Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Pseudospectrum of the discrete Schrödinger operator with a complex step potential and the weak coupling

Pseudospektrum diskrétního Schrödingerova operatorátoru s komplexním schodovitým potenciálem a slabé vazby

Research Project

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ZADÁNÍ VÝZKUMNÉHO ÚKOLU

Student: Bc. Vojtěch Bartoš

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Název práce (anglicky): Pseudospectrum of the discrete Schrödinger operator with a complex step potential and the weak coupling

Pokyny pro vypracování:

1) Nastudujte základní pojmy a obvyklé analytické nástroje používané v analýze psedospektra a slabě vázaných potenciálů [1,2].
2) Proveďte pseudospektrální analýzu diskrétního Schrödingerova oprátoru s komplexním schodovitým potenciálem a výsledky porovnejte se spojitým případem [2].
3) Proveďte analýzu slabě vázaných potenciálů pro případ diskrétního Schrödingerova oprátoru s komplexním schodovitým potenciálem a výsledky porovnejte se spojitým případem [2].
4) Prozkoumejte spektrum diskrétního Schrödingerova oprátoru s komplexním schodovitým potenciálem a Diracovou bodovou interakcí.

Doporučená literatura:

1) J. B. Conway, A course in functional analysis, Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990.
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4) O. O. Ibrogimov, F. Štampach, Spectral enclosures for non-self-adjoint discrete Schrödinger operators, Integr. Equ. Oper. Theory 91, 2019, 1-15.

Jméno a pracoviště vedoucího výzkumného úkolu:

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Author's declaration:
I declare that this Research Project is entirely my own work and I have listed all the used sources in the bibliography.

Prague, May 21, 2023

Bc. Vojtěch Bartoš

Název práce:

Pseudospektrum diskrétního Schrödingerova operatorátoru s komplexním schodovitým potenciálem a slabé vazby

Autor: Bc. Vojtěch Bartoš

Obor: Matematické inženýrství

Druh práce: Výzkumný úkol

Vedoucí práce: Ing. František Štampach, Ph.D., ČVUT, Fakulta jaderná a fyzikálně inženýrská, Katedra matematiky

Abstrakt: Náplní této práce je zkoumání spektrálních vlastností Schrödingerova operátoru H_α na $\ell^2(\mathbb{Z})$ s komplexním schodovitým potenciálem a komplexní vazebnou konstantou α. Zkoumáme pseudospektrum tohoto operátoru. Po nalezení spodního a horního odhadu normy rezolventního operátoru jsme zkonstruovali nadmnožinu a podmnožinu pseudospektra, navíc jsme také odvodili asymptotické chování pseudospektra. Pomocí Birman–Schwingerova principu jsme studovali existenci a jednoznačnost slabě vázaných vlastních hodnot za jistých předpokladů na potenciál V. Operátor H_0+V má jednoznačné vlastní hodnoty. Naproti tomu, pokud $\mathrm{Im}\alpha \neq 0$, potom $H_\alpha+V$ nemá žádné vlastní hodnoty, neboli vykazuje spektralní stabilitu. Tyto výsledky jsme porovnali se spojitým nastavením. Práci jsme zakončili představením problému Diracovy interakce.

Klíčová slova: Birmanův-Schwingerův princip, diskrétní Schrödingerův operátor, nesamosdruženost, pseudospektrum, schodovitý potenciál, slabé vazby, spektrální stabilita

Title:

Pseudospectrum of the discrete Schrödinger operator with a complex step potential and the weak coupling

Author: Bc. Vojtěch Bartoš

*Abstract:*We study spectral properties of a Schrödinger operator H_α on $\ell^2(\mathbb{Z})$ with a step-like potential and complex coupling constant α. We investigate the pseudospectrum of this operator. After obtaining the lower and upper estimates of the resolvent operator's norm, we construct a superset and a subset of the pseudospectrum, in addition, we also derive the asymptotic behavior of the pseudospectrum. Utilizing the Birman–Schwinger principle, we study the existence and uniqueness of weak-coupled eigenvalues under certain assumptions on the potential V. The operator $H_0 + V$ has unique eigenvalues. On the other hand, if $\mathrm{Im}\alpha \neq 0$, then $H_\alpha + V$ has no eigenvalues, i.e. H_α exhibits spectral stability. These results were compared with the continuous setting. The paper concludes by introducing the Dirac interaction.

Key words: Birman-Schwinger principle, discrete Schrödinger operator, non-self-adjointness, pseudospectrum, spectral stability, step-like potential, weak-coupling

Contents

Introduction

Consider the continuous Laplace operator on the real line. Its domain may be defined as the space of twice differentiable functions, and its action is given by

$$\forall f \in C^2(\mathbb{R}) \qquad : \qquad -\Delta f = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

Derivatives, by their nature, are defined by infinitesimal changes, but they can be approximated by finitely small changes. Consider a function f on the real line; we can discretize it simply by restricting its argument to whole numbers. While having at our disposal only the values of f at whole numbers, we can approximate the value of the Laplace operator applied on f at n by

$$-\Delta f(x)\big|_n \approx f(n+1) - 2f(n) + f(n-1).$$

It is important to note that the continuous Laplacian is highly valuable in theoretical analysis, mathematical modeling, and certain scientific disciplines, such as classical physics and differential geometry. It provides a foundation for understanding continuous systems. However, in many practical applications involving discrete data and numerical computations, the discrete Laplacian offers distinct advantages and is more directly applicable.

For our purposes, we will define the discrete Laplacian H_0 as the sum of the forward translation and the backward translation

$$\forall x = \{x_n\}_{n\in\mathbb{Z}} \subset \mathbb{C} \qquad : \qquad (H_0 x)_n := x_{n+1} + x_{n-1}.$$

The difference between this definition and the approximation above is just the identity operator multiplied by two. From the point of view of spectral analysis, this is an insignificant change, as it only shifts everything in the complex plane by 2.

As the title of this project suggests, we study the Schrödinger operator with a complex step potential defined as

$$\forall x = \{x_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{Z}) \ : \ (H_\alpha x)_n = \begin{cases} x_{n-1} + x_{n+1} & n < 0, \\ x_{n-1} + \alpha x_n + x_{n+1} & n \geq 0, \end{cases}$$

where α is a complex parameter. A Schrödinger operator is defined as the Laplace operator with some potential applied, so in our case, the potential can be understood as the discrete Heaviside function multiplied by the complex coupling constant α. Such an operator is non-self-adjoint for any $\alpha \in \mathbb{C} \setminus \mathbb{R}$.

In the first chapter, we state several standard results from functional analysis and recall important results we showed in [14]. These were the resolvent operator and the spectrum of H_α, which is purely continuous and coincides with the two line segments $[-2, 2] \cup [-2 + \alpha, 2 + \alpha]$ in the complex plane.

The aim of the second chapter is to describe ε-pseudospectra of H_α which are nested supersets of the spectrum where the resolvent operator's norm is greater than the reciprocal value of ε. After defining the pseudospectrum, we mention the trivial case, when the operator is self-adjoint. Next, we introduce theorems for estimating the norm of an operator, or more specifically, the resolvent operator's norm, with which we estimate the pseudospectrum.

In the third chapter, we replicated the results obtained by Simon in [8], where he showed the properties of weakly coupled bound states of the continuous Schrödinger operator, for the discrete operator H_0. Then we showed that H_α exhibits spectral stability under some assumptions on the potential and compared it to the continuous case from [10]. The last chapter introduces the problem of the Dirac interaction.

Chapter 1

Preliminaries

1.1 Standard results from functional analysis

Recall several standard results, see [1], [2].

Definition 1.1 (Bounded operator)**:** Let $\mathscr{H}$ be a Hilbert space. An operator $A : \mathscr{H} \to \mathscr{H}$ is said to be *bounded* if

$$\exists M > 0,\ \forall x \in \mathscr{H}\ :\ \|Ax\| < M\|x\|.$$

The smallest of these constants M is called the *norm* of A and may be defined as

$$\|A\| := \sup_{x \in \mathscr{H}} \frac{\|Ax\|}{\|x\|}.$$

We define the linear space of all linear bounded operators on $\mathscr{H}$ by

$$\mathscr{B}(\mathscr{H}) := \big\{A : \mathscr{H} \to \mathscr{H} \ \big|\ \|A\| < \infty\big\}.$$

Definition 1.2 (Compact operator)**:** An operator $A \in \mathscr{B}(\mathscr{H})$ is said to be *compact* if the image of a bounded set is precompact. The set of all compact operators is denoted by $\mathscr{K}(\mathscr{H})$.

Definition 1.3: Let $A \in \mathscr{B}(\mathscr{H})$. The numerical range Num(A) is the image of the unit sphere of H under the quadratic form $x \mapsto \langle x, Ax\rangle$ associated with the operator. More precisely,

$$\mathrm{Num}(A) := \big\{\langle x, Ax\rangle \ \big|\ \|x\| = 1\big\}.$$

Theorem 1.4 (Birman–Schwinger principle)**:** Let $H, V \in \mathscr{B}(\mathscr{H})$, $\lambda \in \rho(H)$. Let us decompose operator V such that $A, B \in \mathscr{B}(\mathscr{H})$ and $V = AB$, next we define the Birman–Schwinger operator

$$K(\lambda) := B(H - \lambda)^{-1}A.$$

Then

1. $\lambda \in \sigma_{\mathrm{p}}(H + V) \qquad \Longrightarrow \qquad -1 \in \sigma_{\mathrm{p}}(K(\lambda)),$
2. $V \in \mathscr{K}(\mathscr{H}) \quad \& -1 \in \sigma_{\mathrm{p}}(K(\lambda)) \qquad \Longrightarrow \qquad \lambda \in \sigma_{\mathrm{p}}(H + V).$

Proof. This theorem was proven in my bachelor's degree project, see [14]. □

Notation 1.5: Assuming $v \equiv \{v_n\}_{n\in\mathbb{Z}}$ is a complex-valued sequence. Let us denote

$$\begin{aligned} & & V &:= \operatorname{diag}(v),\\ |v| &:= \big\{|v_n|\big\}_{n\in\mathbb{Z}}, & |V| &:= \operatorname{diag}(|v|),\\ |v|^{1/2} &:= \big\{\sqrt{|v_n|}\big\}_{n\in\mathbb{Z}}, & |V|^{1/2} &:= \operatorname{diag}(|v|^{1/2}),\\ v_{1/2} &:= \big\{\sqrt{|v_n|}\operatorname{sgn} v_n\big\}_{n\in\mathbb{Z}}, & V_{1/2} &:= \operatorname{diag}(v_{1/2}), \end{aligned}$$

where the signum function for complex inputs is given by

$$\operatorname{sgn} z := \begin{cases} \frac{z}{|z|} & z \neq 0,\\ 0 & z = 0. \end{cases}$$

For the purposes of this paper, we will formulate Theorem 1.4 with the Hilbert space being $\ell^2(\mathbb{Z})$ and some extra assumptions on the potential V, namely diagonality so we may use Notation 1.5. A common choice for decomposing the diagonal operator V in the Birman–Schwinger principle is to set

$$A = |V|^{1/2}, \qquad B = V_{1/2}.$$

Apart from diagonality, if we impart the assumption on V that the sequence v is summable in absolute value, i.e. $v \in \ell^1(\mathbb{Z})$, then necessarily $\lim_{n\to\pm\infty} v_n = 0$. Since

$$V \in \mathscr{K}(\mathscr{H}) \qquad \Longleftrightarrow \qquad \lim_{n\to\pm\infty} v_n = 0,$$

the potential V is a compact operator. With this assumption, the Birman–Schwinger principle states an equivalence between -1 being an eigenvalue of $K(\lambda)$ and λ being an eigenvalue of $H+V$.

Theorem 1.6: Let $H \in \mathscr{B}(\ell^2(\mathbb{Z}))$, $\lambda \in \rho(H)$, $v \in \ell^1(\mathbb{Z})$, then

$$\lambda \in \sigma_{\mathrm{p}}(H+V) \qquad \Longleftrightarrow \qquad -1 \in \sigma_{\mathrm{p}}(K(\lambda)).$$

Let us introduce a set of important statements from spectral analysis theory. For reference, see Chapter XIII.1 in [2].

Definition 1.7 (Bounded from below operator)**:** A densely defined operator A on a Hilbert space $\mathscr{H}$ is said to be *bounded from below* if

$$\exists c \in \mathbb{R},\ \forall \psi \in \operatorname{Dom} A \ :\ \langle \psi, A\psi\rangle \geq c\|\psi\|^2.$$

Theorem 1.8 (Min-Max)**:** Let A be a self-adjoint operator on $\mathscr{H}$ bounded from below. Define

$$\lambda_n(A) := \sup_{\psi_1,\dots,\psi_{n-1}\in\operatorname{Dom}A} \inf\big\{\langle\psi, A\psi\rangle \bigm| \psi \in \operatorname{Dom}A, \psi \perp \psi_1,\dots,\psi_{n-1}, \|\psi\| = 1\big\}.$$

Then for each fixed n, either

(a) there are n eigenvalues(counting degenerate eigenvalues a number of times equal to their multiplicity) bellow the bottom of the essential spectrum, and $\lambda_n(A)$ is the nth eigenvalue counting multiplicity;

(b) λ_n is the bottom of the essential spectrum, i.e. $\lambda_n = \inf\{\lambda \mid \lambda \in \sigma_{\mathrm{ess}}(A)\}$ an in that case $\lambda_n = \lambda_{n+1} = \dots$ and there are at mot $n-1$ eigenvalues (counting multiplicity) below λ_n.

