequence, the following sums can be estimated

$$\left|\sum_{m=n+1}^{+\infty} -b_m \tilde{G}_r(n,m)\right| \leq \left|\frac{k}{k^2-1}\right| R_n^r, \tag{3.19}$$

$$\left|\sum_{m=-\infty}^{n-1} -b_m \tilde{G}_l(n,m)\right| \leq \left|\frac{k}{k^2-1}\right| R_n^l, \tag{3.20}$$

Where

$$R_n^r := \sum_{m=n+1}^{+\infty} |b_m| \underset{n\to+\infty}{\longrightarrow} 0,$$

$$R_n^l := \sum_{m=-\infty}^{n-1} |b_m| \underset{n\to-\infty}{\longrightarrow} 0,$$

since they are remainders of convergent series for sequence b. It follows from these estimates that the series

$$\sum_{m=n+1}^{+\infty} -b_m \tilde{G}_r(n,m) \text{ and } \sum_{m=-\infty}^{n-1} -b_m \tilde{G}_l(n,m)$$

converge absolutely on $\bar{\mathbb{D}} \setminus \{\pm 1\}$. Now it is possible to show that

$$|v_n^+ - k^n| \leq |k|^n \left|\frac{k}{k^2-1}\right| R_n^r \exp\left(\left|\frac{k}{k^2-1}\right| R_n^r\right),$$

$$|v_n^- - k^{-n}| \leq |k|^{-n} \left|\frac{k}{k^2-1}\right| R_n^l \exp\left(\left|\frac{k}{k^2-1}\right| R_n^l\right).$$

It follows directly from the following proposition which can be found as Lemma 7.8 in [5].

Lemma 3.11. Consider the sequences $\{f_n\}$ and $\{f_n\}$ satisfying

$$f_n = g_n + \sum_{m=n+1}^{+\infty} G(n,m) f_m,\ n\mathbb{Z}. \tag{3.21}$$

And consider a kernel $\tilde{G}(n,m)$ such that

$$|G(n,m)| \leq |\tilde{G}(n,m)|, \quad \tilde{G}(n+1,m) \leq \tilde{G}(n,m) \text{ and } \tilde{G}(n,\cdot) \in \ell^\infty.$$

Then for the sequence $g \in \ell^\infty$ there exists a unique solution $f \in \ell^\infty$ of (3.21), fulfilling the estimate

$$|f_n| \leq \left(\sup_{m>n} |g_m|\right) \exp\left(\sum_{m=n+1}^{+\infty} \tilde{G}(n,m)\right).$$

It is clear that the assumptions of this Lemma are fulfilled in our case.

Now we will consider a $k \in \mathbb{T} \setminus \{\pm 1\}$, where $\mathbb{T}$ is the unit circle. We obtained two linearly independent Jost solutions of the equation (3.4) and according to the last result, we know that v^+ behaves as k^n at $+\infty$ and the v^- as k^{-n} at $-\infty$. Moreover any solution s of the equation (3.4) must be in the form $s = \alpha v^+ + \beta v^-$. To get the solution s in $\ell^2(\mathbb{Z})$, the limits of s must be 0. Let us assume that

$$\alpha v_n^+ + \beta v_n^- \underset{n\to\pm\infty}{\longrightarrow} 0.$$

If $\alpha = 0$ then $\beta = 0$ since the limit at $-\infty$ of v^- is not 0. If $\beta = 0$ then $\alpha = 0$, the reason is almost the same. Now consider $\alpha, \beta \neq 0$ and the limit at $+\infty$. Since $k \in \mathbb{T} \setminus \{\pm 1\}$, it follows that

$$\begin{aligned} \alpha v_n^+ + \beta v_n^- &\underset{n\to+\infty}{\longrightarrow} 0, \\ \alpha \underbrace{v_n^+ k^{-n}}_{\to 1} + \beta v_n^- k^{-n} &\underset{n\to+\infty}{\longrightarrow} 0, \\ k^{-n} v_n^- &\underset{n\to+\infty}{\longrightarrow} \frac{\alpha}{\beta}. \end{aligned}$$