Remark. One can define $\lambda_n(A)$ equivalently by

$$\lambda_n(A) = \inf_{\psi_1,\ldots,\psi_n \in \mathrm{Dom} A} \sup \left\{ \langle \psi, A\psi \rangle \;\middle|\; \psi \in \mathrm{span}(\psi_1, \ldots, \psi_n), \|\psi\| = 1 \right\}.$$

If we wish to investigate eigenvalues above the essential spectrum, the operator in question has to be bounded from above and it suffices to set $A \mapsto -A$, since $\lambda \in \sigma(A)$ if and only if $-\lambda \in \sigma(-A)$. Factoring the negative sign out of supremum changes it to infimum, and vice versa. Let us also state an immediate corollary of the Min-Max theorem which allows us to compare eigenvalues of different operators. We will formulate this corollary for bounded operators.

Corollary 1.9: Let $A, B \in \mathscr{B}(\mathscr{H})$ be self-adjoint operators. Then for eigenvalues either below the essential spectra of the respective operators holds the implication

$$A \leq B \qquad \Longrightarrow \qquad \forall n \in \mathbb{N} \;:\; \lambda_n(A) \leq \lambda_n(B).$$

Proof. Let us prove only the case, where the eigenvalues are below the essential spectrum; the other case is analogous. The operator inequality is in the sense of quadratic forms

$$\forall \psi \in \mathscr{H} \;:\; \langle \psi, A\psi \rangle \leq \langle \psi, B\psi \rangle.$$

Choose an arbitrary n-tuple of vectors $\psi_1, \ldots, \psi_n \in \mathscr{H}$, then

$$\sup \left\{ \langle \psi, A\psi \rangle \;\middle|\; \psi \in \mathrm{span}(\psi_1, \ldots, \psi_n), \|\psi\| = 1 \right\} \leq \sup \left\{ \langle \psi, B\psi \rangle \;\middle|\; \psi \in \mathrm{span}(\psi_1, \ldots, \psi_n), \|\psi\| = 1 \right\}.$$

Taking the infimum over all choices of these n-tuples in the inequality above proves the assertion. $\square$

1.2 Main results from bachelor's degree project

This research project is a direct continuation of my bachelor's degree project [14]. Let us reiterate the setting and some of the results which will be used in this text.

We studied the bounded non-self-adjoined operator H_α on the Hilbert space $\ell^2(\mathbb{Z})$ defined as

$$\forall x \equiv \{x_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{Z}) \;:\; (H_\alpha x)_n = \begin{cases} x_{n-1} + x_{n+1} & n < 0, \\ x_{n-1} + \alpha x_n + x_{n+1} & n \geq 0, \end{cases}$$

where α is a free complex parameter. The matrix representation of H_α reads

$$H_\alpha = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & \alpha & 1 & & \\ & & & 1 & \alpha & 1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

To find the spectrum of H_α we utilized the Joukowsky transform, more on this topic in Section 1.3, on λ and $\lambda - \alpha$ separately, i.e.

$$\lambda = \xi + \xi^{-1}, \qquad \lambda - \alpha = \eta + \eta^{-1}.$$

This transformation allowed us to find the solution of the eigenvalue difference equation in a simple form

$$\begin{aligned} \lambda x_n &= x_{n-1} + x_{n+1} \qquad n < 0, \\ (\lambda - \alpha) x_n &= x_{n-1} + x_{n+1} \qquad n \geq 0. \end{aligned} \tag{1.2.1}$$

$$y_n = \begin{cases} \xi^{-n} & n < 0, \\ \frac{\xi - \eta}{\eta^{-1} - \eta} \eta^n + \frac{\xi - \eta^{-1}}{\eta - \eta^{-1}} \eta^{-n} & n \geq 0, \end{cases} \tag{1.2.2}$$

$$z_n = \begin{cases} \frac{\eta^{-1} - \xi}{\xi^{-1} - \xi} \xi^n + \frac{\eta^{-1} - \xi^{-1}}{\xi - \xi^{-1}} \xi^{-n} & n < 0, \\ \eta^n & n \geq 0, \end{cases} \tag{1.2.3}$$

The spectrum of the operator H_α is purely continuous and coincides with the set

$$[-2, 2] + \alpha\{0, 1\}$$

depicted in Figure 1.1. We have also discovered the numerical range of H_α to be

$$\mathrm{Num}(H_\alpha) = [-2, 2] + [0, \alpha].$$

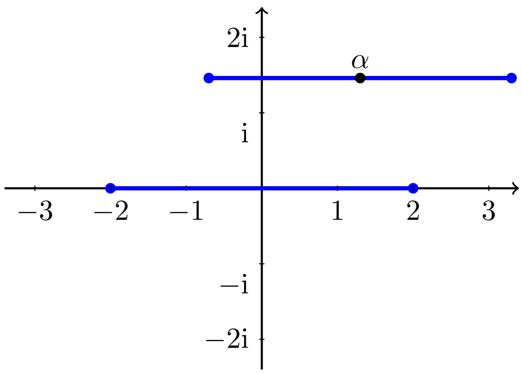


Figure 1.1: Spektrum operátoru H_α

We have also managed to get an explicit description of the resolvent operator using the Green Kernel theorem, thus denoted $G(\lambda) := (H_\alpha - \lambda)^{-1}$, which reads for $\lambda \in \rho(H_\alpha)$

$$G_{m,n}(\lambda) = \frac{1}{w} \begin{cases} \frac{\xi - \eta}{\eta^{-1} - \eta} \eta^{m+n} + \frac{\xi - \eta^{-1}}{\eta - \eta^{-1}} \eta^{|m-n|} & m, n \geq 0, \\ \eta^m \xi^{-n} & m \geq 0, n < 0, \\ \frac{\eta^{-1} - \xi}{\xi^{-1} - \xi} \xi^{|m-n|} + \frac{\eta^{-1} - \xi^{-1}}{\xi - \xi^{-1}} \xi^{-m-n} & m, n < 0, \\ \eta^n \xi^{-m} & m < 0, n \geq 0, \end{cases} \tag{1.2.4}$$

where $w = \xi - \eta^{-1}$.

For ease of notation, it is useful to set

$$A := \frac{\xi - \eta}{\eta^{-1} - \eta}, \qquad B := \frac{\xi - \eta^{-1}}{\eta - \eta^{-1}}, \qquad C := \frac{\eta^{-1} - \xi}{\xi^{-1} - \xi}, \qquad D := \frac{\eta^{-1} - \xi^{-1}}{\xi - \xi^{-1}}. \tag{1.2.5}$$

Thus, writing

$$G_{m,n}(\lambda) = \begin{cases} \frac{A\eta^{m+n}+B\eta^{|m-n|}}{w} & m,n \geq 0, \\ \frac{\eta^m \xi^{-n}}{w} & m \geq 0, n < 0, \\ \frac{C\xi^{|m-n|}+D\xi^{-m-n}}{w} & m,n < 0, \\ \frac{\eta^n \xi^{-m}}{w} & m < 0, n \geq 0, \end{cases}. \tag{1.2.6}$$

By direct calculation

$$\frac{A}{w} = \frac{1}{\eta^{-1}-\eta} + \frac{1}{w}, \qquad \frac{B}{w} = \frac{1}{\eta-\eta^{-1}}, \qquad \frac{C}{w} = \frac{1}{\xi-\xi^{-1}}, \qquad \frac{D}{w} = \frac{1}{\xi^{-1}-\xi} + \frac{1}{w}. \tag{1.2.7}$$

In some cases, it will be more useful to formulate the Green kernel as such

$$G_{m,n}(\lambda) = \begin{cases} \frac{\eta^{m+n}-\eta^{|m-n|}}{\eta^{-1}-\eta} + \frac{\eta^{m+n}}{w} & m,n \geq 0, \\ \frac{1}{w}\eta^m \xi^{-n} & m \geq 0, n < 0, \\ \frac{\xi^{|m-n|}-\xi^{-m-n}}{\xi-\xi^{-1}} + \frac{\xi^{-m-n}}{w} & m,n < 0, \\ \frac{1}{w}\eta^n \xi^{-m} & m < 0, n \geq 0. \end{cases} \tag{1.2.8}$$

1.3 Joukowsky transform

Definition 1.10: Joukowsky transform is a bijective map between the sets

$$\mathbb{C} \setminus [-2,2] \qquad \longleftrightarrow \qquad \big\{\xi \in \mathbb{C} \;\big|\; 0 < |\xi| < 1\big\}$$

given by the equation

$$\lambda = \xi + \xi^{-1}. \tag{1.3.1}$$

Furthermore, (1.3.1) maps $\big\{\xi \in \mathbb{C} \;\big|\; 0 < |\xi| \leq 1\big\}$ onto the whole complex plane, though not injectively.

Remark. Given the two Joukowsky transforms $\lambda = \xi + \xi^{-1}$ and $\lambda - \alpha = \eta + \eta^{-1}$, for any $\lambda \in \mathbb{C} \setminus \big([-2,2] \cup [-2+\alpha, 2+\alpha]\big)$ there exist unique Joukowsky parameters ξ and η such that $0 < |\xi| < 1$ and $0 < |\eta| < 1$.

Proposition 1.11: Let $\lambda \in \mathbb{C} \setminus [-2,2]$. The inverse to the Joukowsky transform $\lambda = \xi + \xi^{-1}$ is given by

$$\xi(\lambda) = \begin{cases} \big(\lambda + \sqrt{\lambda^2-4}\big)/2 & \mathrm{Re}\lambda < 0 \text{ or } \mathrm{Re}\lambda = 0, \mathrm{Im}\lambda \leq 0, \\ \big(\lambda - \sqrt{\lambda^2-4}\big)/2 & \mathrm{Re}\lambda > 0 \text{ or } \mathrm{Re}\lambda = 0, \mathrm{Im}\lambda \geq 0, \end{cases}$$

where $\sqrt{\cdot}$ assumes its principal branch. Furthermore, the reciprocal value of ξ is given by

$$\big(\xi(\lambda)\big)^{-1} = \begin{cases} \big(\lambda - \sqrt{\lambda^2-4}\big)/2 & \mathrm{Re}\lambda < 0 \text{ or } \mathrm{Re}\lambda = 0, \mathrm{Im}\lambda \leq 0, \\ \big(\lambda + \sqrt{\lambda^2-4}\big)/2 & \mathrm{Re}\lambda > 0 \text{ or } \mathrm{Re}\lambda = 0, \mathrm{Im}\lambda \geq 0. \end{cases}$$

Proof. Let us break down the proof of this proposition into three parts.

1. Form of ξ:

   If we multiply the definitory equation of the Joukowsky transform by a non-zero ξ, we obtain a quadratic equation $\xi^2 - \lambda\xi + 1 = 0$, the solutions of which read

$$\xi_\pm = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}.$$

2. Reciprocal value of ξ:

   Since the constant term in a quadratic equation is the product of its roots and the constant term in the studied equation is 1, we have $\xi_+\xi_- = 1$; hence,

$$\xi_\pm^{-1} = \xi_\mp = \frac{\lambda \mp \sqrt{\lambda^2 - 4}}{2}.$$

   From this follows the equivalence

$$|\xi_+| \geq 1 \qquad \Longleftrightarrow \qquad |\xi_-| \leq 1.$$

3. Piece-wise nature of ξ:

   It suffices to show the following

$$\mathrm{Re}\lambda > 0 \text{ or } \mathrm{Re}\lambda = 0, \mathrm{Im}\lambda \geq 0 \qquad \Longrightarrow \qquad \left|\frac{\lambda - \sqrt{\lambda^2 - 4}}{2}\right| \leq \left|\frac{\lambda + \sqrt{\lambda^2 - 4}}{2}\right|. \tag{1.3.2}$$

   - Let $\mathrm{Re}\lambda \geq 0$, then one may write $\lambda = x + \mathrm{i}y$, where $x \geq 0$ and $y \in \mathbb{R}$. Considering the principal square root, we know that $\forall z \in \mathbb{C} \;:\; \mathrm{Re}\sqrt{z} \geq 0$. This allows us to write $\sqrt{\lambda^2 - 4} = u + \mathrm{i}v$, where also $u \geq 0$ and $v \in \mathbb{R}$. Let us rewrite the inequality in (1.3.2)

$$\begin{aligned} \left|\frac{x + \mathrm{i}y - u - \mathrm{i}v}{2}\right| &\leq \left|\frac{x + \mathrm{i}y + u + \mathrm{i}v}{2}\right|, \\ (x - u)^2 + (y - v)^2 &\leq (x + u)^2 + (y + v)^2, \\ 0 &\leq 4xu + 4yv. \end{aligned}$$

     The assumption above and said property of complex square root implies that $4xu \geq 0$. To conclude this proof we need to show that $\mathrm{sgn}\, y = \mathrm{sgn}\, v$, i.e. $\mathrm{sgn}\, \mathrm{Im}\lambda = \mathrm{sgn}\, \mathrm{Im}\sqrt{\lambda^2 - 4}$. Applying the formula for the square root of a complex number

$$\sqrt{a + \mathrm{i}b} = \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}} + \mathrm{i}\, \mathrm{sgn}\,(b) \sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}}$$

     on $\sqrt{\lambda^2 - 4}$ we get

$$\begin{aligned} \mathrm{Im}\sqrt{\lambda^2 - 4} &= \mathrm{Im}\sqrt{(x^2 - y^2 - 4) + \mathrm{i}(2xy)} \\ &= \mathrm{sgn}\,(2xy) \sqrt{\frac{\sqrt{(x^2 - y^2 - 4)^2 + (2xy)^2} - x^2 + y^2 + 4}{2}}. \end{aligned}$$