Thanks to the linear independence of $v^\pm$, the wronskian $W(v^+, v^-) = \text{const} \neq 0$. On the other hand

$$W(v^+, v^-) = (v_{n+1}^+ v_n^- - v_n^+ v_{n+1}^-) = \ = k^{2n+1} \left(\underbrace{k^{-n-1} v_{n+1}^+}_{\to 1} \underbrace{k^{-n} v_n^-}_{\to \frac{\alpha}{\beta}} - \underbrace{k^{-n} v_n^+}_{\to 1} \underbrace{k^{-n-1} v_{n+1}^-}_{\to \frac{\alpha}{\beta}} \right) \underset{n\to+\infty}{\longrightarrow} 0,$$

what is a contradiction. We can proceed the same way for the case of the limit at $-\infty$. Hence we have that only the trivial combination of the Jost solutions is in $\ell^2(\mathbb{Z})$ and thus there can be no eigenvalue for $k \in \mathbb{T} \setminus \{\pm 1\}$.

To conclude, both upper and lower half of the unit circle is mapped onto the interval $[-2, 2]$ by the Joukowski transform. Moreover, $k = \pm 1$ is mapped on $\lambda = \pm 2$ and thus there cannot be the eigenvalue of $J(\{1\}, b, \{1\})$ in the interval $(-2, 2)$ what is an interior of the essential spectrum of J_0.

3.1 Some ideas of generalization for the discrete bilaplacian

In this section, we will point out the key parts of the proof and then discuss possible generalization on the operators from the Laurent class, especially for the bilplacian. In that case, we need to have four linearly independent solutions of the 4-th order difference equation

$$v_{n-2} - 4v_{n-1} + 6v_n - 4v_{n+1} + v_{n-2} = \lambda(k)v_n, \tag{3.22}$$

where $\lambda(k) = k^{-2} - 4k^{-1} + 6 - 4k^1 + k^2$ is the bijection from $\mathbb{D}_\zeta$ onto $\mathbb{C} \setminus [0, 16]$, which was used in [3]. It would be necessary to show that for k in the upper half of the unit circle except ± 1 (which is mapped by the transform onto 0 and 16) only a trivial linear combination is in $\ell^2(\mathbb{Z})$. We would like to proceed the same way as in the beginning of this chapter.

The key parts of the proof for Jacobi operators are as follows. The first is definition of non-symmetric Green kernels (3.6) and (3.7). For T^2 it could be in the similar way

$$G_r(n,m) := \begin{cases} \frac{k}{2(k-1)^2}\left(\frac{k^{m-n}-k^{n-m}}{k-k^{-1}} - \frac{z_-^{m-n}-z_-^{n-m}}{z_- - z_-^{-1}}\right), & m \geq n, \\ 0, & m \leq n, \end{cases}$$

$$G_l(n,m) := \begin{cases} \frac{k}{2(k-1)^2}\left(\frac{k^{n-m}-k^{m-n}}{k-k^{-1}} - \frac{z_-^{n-m}-z_-^{m-n}}{z_- - z_-^{-1}}\right), & m \leq n, \\ 0, & m \geq n. \end{cases}$$

In this case, we have just two kernels, but it seems, that it would be necessary to define four kernels since we want four independent solutions. Thus we can split the kernels and define four new kernels in following way

$$G_r^1(n,m) := \begin{cases} \frac{k^{m-n}-k^{n-m}}{k-k^{-1}}, & m \geq n, \\ 0, & m \leq n, \end{cases}$$

$$G_r^2(n,m) := \begin{cases} \frac{z_-^{m-n}-z_-^{n-m}}{z_- - z_-^{-1}}, & m \geq n, \\ 0, & m \leq n, \end{cases}$$

$$G_l^1(n,m) := \begin{cases} \frac{k^{n-m}-k^{m-n}}{k-k^{-1}}, & m \leq n, \\ 0, & m \geq n, \end{cases}$$

$$G_l^2(n,m) := \begin{cases} \frac{z_-^{n-m}-z_-^{m-n}}{z_- - z_-^{-1}}, & m \leq n, \\ 0, & m \geq n. \end{cases}$$

We have not included the factor $k/(2(k-1)^2)$ into the definition since it seems not to be necessary which will be explained in following paragraph.