     From this one easily sees

$$\mathrm{sgn}\left(\mathrm{Im}\sqrt{\lambda^2 - 4}\right) = \mathrm{sgn}\left(2\underbrace{\mathrm{Re}\lambda}_{\geq 0}\mathrm{Im}\lambda\right) = \mathrm{sgn}\left(\mathrm{Im}\lambda\right).$$

- Let $\mathrm{Re}\lambda = 0, \mathrm{Im}\lambda \geq 0$, then one may write $\lambda = \mathrm{i}c$, where $c \geq 0$. A simple estimation yields (1.3.2); we have

$$\big|\lambda - \sqrt{\lambda^2 - 4}\big| = \big|\mathrm{i}c - \sqrt{-c^2 - 4}\big| \leq c + \sqrt{c^2 + 4} = \big|\mathrm{i}c + \sqrt{-c^2 - 4}\big| = \big|\lambda + \sqrt{\lambda^2 - 4}\big|$$

Similarly, one would show the other implication

$$\mathrm{Re}\lambda < 0 \text{ or } \mathrm{Re}\lambda = 0, \mathrm{Im}\lambda \leq 0 \qquad \Longrightarrow \qquad \left|\frac{\lambda - \sqrt{\lambda^2 - 4}}{2}\right| \geq \left|\frac{\lambda + \sqrt{\lambda^2 - 4}}{2}\right|.$$

□

Proposition 1.12: Assuming the bijective relation between $\lambda \sim \xi$ given by the Joukowsky transform $\lambda = \xi + \xi^{-1}$, the following statements hold:

$$\mathrm{Im}\lambda \neq 0 \qquad \Longrightarrow \qquad \exists \delta < 1 \;:\; |\xi(\lambda)| \leq \delta,$$

$$\lambda \in \mathbb{C} \qquad \Longrightarrow \qquad \exists \rho > 0 \;:\; |\xi(\lambda)| \geq \rho.$$

Proof. Clearly, the function $R : \mathbb{R}_+ \to (0, 1) : t \mapsto \frac{1}{2}\big(\sqrt{t^2 + 4} - t\big)$ is continuous and monotonic. The limits of R at the boundary of its domain are $R(0_+) = 1$ and $R(+\infty) = 0$; therefore, it is also decreasing. Let us describe ξ in polar form $\xi = re^{\mathrm{i}\phi}$, where $\phi \in (-\pi, \pi]$ and $r \in (0, 1]$. For the first statement, we estimate

$$a := |\mathrm{Im}(\lambda)| = |\mathrm{Im}(\xi + \xi^{-1})| = \left|\left(r - \frac{1}{r}\right)\sin\phi\right| \leq \frac{1}{r} - r. \tag{1.3.3}$$

This leads to the quadratic inequality

$$r^2 + ra - 1 \leq 0.$$

The solution of its boundary equation is $r = R(a) = \frac{1}{2}\big(\sqrt{a^2 + 4} - a\big)$. We ignore the other quadratic solution because we assume r to be positive; hence, the solution of the inequality with this assumption is $0 < r \leq \frac{1}{2}\big(\sqrt{a^2 + 4} - a\big)$. Therefore, we may set

$$\delta := \frac{1}{2}\Big(\sqrt{|\mathrm{Im}(\lambda)|^2 + 4} - |\mathrm{Im}(\lambda)|\Big).$$

For the second statement, we estimate

$$b := |\lambda| = |\xi + \xi^{-1}| = \left|re^{\mathrm{i}\phi} + \frac{1}{r}re^{-\mathrm{i}\phi}\right| \geq \left|\frac{1}{r}re^{-\mathrm{i}\phi}\right| - \left|re^{\mathrm{i}\phi}\right| = \frac{1}{r} - r.$$

This leads to a quadratic inequality

$$r^2 + rb - 1 \geq 0,$$

the boundary equation of which is similar to the case above. Its solution of the boundary equation is $r = R(b) = \frac{1}{2}\big(\sqrt{b^2 + 4} - b\big)$. Hence, for any fixed $b > 0$ the value $r(b)$ is also positive. The solution of the inequality is $\frac{1}{2}\big(\sqrt{b^2 + 4} - b\big) \leq r \leq 1$. Therefore,

$$\rho := \frac{1}{2}\Big(\sqrt{|\lambda|^2 + 4} - |\lambda|\Big).$$

□

Remark. Considering the other Joukowsky transform $\lambda - \alpha = \eta + \eta^{-1}$ we get

$$\mathrm{Im}(\lambda - \alpha) \neq 0 \quad \Longrightarrow \quad \exists \delta < 1 \;:\; |\eta(\lambda)| \leq \delta,$$
$$\lambda \in \mathbb{C} \quad \Longrightarrow \quad \exists \rho > 0 \;:\; |\eta(\lambda)| \geq \rho.$$

Proposition 1.13: Let $\mathrm{Im}\alpha \neq 0$ and consider the two Joukowsky transforms $\lambda = \xi + \xi^{-1}$ and $\lambda - \alpha = \eta + \eta^{-1}$, then

(a) $|\xi| > \frac{1}{2}\big(\sqrt{|\mathrm{Im}\alpha|^2 + 4} - |\mathrm{Im}\alpha|\big) \quad \Longrightarrow \quad \exists \delta_2 < 1,\ \exists \rho_2 > 0 \;:\; \rho_2 \leq |\eta| \leq \delta_2.$

(b) $|\eta| > \frac{1}{2}\big(\sqrt{|\mathrm{Im}\alpha|^2 + 4} - |\mathrm{Im}\alpha|\big) \quad \Longrightarrow \quad \exists \delta_1 < 1,\ \exists \rho_1 > 0 \;:\; \rho_1 \leq |\xi| \leq \delta_1,$

Proof. Since the two statements are analogous to each other, we shall prove only one; let us choose (a). Consider ξ in polar form $\xi = re^{\mathrm{i}\phi}$, where $\phi \in (-\pi, \pi]$ and $r \in (0, 1]$. Let us show a series of implications and justify them individually

$$|\xi| > \frac{1}{2}\big(\sqrt{|\mathrm{Im}\alpha|^2 + 4} - |\mathrm{Im}\alpha|\big) \quad \Longrightarrow \quad \frac{1}{r} - r < |\mathrm{Im}\alpha| \quad \Longrightarrow \quad |\mathrm{Im}\lambda| < |\mathrm{Im}\alpha|$$
$$\Longrightarrow \quad |\mathrm{Im}(\lambda - \alpha)| \neq 0 \quad \Longrightarrow \quad \exists \delta_2 < 1,\ \exists \rho_2 > 0 \;:\; \rho_2 \leq |\eta| \leq \delta_2.$$

If we denote $c := |\mathrm{Im}\alpha|$, the first implication follows from the fact that the right side of the implication is equivalent to $r^2 + cr - 1 > 0$. Similarly to the proof above, the solution is $\frac{1}{2}\big(\sqrt{c^2 + 4} - c\big) < r \leq 1$. The second implication follows from the inequality (1.3.3), and the last implication follows from Proposition 1.12. ☐

Lemma 1.14: Let $\mathrm{Im}\alpha \neq 0$. Then there exists $\widehat{w} > 0$ such that

$$\forall \lambda \in \mathbb{C} \;:\; w(\lambda) \geq \widehat{w},$$

where $w = \xi - \eta^{-1}$ is the Wronskian.

Proof. Let us decompose the complex plane into two sets U^η and U^ξ. If $\mathrm{Im}\alpha > 0$, we set

$$U^\eta := \big\{\lambda \in \mathbb{C} \;\big|\; |\mathrm{Im}\lambda| > |\mathrm{Im}\alpha|/2\big\} \qquad \text{and} \qquad U^\xi := \big\{\lambda \in \mathbb{C} \;\big|\; |\mathrm{Im}\lambda| \leq |\mathrm{Im}\alpha|/2\big\}.$$

Notice that

$$\big\{\lambda \in \mathbb{C} \;\big|\; |\eta| = 1\big\} \subset U^\eta \qquad \text{and} \qquad \big\{\lambda \in \mathbb{C} \;\big|\; |\xi| = 1\big\} \subset U^\xi.$$

This decomposition is useful because we can use the proof of Proposition 1.12 for all ξ corresponding to $\lambda \in U^\eta$

$$|\xi| \leq \frac{1}{2}\Big(\sqrt{|\mathrm{Im}\lambda|^2 + 4} - |\mathrm{Im}\lambda|\Big) \leq \frac{1}{2}\left(\sqrt{\frac{|\mathrm{Im}\alpha|^2}{4} + 4} - \frac{|\mathrm{Im}\alpha|}{2}\right) =: \delta,$$

where the second inequality follows from the fact that the function $t \mapsto \big(\sqrt{t^2 + 4} - t\big)/2$ is decreasing and the definitory property of the set U^η. Similarly, we get the same estimate for all η corresponding to $\lambda \in U^\xi$. For the Joukowsky parameter η the proof of Proposition 1.12 we have

$$\mathrm{Im}(\lambda - \alpha) \neq 0 \quad \Longrightarrow \quad |\eta| \leq \frac{1}{2}\Big(\sqrt{|\mathrm{Im}(\lambda - \alpha)|^2 + 4} - |\mathrm{Im}(\lambda - \alpha)|\Big). \tag{1.3.4}$$

With the inequality defining the set U^ξ in mind let us estimate

$$\big|\mathrm{Im}(\lambda-\alpha)\big| = \big|\mathrm{Im}\lambda - \mathrm{Im}\alpha\big| \geq |\mathrm{Im}\alpha| - |\mathrm{Im}\lambda| \geq |\mathrm{Im}\alpha| - \frac{1}{2}|\mathrm{Im}\alpha| = \frac{1}{2}|\mathrm{Im}\alpha|.$$

Plugging this estimate into (1.3.4) we get $|\eta| \leq \delta$. With these estimates established, we may move to estimate the Wronskian $w = \xi - \eta^{-1}$ itself

$$|w| = |\xi - \eta^{-1}| = \frac{|\xi\eta - 1|}{|\eta|} \geq \frac{1 - |\eta||\xi|}{1} \geq \begin{cases} 1 - \delta|\xi| \geq 1 - \delta & \text{on } U^\xi \\ 1 - \delta|\eta| \geq 1 - \delta & \text{on } U^\eta \end{cases} \geq 1 - \delta =: \widehat{w}$$

□

Chapter 2

Pseudospectrum

In this chapter, we will almost solely rely on the book *Spectra and pseudospectra* by Lloyd Nicholas Trefethen and Mark Embree, see [5]. We will define what pseudospectra are and state their basic properties before we commence pseudospectral analysis of the studied operator H_α. Since these properties are not crucial for this paper and serve only the purpose to familiarize us with this new notion, we will not be providing proofs; one may find them in [5].

Definition 2.1: Let $A \in \mathscr{B}(\mathscr{H})$ and $\varepsilon > 0$ be arbitrary. The *ε-pseudospectrum* of operator A is defined as the set

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{\lambda \in \mathbb{C} \setminus \sigma(A) \;\middle|\; \|(A-\lambda)^{-1}\| \geq \varepsilon^{-1}\right\}.$$

Theorem 2.2: Given $A \in \mathscr{B}(\mathscr{H})$, the pseudospectra $\left\{\sigma_\varepsilon(A)\right\}_{\varepsilon>0}$ have the following properties.

- Each $\sigma_\varepsilon(A)$ is a nonempty subset of $\mathbb{C}$.
- Any bounded connected component of $\sigma_\varepsilon(A)$ has a nonempty intersection with $\sigma(A)$.
- The pseudospectra are strictly nested supersets of the spectrum.

Describing the pseudospectra of self-adjoint operators is a relatively trivial endeavor because we can describe the norm of the resolvent in terms of the distance between its spectral parameter and the spectrum of the operator.

Theorem 2.3: Let $A \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator and $\lambda \in \varrho(A)$, then

$$\|(A-\lambda)^{-1}\| = \frac{1}{\operatorname{dist}(\sigma(A), \lambda)}.$$

Therefore, given a bounded self-adjoint operator with a known spectrum, one can explicitly describe the ε-pseudospectra as the ε-neighborhood of $\sigma(A)$.

Corollary 2.4: Let $A \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator, then

$$\varepsilon > 0: \qquad \sigma_\varepsilon(A) = \sigma(A) \cup \left\{\lambda \in \mathbb{C} \setminus \sigma(A) \;\middle|\; \operatorname{dist}(\sigma(A), \lambda) \leq \varepsilon\right\} = \sigma(A) + D(\varepsilon),$$

where $D(\varepsilon)$ denotes a centered disc in the complex plane with radius ε.

Let us now apply these results to operator H_α. The following proposition immediately follows from the corollary above.

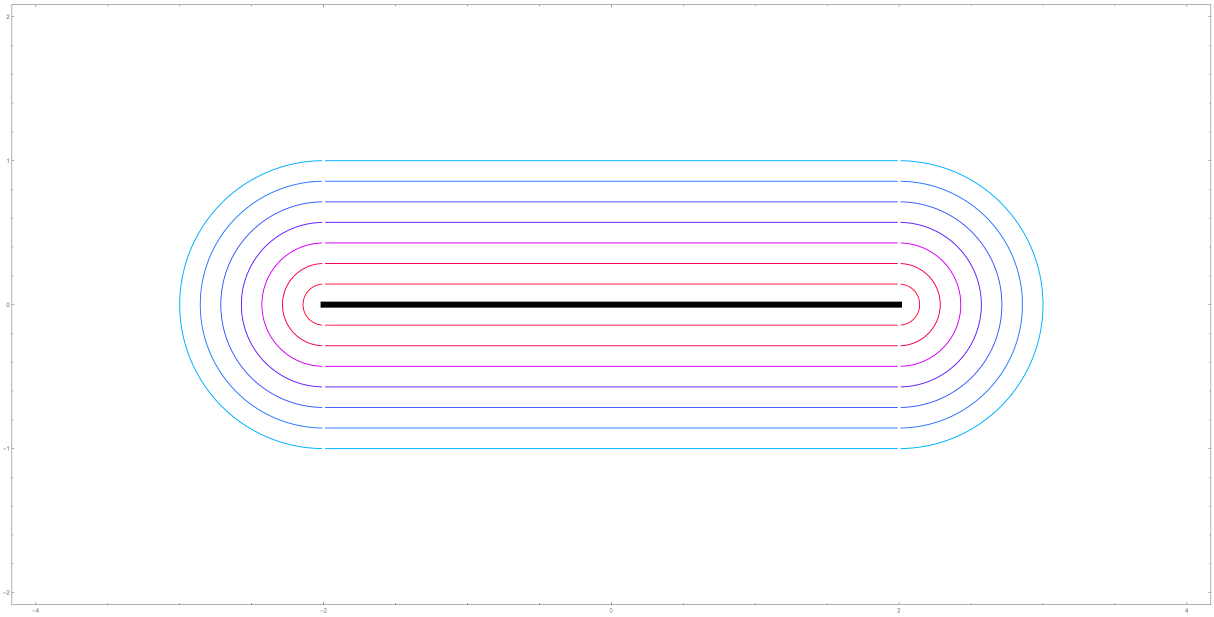


Figure 2.1: $\sigma_\varepsilon(H_0)$ for $\varepsilon = \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{7}{7}$

Proposition 2.5: Let α be a real number. Then

$$\varepsilon > 0: \qquad \sigma_\varepsilon(H_\alpha) = [-2, 2] \cup [-2+\alpha, 2+\alpha] \; + \; D(\varepsilon).$$

Visualizations of the ε-pseudospectra of the self-adjoint operator H_0 are in Fig. 2.1 for various values of ε.

Setting aside the trivial case for a real parameter α, we will discuss the general case of α being any complex number. Describing the pseudospectra of non-self-adjoint operators is no more a trivial endeavor. Still, we can get an explicit expression for the pseudospectrum intersected with a certain set. This is justified by the following propositions.

Proposition 2.6: Let $A \in \mathscr{B}(\mathscr{H})$ and $\lambda \in \rho(A)$. Then

$$\frac{1}{\operatorname{dist}\big(\lambda, \sigma(A)\big)} \leq \big\|(A-\lambda)^{-1}\big\|.$$

A lower bound can also be obtained simply from the definition of operator norm. Let A be a bounded operator on a Hilbert space $\mathscr{H}$, then

$$\forall y \in \mathscr{H}, \; \|y\| = 1: \qquad \|A\| = \sup_{\|x\|=1} \|Ax\| \geq \|Ay\|. \tag{2.0.1}$$

We shall refer to such a vector y as a *test vector*. However, in our case a more suitable lower bound is given by the proposition above.

Proposition 2.7 ([6, Proposition 2.8])**:** Let A be a closed operator on $\mathscr{H}$. Let U be a connected open subset of $\mathbb{C} \setminus \overline{\operatorname{Num}(A)}$. If there exists a number $\lambda_0 \in U$ which is contained in $\rho(A)$, then $U \subset \rho(A)$. Moreover,

$$\forall \lambda \in U \;:\; \big\|(A-\lambda)^{-1}\big\| \leq \frac{1}{\operatorname{dist}\big(\lambda, \operatorname{Num}(A)\big)}.$$

Lemma 2.8: Let $K \subset \mathbb{C}$ be bounded closed convex set, then its complement $\mathbb{C} \setminus K$ is connected.

Proof. Let us establish notation for lines, rays, and line segments:

$$\langle a, b\rangle := \big\{(1-t)a + tb \;\big|\; t \in \mathbb{R}\big\} \qquad (a, b\rangle := \big\{(1-t)a + tb \;\big|\; t \in (0, +\infty)\big\}$$
$$[a, b] := \big\{(1-t)a + tb \;\big|\; t \in [0, 1]\big\} \qquad [a, b) := \big\{(1-t)a + tb \;\big|\; t \in [0, 1)\big\}$$

We will show that $\mathbb{C} \setminus K$ is path-connected since it implies connectedness. We need to construct a path connecting any two different points $v_1, v_2 \in \mathbb{C} \setminus K$. The set K is bounded; therefore, we can find $r > 0$ such that $K \subset B(0, r)$. If $\langle v_1, v_2\rangle \cap K = \emptyset$, we connect v_1 and v_2 with the line segment $[v_1, v_2]$. If the line $\langle v_1, v_2\rangle$ has a non-empty intersection with K, then we denote $\langle v_1, v_2\rangle \cap K = \{k_1, k_2\}$, where k_1 and k_2 may equal each other if the intersection is a singleton. Moreover, $\langle v_1, v_2\rangle \cap K$ is necessarily a convex set. Therefore, the line $\langle v_1, v_2\rangle$ decomposes into exactly three disjoint parts which read

$$\langle v_1, v_2\rangle = \langle v_1, k_1) \cup [k_1, k_2] \cup (k_2, v_2\rangle.$$

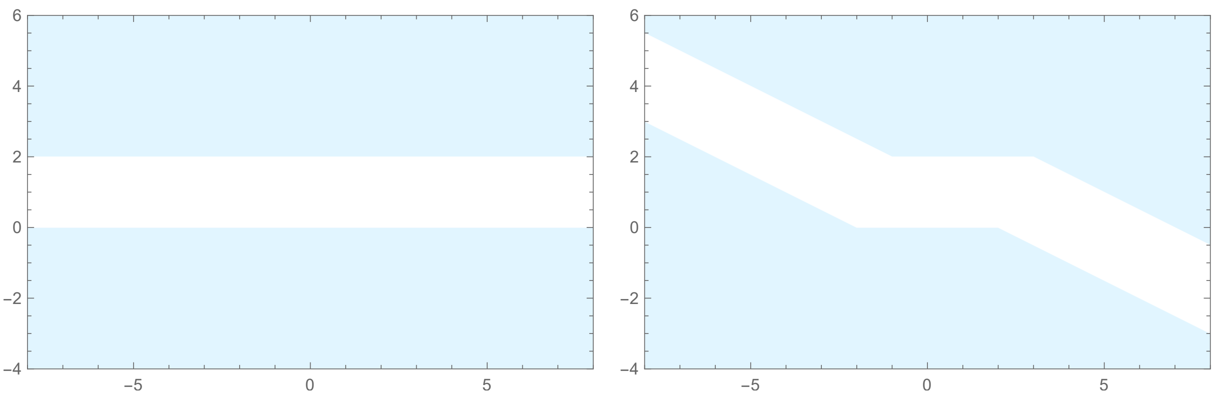
Let us denote $\{b_1\} = \partial B(0, r) \cap [k_1, v_1\rangle$ and $\{b_2\} = \partial B(0, r) \cap [k_2, v_2\rangle$. Now we can connect $v_1, v_2 \in \mathbb{C} \setminus K$ with either one of the paths given by the set $[v_1, b_1) \cup \partial B(0, r) \cup (b_1, v_2]$. □

If an operator is bounded the closure of its numerical range is convex; therefore $\mathbb{C} \setminus \overline{\mathrm{Num}(A)}$ is connected. The restriction to bounded operators simplifies Proposition 2.7 in such a way that the inequality holds for all $\lambda \in \mathbb{C} \setminus \overline{\mathrm{Num}(A)}$.

Let us define

$$\Omega(\alpha) := \big\{z \in \mathbb{C} \;\big|\; \mathrm{dist}\big(\lambda, \mathrm{Num}(H_\alpha)\big) = \mathrm{dist}\big(\lambda, \sigma(H_\alpha)\big)\big\}.$$

In Fig. 2.2, we have depicted the set $\Omega(\alpha)$ for certain values of α. For all $\lambda \in \Omega(\alpha)$ the upper



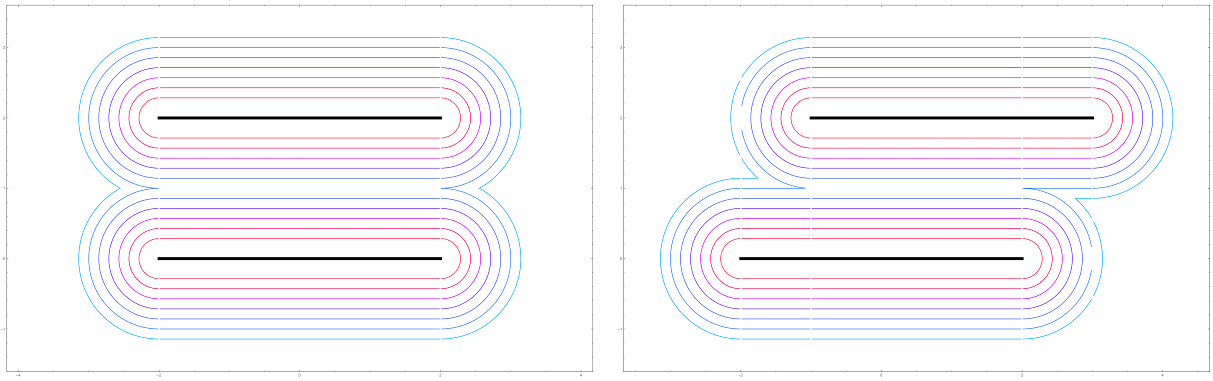
(a) $\alpha = 2\mathrm{i}$ (b) $\alpha = 2\mathrm{i} + 1$

Figure 2.2: Depictions of the set $\Omega(\alpha)$ for certain choices of α.

and lower estimate of $\big\|(A-\lambda)^{-1}\big\|$ equal each other on; therefore, we can describe the ε-spectrum explicitly as

$$\Omega(\alpha) \cap \sigma_\varepsilon(H_\alpha) \;=\; \Omega(\alpha) \cap \big(\sigma(H_\alpha) + D(\varepsilon)\big).$$

To our knowledge, there is no way to describe the pseudospectra of H_α explicitly outside of $\Omega(\alpha)$; therefore, our task is to obtain a superset and a subset of each ε-pseudospectrum. This



(a) $\alpha = 2\mathrm{i}$ (b) $\alpha = 2\mathrm{i}+1$

Figure 2.3: Depictions $\operatorname{dist}\big(\lambda, \sigma(H_\alpha)\big) = \varepsilon$ for $\varepsilon = \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{7}{7}, \frac{8}{7}$

way we get a region where the boundary of the ε-pseudospectrum must reside; this region is the set difference of the superset and the subset. We will obtain these supersets and subsets by estimating the norm of the resolvent above and below. As stated above we obtain a lower bound by virtue of Proposition 2.6. These lower bounds are depicted in Figures 2.3. The tool we will use for obtaining upper estimates of an operator's norm is the Schur test, though it is usually formulated for continuous operators, i.e. operators on the Hilbert space L^2. A discrete formulation (for $\ell^2(\mathbb{N})$) can be found in [7] as an exercise. Let us now present a formulation of the Schur test for an operator on $\ell^2(\mathbb{Z})$.

Theorem 2.9 (Schur test)**:** Let $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ be a doubly infinite matrix of complex entries. If there exists a positive number p_j for all $j \in \mathbb{Z}$ such that

$$\sup_{j\in\mathbb{Z}} \frac{1}{p_j} \sum_{i\in\mathbb{Z}} |a_{i,j}| p_i =: \alpha < \infty \qquad \& \qquad \sup_{i\in\mathbb{Z}} \frac{1}{p_i} \sum_{j\in\mathbb{Z}} |a_{i,j}| p_j =: \beta < \infty,$$

then $A \in \mathscr{B}(\ell^2(\mathbb{Z}))$ and $\|A\| \leq \sqrt{\alpha\beta}$.