The second key part is the formula in (3.8) and (3.9). It is necessary to define the kernel G in the way that they fulfill following identity

$$\begin{aligned} G(n,m-2) - 4G(n,m-1) + 6G(n,m) - 4G(n,m+1) + G(n,m+2) \\ - \left(k^2 - 4k + 6 - 4k^{-1} + k^{-2}\right) G(n,m) = \delta_{n,m}. \end{aligned} \tag{3.23}$$

This identity is given according to (3.8) with appropriate shift of indexes for bilplacian. The proof of the Theorem 3.9 is based on these two key properties, standard algebraic manipulations and basic operations of mathematical analysis. It is not hard to verify that the kernels G_r and G_l do not fulfill this identity. It is caused by the fraction

$$\frac{z_-^{m-n} - z_-^{n-m}}{z_- - z_-^{-1}}.$$

Indeed, we consider $m > n$, then the formula on the LHS of (3.23) should be equal to 0. Thanks to the factor $\left(k^2 - 4k + 6 - 4k^{-1} + k^{-2}\right) G(n, m)$ there is the same shift in the powers of k as in the part $G(n, m-2) - 4G(n, m-1) + 6G(n, m) - 4G(n, m+1) + G(n, m+2)$ and all the powers of k sum to zero. But, in the case of z_- there is no shift in the powers of z_- in the part $\left(k^2 - 4k + 6 - 4k^{-1} + k^{-2}\right) G(n, m)$ thus it cannot be subtracted from any power of z_- in the part $G(n, m-2) - 4G(n, m-1) + 6G(n, m) - 4G(n, m+1) + G(n, m+2)$.
Using the same ideas, it is clear that the kernels $G_{r,l}^1$ fulfill the identity (3.23) and the kernels $G_{r,l}^2$ do not. In these ideas there is most of all important the shift in the powers. Thus we have omitted the factor $k/(2(k-1)^2)$ in the definitions of some kernels. To get $\delta_{m,n}$ on the RHS of the identity (3.23) we can multiply the kernel by any constant after getting the right shift of m, n in the powers. Moreover, the kernels $G_{r,l}$ do not satisfy the property $G(n, n+1) = 1$.

It is necessary to find enough kernels to finish the generalization for the bilplacian. Even if we had enough kernels we need to figure out, how to do the algebraic manipulations in the proof, which is not easy and I do not know how to do them even with kernels $G_{r,l}^1$. Finding such a kernels and the way of finishing the proof with these kernels will be the goal of my future work.

Conclusion

In this project we have studied deeper details of the spectrum of the discrete bilaplace operator with a complex potential. Main result is the proof of the conjecture on the estimate of the green kernel

$$\left|(T^2-\lambda)^{-1}_{m,n}\right| \leq \left|(T^2-\lambda)^{-1}_{0,0}\right|, \ \forall m,n \in \mathbb{Z}.$$

The conjecture was set earlier using some numerical simulations and now it was proved analytically. By denoting $s := |m-n|$ the problem turns into a problem with one non-negative integer parameter. First we have used the bijective transformation of λ which maps the upper half of the unit circle with interval $(0,1)$ onto $\mathbb{C} \setminus [0,16]$. Then, using the Maximum modulus principle, we moved the problem onto the boundary of the unit circle and proved the conjecture there.

In the second chapter, having the proof of the conjecture and thus the optimal spectral enclosures, we have analyzed the boundary curves of these enclosures using some fundamental theorems of the complex analysis. We have shown that their topological and analytical properties are consistent with the results coming from the numerical simulations.

At the end, we have made the the recherche of the proof of the absence of the eigenvalue of discrete Laplace operator with a complex potential in the interior of its essential spectrum. The proof is based on the turning the equation for eigenvalues $J_0 u = \lambda u$ into the discrete Volterra-type integral equation. Using this, we can conveniently analyze the Jost solutions of this equation and show that there exist no $\ell^2(\mathbb{Z})$ solution for λ in the interior of the essential spectrum. The key step of this proof is the definition of kernels for iteration, which fulfills some identity. The identity was modified according to definition of the bilaplacian and enough fitting kernels were unsuccessfully being searched for.

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