Proof. Choose an arbitrary $\psi \in \operatorname{span}\{e_n \mid n \in \mathbb{Z}\}$, then

$$\begin{aligned} \|A\psi\|^2 &= \sum_{i\in\mathbb{Z}} \Big| \sum_{j\in\mathbb{Z}} a_{i,j}\psi_j \Big|^2 \leq \sum_{i\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} \sqrt{|a_{i,j}|p_j} \sqrt{\frac{|a_{i,j}|}{p_j}} |\psi_j| \right)^2 \\ &\overset{\text{C.–S.}}{\leq} \sum_{i\in\mathbb{Z}} \underbrace{\left(\sum_{j\in\mathbb{Z}} |a_{i,j}| p_j \right)}_{\leq \alpha p_i,\ \forall i} \left(\sum_{j\in\mathbb{Z}} \frac{|a_{i,j}|}{p_j} |\psi_j|^2 \right) \overset{\star}{\leq} \alpha \sum_{j\in\mathbb{Z}} \underbrace{\left(\frac{1}{p_j} \sum_{i\in\mathbb{Z}} |a_{i,j}| p_i \right)}_{\leq \beta} |\psi|_i^2 \leq \alpha\beta \|\psi\|^2. \end{aligned}$$

C.–S. signifies the Cauchy–Schwarz inequality and after starred inequality, the interchange of summation is justified by Tonneli's theorem. □

Remark. Let us state a couple of notes on this operator norm inequality theorem:

1. If the given doubly infinite matrix A is symmetric in terms of the absolute value of its entries, then

$$\alpha = \beta \qquad \& \qquad \|A\| \leq \sup_{j\in\mathbb{Z}} \frac{1}{p_j} \sum_{i\in\mathbb{Z}} |a_{i,j}| p_i.$$

2. In some cases we may get estimates in a simple form using $p_j = 1$ for all $j \in \mathbb{Z}$. However, these estimates may be rougher than ones using carefully optimized weights p_j.

2.1 Global behavior

As we have indicated before, to obtain an upper bound we will use the Schur test, Theorem 2.9. We set $p_j = 1$ for all $j \in \mathbb{Z}$. This way we get a readily available, albeit possibly rough, estimate. In this setting, the Schur test implies

$$\|(H_\alpha - \lambda)^{-1}\| \leq \sup_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \big|G_{m,n}(\lambda)\big|.$$

Finding the actual supremum proves to be quite difficult; therefore, we will be taking an estimate of $\sum_{n\in\mathbb{Z}} \big|G_{m,n}(\lambda)\big|$ from above while ridding it of dependence on m. Recall the constants A, B, C, D from (1.2.5). Due to the form that $G(\lambda)$ takes, we need to investigate two branches:

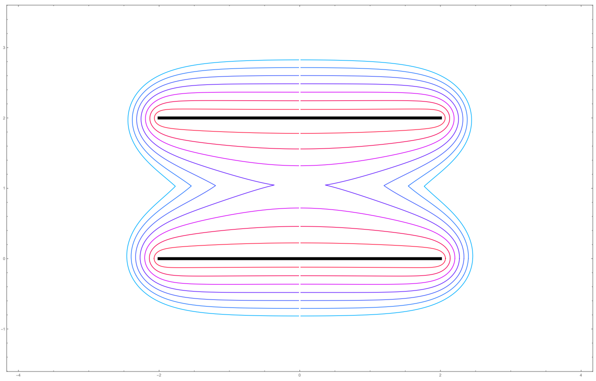
$$\begin{aligned}
m \geq 0: \quad \sum_{n\in\mathbb{Z}} \big|G_{m,n}(\lambda)\big| &= \frac{|\eta|^m}{|w|} \sum_{n=-\infty}^{-1} |\xi|^{-n} + \frac{|\eta|^m}{|w|} \sum_{n=0}^{m-1} \big|A\eta^n + B\eta^{-n}\big| + \frac{\big|A\eta^m + B\eta^{-m}\big|}{|w|} \sum_{n=m}^{\infty} |\eta|^n \\
&\leq \frac{|\eta|^m|\xi|}{|w|(1-|\xi|)} + \frac{|\eta|^m}{|w|}\Big(|A| \underbrace{\sum_{n=0}^{m-1} |\eta|^n}_{=\frac{1-|\eta|^m}{1-|\eta|}} + |B| \underbrace{\sum_{n=0}^{m-1} |\eta^{-1}|^n}_{|\eta|\frac{|\eta|^{-m}-1}{1-|\eta|}}\Big) + \frac{\big|A\eta^m + B\eta^{-m}\big|}{|w|} \frac{|\eta|^m}{1-|\eta|} \\
&= \frac{|\eta|^m|\xi|}{|w|(1-|\xi|)} + \frac{1}{|w|(1-|\eta|)}\Big(|A||\eta|^m - |A||\eta|^{2m} + |B||\eta| - |B||\eta|^{1+m} + |A||\eta|^{2m} + |B|\Big) \\
&\leq \frac{|\xi|}{|w|(1-|\xi|)} + \frac{1}{|w|(1-|\eta|)}\big(|A| + (1+|\eta|)|B|\big) =: U_1(\lambda),
\end{aligned}$$

$$\begin{aligned}
m < 0: \quad \sum_{n\in\mathbb{Z}} \big|G_{m,n}(\lambda)\big| &= \frac{|\xi|^{-m}}{|w|} \sum_{n=1}^{\infty} |\eta|^n + \frac{|\xi|^{-m}}{|w|} \sum_{n=m+1}^{0} \big|C\xi^n + D\xi^{-n}\big| + \frac{\big|C\xi^m + D\xi^{-m}\big|}{|w|} \sum_{n=-\infty}^{m} |\xi|^{-n} \\
&\leq \frac{|\xi|^{-m}|\eta|}{|w|(1-|\eta|)} + \frac{|\xi|^{-m}}{|w|}\Big(|C| \underbrace{\sum_{n=0}^{-m-1} |\xi|^{-n}}_{=|\xi|\frac{|\xi|^{m}-1}{1-|\xi|}} + |D| \underbrace{\sum_{n=0}^{-m-1} |\xi|^n}_{\frac{1-|\xi|^{-m}}{1-|\xi|}}\Big) + \frac{\big|C\xi^m + D\xi^{-m}\big|}{|w|} \frac{|\xi|^{-m}}{1-|\xi|} \\
&\leq \frac{|\xi|^{-m}|\eta|}{|w|(1-|\eta|)} + \frac{1}{|w|(1-|\xi|)}\Big(|C||\xi| - |C||\xi|^{-m+1} + |D||\xi|^{-m} - |D||\xi|^{-2m} + |C| + |D||\xi|^{-2m}\Big) \\
&\leq \frac{|\xi\eta|}{|w|(1-|\eta|)} + \frac{1}{|w|(1-|\xi|)}\big(|D||\xi| + (1+|\xi|)|C|\big) =: U_2(\lambda).
\end{aligned}$$

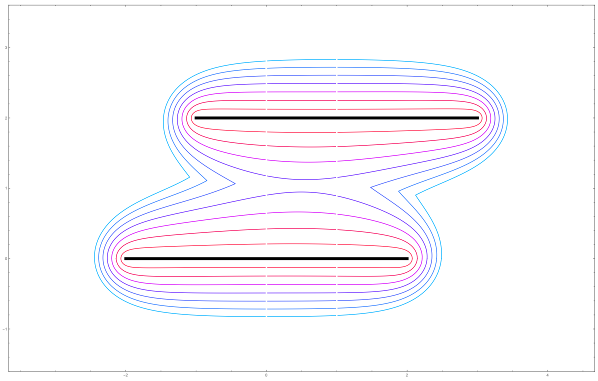
Proposition 2.10: An upper bound of the norm of the resolvent operator takes the form

$$\forall \lambda \in \rho(H_\alpha) \ : \quad \|(H_\alpha - \lambda)^{-1}\| \leq \max\big\{U_1(\lambda), U_2(\lambda)\big\} =: U(\lambda)$$

For illustrations, see Figures 2.4.



(a) $\alpha = 2\mathrm{i}$



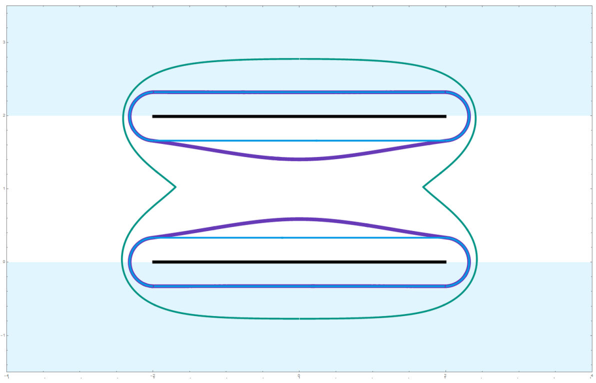
(b) $\alpha = 2\mathrm{i}+1$

Figure 2.4: Depictions $U(\lambda) = \varepsilon$ for $\varepsilon = \frac{20}{1}, \frac{20}{2}, \frac{20}{3}, \frac{20}{4}, \frac{20}{5}, \frac{20}{6}, \frac{20}{7}$

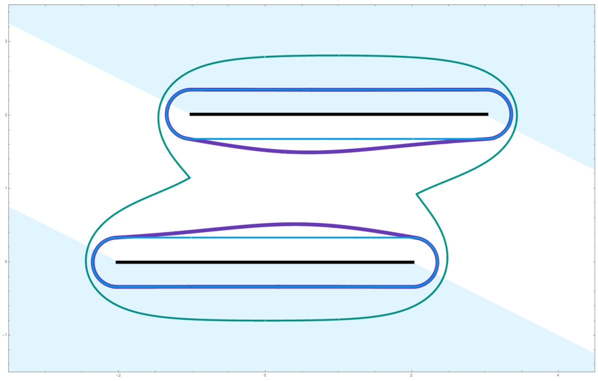
Theorem 2.11: Let $\mathrm{Im}\alpha \neq 0$. Then the ε-pseudospectra of H_α follow these inclusions:

$$\left\{\lambda \in \mathbb{C} \;\middle|\; \frac{1}{\operatorname{dist}\big(\lambda, \sigma(H_\alpha)\big)} \geq \varepsilon^{-1}\right\} \subset \sigma_\varepsilon(H_\alpha) \subset \left\{\lambda \in \mathbb{C} \;\middle|\; U(\lambda) \geq \varepsilon^{-1}\right\}.$$

These estimates are used when we do not have an explicit expression for the resolvent operator's norm, i.e. on $\mathbb{C} \setminus \Omega(\alpha)$. In Fig. 2.5 we ilustrate our findings.



(a) $\alpha = 2\mathrm{i}$



(b) $\alpha = 2\mathrm{i}+1$

Figure 2.5: In these pictures $\varepsilon = 1/3$. The green line is given by $U(\lambda) = \varepsilon^{-1}$. The blue line is given by $\operatorname{dist}\big(\lambda, \sigma(H_\alpha)\big) = \varepsilon$. The purple line is the boundary of the ε-pseudospectrum of a finite-dimensional approximation of H_α. The light blue region is the set $\Omega(\alpha)$ on which we can describe the pseudospectrum exactly by the blue line.

2.2 Asymptotic behavior

In this section we will try to describe the behavior of pseudospectra near the spectrum of H_α. Since the global estimates of $\|(H_\alpha - \lambda)^{-1}\|$ are quite cumbersome, it may be helpful to derive asymptotic formulas for these estimates. We will be heavily using big $\mathcal{O}$ notation so let us first define it for the sake of clarity.

Definition 2.12 (Big $\mathcal{O}$ notation)**:** We say

$$f(x) = \mathcal{O}(g(x)) \text{ as } x \to a \qquad \text{if } \exists \delta, M > 0;\ \forall x \in (a-\delta, a) \cup (a, a+\delta)\ :\ |f(x)| \leq M g(x).$$

Because there are two connected components of $\sigma(H_\alpha)$ (assuming $\mathrm{Im}\alpha \neq 0$), we will be showing asymptotic behavior around each of them. The component $(-2, 2]$ corresponds to $|\xi| = 1$ and the component $[-2+\alpha, 2+\alpha]$ corresponds to $|\eta| = 1$. Let us quickly note that the previous sentence implies that if one of the parameters ξ or η is equal to 1 in absolute value, then the other is strictly less than 1 in absolute value; this is discussed in greater detail in Proposition 1.13. Assuming $\phi \in [-\pi, \pi]$, we will denote these two cases as such

$$\begin{aligned} |\xi| \longrightarrow 1 \quad &: \quad \varepsilon \to 0_+, \text{ where } \xi = (1-\varepsilon)e^{\mathrm{i}\phi}, \\ |\eta| \longrightarrow 1 \quad &: \quad \varepsilon \to 0_+, \text{ where } \eta = (1-\varepsilon)e^{\mathrm{i}\phi}. \end{aligned}$$

The number ϕ describes which point in a particular component are we approaching and from which side. E.g. if $\phi = 0$, ξ or η is approaching 1; therefore, λ is approaching 2 or $2 + \alpha$, respectively.

For easier readability we will divide this section into three parts; in the first part, we will prepare the upper and lower bounds of $\|(H_\alpha - \lambda)^{-1}\|$ in a way that is more suitable for asymptotic analysis than bounds derived in the previous section. Next, we shall derive asymptotic formulas for each of the expressions in the upper and lower bounds. In the last part, we will put it all together and evaluate the result.

2.2.1 Preparation of bounds

A trick we can use while evaluating asymptotic pseudospectra is to separate the quadrants of the matrix representation of the resolvent operator

$$G(\lambda) = \begin{pmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{pmatrix} = \begin{pmatrix} G^{--} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & G^{-+} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ G^{+-} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & G^{++} \end{pmatrix}. \tag{2.2.1}$$

Taking the norm of the resolvent operator and estimating it with triangle inequality may be too rough of an estimate for global estimation but if we show that all but one term are asymptotically insignificant, we can make the asymptotic formula simpler. To make the quadrant separation more clear, let us read them out individually

$$\begin{aligned} G^{++}_{m,n}(\lambda) &= \frac{1}{w}\left(A\eta^{m+n} + B\eta^{|m-n|}\right) & m, n &\geq 0, \\ G^{+-}_{m,n}(\lambda) &= \frac{1}{w}\eta^m \xi^{-n} & m &\geq 0, n < 0, \\ G^{--}_{m,n}(\lambda) &= \frac{1}{w}\left(C\xi^{|m-n|} + D\xi^{-m-n}\right) & m, n &< 0, \\ G^{-+}_{m,n}(\lambda) &= \frac{1}{w}\eta^n \xi^{-m} & m &< 0, n \geq 0. \end{aligned}$$

Since it is not ideal to work with matrices whose indices are on the negative half-line, let us rearrange them appropriately

$$\widetilde{G}^{+-}_{m,n}(\lambda) = \frac{1}{\eta w}\eta^m \xi^n \qquad m,n \geq 1,$$

$$\widetilde{G}^{--}_{m,n}(\lambda) = \frac{1}{w}\left(C\xi^{|m-n|} + D\xi^{m+n}\right) \qquad m,n \geq 1,$$

$$\widetilde{G}^{-+}_{m,n}(\lambda) = \frac{1}{\eta w}\eta^n \xi^m \qquad m,n \geq 1.$$

In [14] we showed that

$$\left\|\begin{pmatrix} G^{--} & 0 \\ 0 & 0 \end{pmatrix}\right\|_{\ell^2(\mathbb{Z})} = \|\widetilde{G}^{--}\|_{\ell^2(\mathbb{N})}, \qquad \left\|\begin{pmatrix} 0 & G^{-+} \\ 0 & 0 \end{pmatrix}\right\|_{\ell^2(\mathbb{Z})} = \|\widetilde{G}^{-+}\|_{\ell^2(\mathbb{N})},$$

$$\left\|\begin{pmatrix} 0 & 0 \\ G^{+-} & 0 \end{pmatrix}\right\|_{\ell^2(\mathbb{Z})} = \|\widetilde{G}^{+-}\|_{\ell^2(\mathbb{N})}, \qquad \left\|\begin{pmatrix} 0 & 0 \\ 0 & G^{++} \end{pmatrix}\right\|_{\ell^2(\mathbb{Z})} = \|G^{++}\|_{\ell^2(\mathbb{N}_0)}.$$

With this established let us evaluate or estimate the norm of these operators.

The norms of G^{+-} and G^{-+} can be evaluated exactly by applying the following lemma.

Lemma 2.13: Let $p, q \in \mathbb{C}$, $|p|$, $|q| < 1$. We define an operator $A = A(p,q)$ on $\ell^2(\mathbb{N})$ whose matrix representation reads

$$\forall m,n \in \mathbb{N} \quad : \quad A_{m,n} := p^m q^n.$$

Then $A \in \mathscr{B}\big(\ell^2(\mathbb{N})\big)$. Moreover, we can calculate its norm

$$\|A\| = \sqrt{\frac{|p|^2}{1-|p|^2}\frac{|q|^2}{1-|q|^2}}.$$

Proof. Choose an arbitrary vector $\psi \in \ell^2(\mathbb{N})$, then the mth element of $A\psi$ reads

$$(A\psi)_m = \sum_{n=1}^{\infty} A_{m,n}\psi_n = \sum_{n=1}^{\infty} p^m q^n \psi_n = p^m \sum_{n=1}^{\infty} q^n \psi_n.$$

Next, we estimate the square of its norm

$$\|A\psi\|^2 = \sum_{m=1}^{\infty} |(A\psi)_m|^2 = \sum_{m=1}^{\infty} |p|^{2m}\Big|\sum_{n=1}^{\infty} q^n\psi_n\Big|^2 \leq \sum_{m=1}^{\infty}|p|^{2m}\Big(\sum_{n=1}^{\infty}|q|^n|\psi_n|\Big)^2$$
$$\overset{\text{C.–S.}}{\leq} \sum_{m=1}^{\infty}|p|^{2m}\sum_{n=1}^{\infty}|q|^{2n}\sum_{k=1}^{\infty}|\psi_k|^2 = \frac{|p|^2}{1-|p|^2}\frac{|q|^2}{1-|q|^2}\|\psi\|^2,$$

where the marked inequality denotes the Cauchy–Schwarz inequality. This calculation proves the assertion that A is a bounded operator on $\ell^2(\mathbb{N})$ and provides an upper bound for the norm of A. For the lower bound estimation let us choose a test vector $\psi_n := (\overline{q})^n$, then

$$\|A\psi\|^2 = \sum_{m=1}^{\infty} |(A\psi)_m|^2 = \sum_{m=1}^{\infty} |p|^{2m}\Big|\sum_{n=1}^{\infty} q^n(\overline{q})^n\Big|^2 = \sum_{m=1}^{\infty}|p|^{2m}\Big(\sum_{n=1}^{\infty}|q|^{2n}\Big)^2$$
$$= \sum_{m=1}^{\infty}|p|^{2m}\sum_{n=1}^{\infty}|q|^{2n}\sum_{k=1}^{\infty}|\overline{q}|^{2k} = \sum_{m=1}^{\infty}|p|^{2m}\sum_{n=1}^{\infty}|q|^{2n}\sum_{k=1}^{\infty}|\psi_k|^2 = \frac{|p|^2}{1-|p|^2}\frac{|q|^2}{1-|q|^2}\|\psi\|^2.$$

Since these two estimates are equal, we know it is the norm of A itself. □

Moreover, as it turns out they are equal to each other

$$\|\widetilde{G}^{+-}\| = \|\widetilde{G}^{-+}\| = \frac{1}{|w|}\frac{1}{\sqrt{1-|\eta|^2}}\frac{|\xi|}{\sqrt{1-|\xi|^2}}.$$

The norms of the other quadrants are more complicated. We were not able to evaluate them exactly; hence, we give estimates. Multiple estimation techniques were used to take the upper bound, though not one yielded a better asymptotic formula than the others; therefore, we will show the most conspicuous one – the Schur test. We will get the lower bound by choosing suitable test vectors.

Proposition 2.14: Let us denote

$$U_\xi := \frac{1}{1-|\xi|}\left(\frac{1}{\widehat{w}} + \frac{3}{|\xi - \xi^{-1}|}\right),$$

$$L_\xi := \frac{\sqrt{(1-|\xi|^2)^2|\xi| + 2\mathrm{Re}(-\xi^{-1}w)|\xi|(1-|\xi|^2) + |w|^2(1+|\xi|^2)}}{|w|\,|\xi^{-1}-\xi|\,(1-|\xi|^2)},$$

$$U_\eta := \frac{1}{1-|\eta|}\left(\frac{1}{\widehat{w}} + \frac{3}{|\eta - \eta^{-1}|}\right),$$

$$L_\eta := \frac{\sqrt{(1-|\eta|^2)^2 + 2\mathrm{Re}(-\eta w)|\eta|^2(1-|\eta|^2) + |\eta|^4|w|^2(1+|\eta|^2)}}{|w|\,|\eta|\,|\eta^{-1}-\eta|\,(1-|\eta|^2)}.$$

The upper and lower bounds of the norm of $\widetilde{G}^{--}$ and G^{++} take the forms

$$U_\xi \geq \|\widetilde{G}^{--}\| \geq L_\xi, \qquad U_\eta \geq \|G^{++}\| \geq L_\eta,$$

where $\widehat{w}$ is the uniform lower bound of $w = \xi - \eta^{-1}$ from Proposition 1.14.

Proof. Upper bounds U_ξ and U_η were obtained in a similar fashion as the upper bound for the global estimate in Proposition 2.10 using the Schur test. Hence, we omit the details of this calculation. For the lower bound we will be using estimation by a test vector, see (2.0.1). For $G^{++}(\lambda)$ and $\widetilde{G}^{--}(\lambda)$ we define test vectors

$$\psi = \{\eta^n\}_{n\in\mathbb{N}_0} \qquad \text{and} \qquad \varphi = \{\xi^n\}_{n\in\mathbb{N}},$$

respectively. Their norms are

$$\|\psi\| = \frac{1}{\sqrt{1-|\eta|^2}} \qquad \text{and} \qquad \|\varphi\| = \frac{|\xi|}{\sqrt{1-|\xi|^2}}.$$

Let us calculate the m element of $G^{++}(\lambda)\psi$. Let $m \in \mathbb{N}_0$, then

$$\big(G^{++}(\lambda)\psi\big)_m = \eta^m \sum_{n=0}^{m-1}(A\eta^{2n} - B) + (A\eta^m + B\eta^{-m})\sum_{n=m}^{\infty}\eta^{2n} = \frac{\eta^m}{w(1-\eta^2)}\big(1 - m\eta w\big),$$

where we substituted for A and B from (1.2.5); from these formulas it is easy to see that $A+B=1$ and $(1-\eta^2)B=-w\eta$. If we calculate

$$|1-m\eta w|^2 = \big(1-m\mathrm{Re}(\eta w)\big)^2 + \big(m\mathrm{Im}(\eta w)\big)^2 = 1-2m\mathrm{Re}(\eta w)+m^2|\eta w|^2.$$

then the squared absolute value of $\big(G^{++}(\lambda)\psi\big)_m$ reads

$$\left|\big(G^{++}(\lambda)\psi\big)_m\right|^2 = \frac{1}{|w|^2|1-\eta^2|^2}\Big(|\eta|^{2m} - m|\eta|^{2m}2\mathrm{Re}(\eta w) + m^2|\eta|^{2m}|\eta w|^2\Big). \tag{2.2.2}$$

In order to calculate the sum over indices m of the expression above, let us show special power series, so-called *low-order polylogarithms*, a special case of which is the geometric series. Let $z\in(0,1)$, then

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \qquad \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2}, \qquad \sum_{n=1}^{\infty} n^2z^n = \frac{z(1+z)}{(1-z)^3}.$$

After we take the square root of the sum of (2.2.2) and divide it by the norm of ψ we get $L\eta$ after simple manipulations. The calculation of the lower bound of $\|\widetilde{G}^{--}(\lambda)\|$ denoted by L_ξ is analogous and we also omit details. □

2.2.2 Asymptotic formulas

As this section heavily relies on Proposition 1.13, let us reiterate it in a more intuitive though less precise way. Because we are studying the immediate neighborhood of the spectrum, the left side of the implication will always hold.

Let $\mathrm{Im}\alpha \neq 0$. If λ is sufficiently close to

(a) $[-2,2] \quad\Longrightarrow\quad \exists\delta_1<1 \ :\leq|\eta|\leq\delta_1.$

(b) $[-2+\alpha,2+\alpha] \quad\Longrightarrow\quad \exists\delta_2<1 \ :\leq|\xi|\leq\delta_2,$

and if λ is close to either connected component of the spectrum, there exists $\rho>0$ such that $|\xi|,|\eta|>\rho$.

In Table 2.1 we provide asymptotic formulas for the expressions inside Proposition 2.14. Let us prove them for the case $|\xi|\longrightarrow 1$; the other case is analogous. Let us write $\xi=(1-\varepsilon)e^{\mathrm{i}\phi}$.

- From Proposition 1.13 directly follows that the absolute value of η is inside a closed interval, subset of $(0,1)$, when λ is sufficiently close to the interval $[-2,2]$. Therefore,

$$|\eta| = \mathcal{O}(1) \qquad \text{as } |\xi|\to 1.$$

- The absolute value of w can be written as

$$|w| = |\xi-\eta^{-1}| = \frac{|1-\xi\eta|}{|\eta|}.$$

This expression can be easily estimated from above and below in the following way

$$\frac{1+\delta_1}{\rho} \geq \frac{1+|\xi||\eta|}{|\eta|} \geq \frac{|1-\xi\eta|}{|\eta|} \geq \frac{1-|\xi||\eta|}{|\eta|} \geq \frac{1-\delta_1}{\delta_1}.$$

As we can see $|w|$ has values inside a compact interval; therefore,

$$|w| = \mathcal{O}(1) \qquad \text{as } |\xi| \to 1.$$

Since the estimation above shows that $|w|$ can never be 0, we can take its reciprocal value

$$\frac{1}{|w|} = \mathcal{O}(1) \qquad \text{as } |\xi| \to 1.$$

Moreover, this reciprocal value can be uniformly estimated from above by $1/\widehat{w}$ as was already used in Proposition 2.14 for its reciprocal value.

- For the next expression let us assume that $\phi \neq 0, \pi$, then

$$\frac{1}{|\xi - \xi^{-1}|} = \frac{1}{\left|(1-\varepsilon)e^{\mathrm{i}\phi} - \frac{1}{1-\varepsilon}e^{-\mathrm{i}\phi}\right|} \xrightarrow{\varepsilon \longrightarrow 0} \frac{1}{\left|e^{\mathrm{i}\phi} - e^{-\mathrm{i}\phi}\right|} = \frac{1}{2|\sin(\phi)|}.$$

If $\phi = 0, \pi$, then $e^{\mathrm{i}\phi} = e^{-\mathrm{i}\phi} = \pm 1$. With this assumption, we have

$$\frac{1}{|\xi - \xi^{-1}|} = \frac{1}{\left|1 - \varepsilon - \frac{1}{1-\varepsilon}\right|} = \frac{1}{2\varepsilon} + \mathcal{O}(1) \qquad \text{as } |\xi| \to 1$$

- We have

$$\frac{1}{|\eta - \eta^{-1}|} = \mathcal{O}(1) \qquad \text{as } |\xi| \to 1.$$

Indeed, one can estimate

$$\frac{\rho}{1+\delta_1\rho} = \frac{1}{\delta_1 + \frac{1}{\rho}} \leq \frac{1}{|\eta| + |\eta|^{-1}} \leq \frac{1}{|\eta - \eta^{-1}|} \leq \frac{1}{|\eta|^{-1} - |\eta|} \leq \frac{1}{\frac{1}{\delta_1} - \delta_1} = \frac{\delta_1}{1-\delta_1^2}.$$

- If we estimate

$$\delta_1 - 1 \leq |\xi||\eta| - 1 \leq -|\eta||w| \leq \operatorname{Re}(-\eta w) \leq |w||\eta| = |1 - \xi\eta| \leq 1 + |\xi||\eta| \leq 1 + \delta_1,$$

we can conclude that

$$\operatorname{Re}(-\eta w) = \mathcal{O}(1) \qquad \text{as } |\xi| \to 1.$$

- Similarly,

$$-\frac{1+\delta_1}{\rho} \leq -|w| = -|\xi^{-1}w||\xi| \leq \operatorname{Re}(-\xi^{-1}w)|\xi| \leq |\xi^{-1}w||\xi| = |w| \leq \frac{1+\delta_1}{\rho}.$$

Hence,

$$\operatorname{Re}(-\xi^{-1}w)|\xi| = \mathcal{O}(1) \qquad \text{as } |\xi| \to 1.$$

	$\lvert\xi\rvert \longrightarrow 1 \quad : \quad \xi = (1-\varepsilon)e^{\mathrm{i}\phi}$	$\lvert\eta\rvert \longrightarrow 1 \quad : \quad \eta = (1-\varepsilon)e^{\mathrm{i}\phi}$
$\lvert\xi\rvert$	$1-\varepsilon$	$\mathcal{O}(1)$
$\lvert\eta\rvert$	$\mathcal{O}(1)$	$1-\varepsilon$
$\frac{1}{\lvert w\rvert}$	$\mathcal{O}(1)$	$\mathcal{O}(1)$
$\frac{1}{\lvert\xi-\xi^{-1}\rvert}$	$\frac{1}{2\lvert\sin(\phi)\rvert}+\mathcal{O}(\varepsilon)$ $\frac{1}{2\varepsilon}+\mathcal{O}(1), \quad \phi=\pm\pi, 0$	$\mathcal{O}(1)$
$\frac{1}{\lvert\eta-\eta^{-1}\rvert}$	$\mathcal{O}(1)$	$\frac{1}{2\lvert\sin(\phi)\rvert}+\mathcal{O}(\varepsilon)$ $\frac{1}{2\varepsilon}+\mathcal{O}(1), \quad \phi=\pm\pi, 0$
$\mathrm{Re}(\eta w)$	$\mathcal{O}(1)$	$\mathcal{O}(1)$
$\mathrm{Re}(\xi^{-1}w)\lvert\xi\rvert$	$\mathcal{O}(1)$	$\mathcal{O}(1)$

Table 2.1: Table of asymptotic formulas

2.2.3 Asymptotic pseudospectrum

We will calculate asymptotic behavior for each quadrant of $G(\lambda)$ individually.

Lemma 2.15:
$$\|G^{+-}\| = \|G^{-+}\| = \mathcal{O}(\varepsilon^{-\frac{1}{2}}) \qquad \text{as} \;\; |\xi|, |\eta| \to 1$$

Proof. From the calculation it will be clearly visible that is the same in both cases $|\xi| \to 1$ and $|\eta| \to 1$. Let us show it only for $|\xi| \to 1$. It suffices to show

$$\frac{1-\varepsilon}{\sqrt{1-(1-\varepsilon)^2}} = \mathcal{O}(\varepsilon^{-\frac{1}{2}}) \qquad \text{as} \;\; \varepsilon \to 0 \tag{2.2.3}$$

because

$$\|G^{+-}\| = \frac{1}{|w|}\frac{1}{\sqrt{1-|\eta|^2}}\frac{|\xi|}{\sqrt{1-|\xi|^2}} = \mathcal{O}(1)\frac{1}{\sqrt{1-\big(\mathcal{O}(1)\big)^2}}\frac{1-\varepsilon}{\sqrt{1-(1-\varepsilon)^2}} = \mathcal{O}(1)\frac{1-\varepsilon}{\sqrt{1-(1-\varepsilon)^2}}.$$

If we calculate

$$\frac{\sqrt{\varepsilon}(1-\varepsilon)}{\sqrt{1-(1-\varepsilon)^2}} = \frac{\sqrt{\varepsilon}(1-\varepsilon)}{\sqrt{2\varepsilon-\varepsilon^2}} = \frac{(1-\varepsilon)}{\sqrt{2-\varepsilon}} \xrightarrow{\varepsilon\to 0} \frac{1}{\sqrt{2}},$$

we prove (2.2.3) as well as the lemma itself. □

Lemma 2.16: The asymptotic behavior of upper bounds from Proposition 2.14 reads

$$U_\xi = \mathcal{O}(1) \quad \text{as } |\eta| \to 1, \qquad\qquad U_\xi = \mathcal{O}(1) \quad \text{as } |\xi| \to 1.$$

If $\phi \neq \pi, 0$, then

$$U_\xi = \frac{1}{\varepsilon}\Big(\frac{1}{\widehat{w}} + \frac{3}{2|\sin\phi|}\Big) + \mathcal{O}(1) = \mathcal{O}(\varepsilon^{-1}) \qquad \text{as } |\xi| \to 1,$$

$$U_\eta = \frac{1}{\varepsilon}\Big(\frac{1}{\widehat{w}} + \frac{3}{2|\sin\phi|}\Big) + \mathcal{O}(1) = \mathcal{O}(\varepsilon^{-1}) \qquad \text{as } |\eta| \to 1,$$

If $\phi \neq \pi, 0$, then

$$U_\xi = \frac{3}{2\varepsilon^2} + \mathcal{O}(\varepsilon^{-1}) = \mathcal{O}(\varepsilon^{-2}) \qquad \text{as } |\xi| \to 1,$$

$$U_\eta = \frac{3}{2\varepsilon^2} + \mathcal{O}(\varepsilon^{-1}) = \mathcal{O}(\varepsilon^{-2}) \qquad \text{as } |\eta| \to 1.$$

Proof. We get these formulas simply by plugging in asymptotic formulas from Table 2.1 into the expressions in Proposition 2.14. □

Lemma 2.17: The asymptotic behavior of lower bounds from Proposition 2.14 reads

$$L_\xi = \mathcal{O}(1) \quad \text{as } |\eta| \to 1, \qquad\qquad L_\xi = \mathcal{O}(1) \quad \text{as } |\xi| \to 1.$$

If $\phi \neq \pi, 0$, then

$$L_\xi = \frac{1}{\varepsilon}\frac{\sqrt{2}}{4|\sin\phi|} + \mathcal{O}(1) = \mathcal{O}(\varepsilon^{-1}) \qquad \text{as } |\xi| \to 1,$$

$$L_\eta = \frac{1}{\varepsilon}\frac{\sqrt{2}}{4|\sin\phi|} + \mathcal{O}(1) = \mathcal{O}(\varepsilon^{-1}) \qquad \text{as } |\eta| \to 1.$$

If $\phi \neq \pi, 0$, then

$$L_\xi = \frac{\sqrt{2}}{4\varepsilon^2} + \mathcal{O}(\varepsilon^{-1}) = \mathcal{O}(\varepsilon^{-2}) \qquad \text{as } |\xi| \to 1,$$

$$L_\eta = \frac{\sqrt{2}}{4\varepsilon^2} + \mathcal{O}(\varepsilon^{-1}) = \mathcal{O}(\varepsilon^{-2}) \qquad \text{as } |\eta| \to 1.$$

Proof. Again, we get these formulas by plugging in asymptotic formulas from Table 2.1 into the expressions in Proposition 2.14. □

We may conclude this section by stating the following theorem.

Theorem 2.18: The asymptotic behavior of the resolvent operator's norm reads

$$\frac{1}{\varepsilon}\frac{\sqrt{2}}{4|\sin\phi|} + \mathcal{O}\big(\varepsilon^{-\frac{1}{2}}\big) \leq \|(H_\alpha - \lambda)^{-1}\| \leq \frac{1}{\varepsilon}\Big(\frac{1}{\widehat{w}} + \frac{3}{2|\sin\phi|}\Big) + \mathcal{O}\big(\varepsilon^{-\frac{1}{2}}\big) \qquad \text{as } |\xi|, |\eta| \to 1,$$

if it does not approach the spectrum along the line on which a given connected component resides, i.e. $\phi \neq \pi, 0$. Otherwise, the norm follows

$$\frac{\sqrt{2}}{4\varepsilon^2} + \mathcal{O}(\varepsilon^{-1}) \leq \|(H_\alpha - \lambda)^{-1}\| \leq \frac{3}{2\varepsilon^2} + \mathcal{O}(\varepsilon^{-1}) \qquad \text{as } |\xi|, |\eta| \to 1.$$

Proof. Taking the triangle inequality of the decomposed resolvent operator, see (2.2.1), we estimate $\|(H_\alpha - \lambda)^{-1}\|$ from above and below. If we then estimate further using Proposition 2.14, we get

$$\|(H_\alpha - \lambda)^{-1}\| \leq \|\widetilde{G}^{--}\| + \|\widetilde{G}^{+-}\| + \|\widetilde{G}^{-+}\| + \|G^{++}\| \leq U_\xi + 2\|\widetilde{G}^{-+}\| + U_\eta, \tag{2.2.4}$$

$$\|(H_\alpha - \lambda)^{-1}\| \geq \|\widetilde{G}^{--}\| - \|\widetilde{G}^{+-}\| - \|\widetilde{G}^{-+}\| - \|G^{++}\| \geq L_\xi - 2\|\widetilde{G}^{-+}\| - U_\eta, \tag{2.2.5}$$

$$\|(H_\alpha - \lambda)^{-1}\| \geq \|G^{++}\| - \|\widetilde{G}^{+-}\| - \|\widetilde{G}^{-+}\| - \|\widetilde{G}^{--}\| \geq L_\eta - 2\|\widetilde{G}^{-+}\| - U_\xi. \tag{2.2.6}$$

These estimates allowed us to separate asymptotically insignificant terms. We get the assertion of the theorem simply by plugging in the asymptotic expressions from Lemmas 2.15, 2.16, and 2.17. For all upper estimates, we use (2.2.4). If we are evaluating the lower bound as $|\xi| \to 1$, we use the inequality 2.2.5. Similarly, if we are evaluating the lower bound as $|\eta| \to 1$, we use the inequality 2.2.6. We also need to separate the case when $\phi = 0, \pi$. □

Chapter 3

Weak-coupling & spectral stability

The weak-coupling analysis consists of showing the existence, uniqueness, and asymptotic behavior of *bound states*, i.e. eigenvalues, while a small potential V is applied.

3.1 Weak-coupling of the discrete Laplace operator

In this section we work out e a discrete analog to some results from [8], see also [9]. The article showed that under some assumptions posed upon the potential V, there exists a unique negative bound state of $-\mathrm{d}/\mathrm{d}x + V$ on $L^2(\mathbb{R})$.

Consider the discrete Laplace operator, i.e. discrete Schrödinger operator with zero potential. Let us first describe some basic properties of this operator, since they do not immediately arise from the case of the Schrödinger operator with complex step potential. Though our notation H_α is consistent with this case, we denote the Laplace operator H_0. Its action on $x \equiv \{x_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ is given by

$$(H_0 x)_n = x_{n-1} + x_{n+1} \qquad n \in \mathbb{Z}.$$

Honoring the notation from [3], we denote the Joukowsky transform of the spectral parameter λ by the letter k, i.e. $\lambda = k + k^{-1}$. The operator H_0 is self-adjoint. Furthermore, in this section, we will consider only real-valued potential V. Hence, the spectrum of the operator $H + V$ is purely real and we will be looking for bound states only on the real line. As a well-known fact, we state that the spectrum of H_0 coincides with the interval $[-2, 2]$. From this and the Definition 1.10, it is easy to see that the Joukowsky transform bijectively maps the resolvent set $\rho(H_0)$ to the set $\{k \in \mathbb{C} \mid 0 < |k| < 1\}$. If we set $\alpha = 0$ in (1.2.8), we get the matrix representation of the resolvent operator which reads

$$(H_0 - \lambda)^{-1}_{m,n} = (H_0 - k - k^1)^{-1}_{m,n} = \frac{k^{|m-n|}}{k - k^{-1}}, \qquad m, n \in \mathbb{Z}, \quad 0 < |k| < 1.$$

Since the discrete version of the Laplacian is bounded, there arises a significant difference to the continuous version. The spectrum of the continuous Laplacian is $[0, \infty)$; therefore, one needs to consider only the left neighborhood of zero while looking for bound states. In the case of the discrete Laplace operator, we need to investigate both the right neighborhood of 2 and the left neighborhood of -2. Though, the following proposition allows us to investigate, in fact, only one of these cases because the behavior is equivalent.

Proposition 3.1: Consider the Laplace operator H_0 and let $V = \operatorname{diag}\big(\{v_n\}_{n\in\mathbb{Z}}\big)$ be a bounded potential. Then the spectrum of the Schrödinger operator $H_V := H_0 + V$ satisfies

$$\sigma(H_V) = -\sigma(H_{-V}).$$

Proof. Let $x = \{x_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and consider the unitary operator U given by $(Ux)_n = (-1)^n x_n$. Then

$$\begin{aligned} U^{-1}H_V U x &= UH_V U x = UH_V U x = UH_V U\{x_n\} = UH_V\{(-1)^{-1}x_n\} \\ &= U\{(-1)^{n-1}x_{n-1} + (-1)^n v_n x_n + (-1)^{n-1}x_{n-1}\} = \{-x_{n-1} + v_n x_n - x_{n+1}\} = -H_{-V}x. \end{aligned}$$

Since this operation does not change the spectrum and scaling the operator by -1 also scales the spectrum as a subset of $\mathbb{C}$ by -1 we have

$$\sigma(H_V) = \sigma(-H_{-V}) = -\sigma(H_{-V}).$$

More specifically, the equality holds for point spectra as well. □

Let us restrict ourselves to investigating the right neighborhood of 2 when looking for bound states. Considering the Joukowsky transform $\lambda = k + k^{-1}$, one can easily see that

$$\lambda \longrightarrow 2_+ \qquad \Longleftrightarrow \qquad k \longrightarrow 1_-.$$

Before we move further, let us reiterate the Birman–Schwinger principle (Theorem 1.6) which says that, whenever $v \in \ell^1(\mathbb{Z})$, any $\lambda \in \rho(H_0)$ is an eigenvalue of $H_0 + V$ if and only if -1 is an eigenvalue of the Birman–Schwinger operator $K(\lambda)$. Its matrix entries read

$$K_{m,n}(\lambda) = \sqrt{|v_m|}\,\frac{k^{|m-n|}}{k-k^{-1}}\,\sqrt{|v_n|}\,\operatorname{sgn} v_n, \qquad m,n \in \mathbb{Z}.$$

Later, we will be scaling the potential V by some small $\varepsilon > 0$, for this purpose, we will denote $H_\varepsilon := H_0 + \varepsilon V$. The Birman–Schwinger principle still holds in the following sense. Any $\lambda \in \rho(H_0)$ is an eigenvalue of H_ε if and only if -1 is an eigenvalue of $\varepsilon K(\lambda)$.

The trick is to decompose the Birman–Schwinger operator into the sum of a well-behaved operator and a rank-one operator as $\lambda \longrightarrow 2_+$ as follows

$$K(\lambda) = L(\lambda) + M(\lambda),$$

where the matrix entries of these operators read

$$\begin{aligned} L_{m,n}(\lambda) &:= \sqrt{|v_m|}\,\frac{1}{k-k^{-1}}\,\sqrt{|v_n|}\,\operatorname{sgn} v_n, \\ M_{m,n}(\lambda) &:= \sqrt{|v_m|}\,\frac{k^{|m-n|}-1}{k-k^{-1}}\,\sqrt{|v_n|}\,\operatorname{sgn} v_n. \end{aligned}$$

The operator $L(\lambda)$ is rank-one. Indeed, if we take $\psi \in \ell^2(\mathbb{Z})$, we have for $m \in \mathbb{Z}$

$$\begin{aligned} \big(L(\lambda)\psi\big)_m &= \sum_{n\in\mathbb{Z}} L_{m,n}(\lambda)\psi_n = \sum_{n\in\mathbb{Z}} \sqrt{|v_m|}\,\frac{1}{k-k^{-1}}\,\sqrt{|v_n|}\,\operatorname{sgn} v_n\psi_n \\ &= \frac{1}{k-k^{-1}}\Big(\sum_{n\in\mathbb{Z}}\sqrt{|v_n|}\,\operatorname{sgn} v_n\psi_n\Big)\sqrt{|v_m|} = \frac{1}{k-k^{-1}}\langle v_{1/2},\psi\rangle\sqrt{|v_m|}. \end{aligned}$$

From this, one can easily notice that $\mathrm{Ran}L(\lambda) = \mathrm{span}(|v|^{1/2})$, i.e. $L(\lambda)$ is rank-one. This operator removes the singular expression $(k - k^{-1})^{-1}$ from the operator $K(\lambda)$. Now let us prove that $M(\lambda)$ is well-behaved as $\lambda \longrightarrow 2_+$, and state what exactly we mean by that.

The matrix entries operator $M(\lambda)$ converge to

$$\forall m, n \in \mathbb{Z} \quad : \quad M_{m,n}(2) := \sqrt{|v_m|}\, \frac{|m-n|}{2}\, \sqrt{|v_n|}\, \mathrm{sgn}\, v_n.$$

$$\begin{aligned}\frac{k^{|m-n|}-1}{k-k^{-1}} &= \frac{(1-\varepsilon)^{|m-n|}-1}{1-\varepsilon-\frac{1}{1-\varepsilon}} = \frac{1-|m-n|\varepsilon+\mathcal{O}(\varepsilon^2)-1}{1-\varepsilon-1-\varepsilon+\mathcal{O}(\varepsilon^2)} = \frac{-|m-n|\varepsilon+\mathcal{O}(\varepsilon^2)}{-2\varepsilon+\mathcal{O}(\varepsilon^2)} \\ &= \frac{\varepsilon}{\varepsilon}\,\frac{|m-n|+\mathcal{O}(\varepsilon)}{2+\mathcal{O}(\varepsilon)} = \frac{|m-n|}{2}\,\frac{1+\mathcal{O}(\varepsilon)}{1+\mathcal{O}(\varepsilon)} = \frac{|m-n|}{2} + \mathcal{O}(\varepsilon).\end{aligned}$$

Lemma 3.2:

$$\forall\lambda \geq 2,\ \forall m, n \in \mathbb{Z}\ :\ |M_{m,n}(\lambda)| \leq |M_{m,n}(2)|.$$

Proof. Since $\lambda \geq 2$, we have $k \in (0, 1]$ Let us set $N := |m - n|$ and prove that

$$\frac{k^N - 1}{k - k^{-1}} \leq \frac{N}{2}.$$

The inequality is equivalent to $k^{N+1} - k \geq N(k^2 - 1)/2$; therefore, it suffices to show that the function $g(k) := k^{N+1} - k + N(1 - k^2)/2$ is greater than or equal to 0 for $k \in (0, 1)$. Let us take the derivative of g

$$g'(k) = (N+1)k^N - 1 - Nk = Nk\,\underbrace{(k^{N-1} - 1}_{<0} + \underbrace{(k^2 - 1)}_{<0} < 0.$$

We can see that g is decreasing and clearly $g(k) \xrightarrow{k\to 1} 0$; this proves the inequality above and we can write

$$|M_{m,n}(\lambda)| = \sqrt{|v_m|}\left|\frac{k^{|m-n|}-1}{k-k^{-1}}\right|\sqrt{|v_n|} \leq \sqrt{|v_m|}\,\frac{|m-n|}{2}\,\sqrt{|v_n|} = |M_{m,n}(2)|$$

□

Remark. Numerical calculation showed that this inequality may hold more generally for $\mathrm{Re}\lambda \geq 2$. This would simplify one argument later, though it is not necessary. One could surely use the maximum modulus principle to prove the more general inequality, but expressing the region $\mathrm{Re}\lambda \geq 2$ and its boundary in terms of the Joukowsky parameter k is quite complicated.

Notation 3.3: Let $v \in \ell^1(\mathbb{Z})$ and $k \in \mathbb{N}$. We define the weighted ℓ^1 norm of v by

$$\|v\|_{\ell^1(\mathbb{Z},m^k)} := \sum_{m\in\mathbb{Z}} |m|^k |v_m| + |v_0|.$$

We also define subspaces of $\ell^1(\mathbb{Z})$ by

$$\ell^1(\mathbb{Z}, m^k) := \left\{v \in \ell^1(\mathbb{Z}) \;\middle|\; \|v\|_{\ell^1(\mathbb{Z},m^k)} < \infty\right\}.$$

It is easy to see that $\|\cdot\|_{\ell^1(\mathbb{Z},m^k)}$ is, in fact, a norm. Also, one can easily notice that $\|v\|_{\ell^1(\mathbb{Z})} \leq \|v\|_{\ell^1(\mathbb{Z},m^k)} \leq \|v\|_{\ell^1(\mathbb{Z},m^l)}$ whenever $k \leq l$.

If we pose some extra assumption upon the potential V, $M(\lambda)$ converges to $M(2)$ in the operator norm.

Proposition 3.4: Let the sequence $v \in \ell^1(\mathbb{Z}, m^2)$, then

$$\lim_{\lambda \to 2_+} \|M(\lambda) - M(2)\|,$$

where $\|\cdot\|$ denotes the operator norm on $\ell^2(\mathbb{Z})$. Moreover,

$$\forall \lambda \in \mathbb{C}, \mathrm{Re}\lambda \geq 0 \ : \ \|M(\lambda)\| \leq \|v\|_{\ell^1(\mathbb{Z},m^2)}.$$

Proof. The convergence in the space of Hilbert–Schmidt operators implies the standard operator convergence because $\|A\| \leq \|A\|_{HS}$ for any Hilbert–Schmidt operator A, it suffices to show that $\lim_{\lambda\to 2_+} \|M(\lambda) - M(2)\|_{HS}$. Under the stated assumption on potential V, the operator $M(2)$ is, in fact, a Hilbert–Schmidt operator. Indeed,

$$\begin{aligned}
\|M(2)\|_{HS}^2 &= \sum_{m,n\in\mathbb{Z}} |M_{m,n}(2)|^2 = \sum_{m,n\in\mathbb{Z}} |v_m| \left(\frac{|m-n|}{2}\right)^2 |v_n| \\
&\leq \frac{1}{2} \sum_{m,n\in\mathbb{Z}} |v_m|(m^2+n^2)|v_n| = \frac{1}{2} \sum_{m,n\in\mathbb{Z}} m^2 |v_m||v_n| + \frac{1}{2} \sum_{m,n\in\mathbb{Z}} n^2 |v_m||v_n| \\
&= \sum_{m,n\in\mathbb{Z}} m^2 |v_m||v_n| = \sum_{m\in\mathbb{Z}} m^2 |v_m| \sum_{n\in\mathbb{Z}} |v_n| \leq \left(\sum_{n\in\mathbb{Z}} |v_n| n^2 + |v_0| \right)^2 = \|v\|_{\ell^1(\mathbb{Z},m^2)}^2.
\end{aligned}$$

We have already discussed two important properties of the relation between $M(\lambda)$ and $M(2)$; these being

- inequality of matrix entries $\quad \forall \lambda > 2, \ \forall m,n \in \mathbb{Z} \ : \ |M_{m,n}(\lambda)| \leq |M_{m,n}(2)|$,
- pointwise convergence $\quad \forall m,n \in \mathbb{Z} \ : \ M_{m,n}(\lambda) \xrightarrow{\lambda \to 2_+} M_{m,n}(2)$.

This satisfies the assumptions of Dominated convergence theorem; therefore

$$\|M(\lambda) - M(2)\| \leq \|M(\lambda) - M(2)\|_{HS} \xrightarrow{\lambda \to 2_+} 0.$$

□

Proposition 3.5 (Basic criterion for the existence of a bound state)**:** Let $v \in \ell^1(\mathbb{Z}, m^2)$ and let $\varepsilon > 0$ satisfy $\varepsilon \|v\|_{\ell^1(\mathbb{Z},m^2)} < 1$. Then any $k + k^{-1} = \lambda \in \{z \in \rho(H_0) \mid \mathrm{Re} z \geq 0\}$ is an eigenvalue of H_ε if and only if

$$k - k^{-1} = -\varepsilon \langle v_{1/2}, (I + \varepsilon M(\lambda))^{-1} |v|^{1/2} \rangle. \tag{3.1.1}$$

Proof. From Proposition 3.4 we have $\|M(\lambda)\| \leq \|v\|_{\ell^1(\mathbb{Z},m^2)} < \infty$. The assumed choice of ε implies invertibility of $I + \varepsilon M(\lambda)$. If we write

$$\begin{aligned}
(I - \varepsilon K(\lambda))^{-1} &= \Big((I + \varepsilon M(\lambda)) \Big(I + (I + \varepsilon M(\lambda))^{-1} \varepsilon L(\lambda) \Big) \Big)^{-1} \\
&= \Big(I + (I + \varepsilon M(\lambda))^{-1} \varepsilon L(\lambda) \Big)^{-1} (I + \varepsilon M(\lambda))^{-1},
\end{aligned}$$

we see that -1 is an eigenvalue of $\varepsilon K(\lambda)$ if and only if -1 is an eigenvalue of $(I+\varepsilon M(\lambda))^{-1}\varepsilon L(\lambda)$. The latter operator is rank-one because $L(\lambda)$ is rank-one. Hence, the action of this operator may be written using the inner product

$$\forall x \in \ell^2(\mathbb{Z}) \quad : \quad (I+\varepsilon M(\lambda))^{-1}\varepsilon L(\lambda)x = \langle \psi, x\rangle \phi,$$

where

$$\psi := \frac{\varepsilon}{k-k^{-1}} v_{1/2}, \qquad \phi := (I+\varepsilon M(\lambda))^{-1}|v|^{1/2}.$$

Since it is a rank one operator, it has only one eigenvalue, namely $\langle \psi, \phi\rangle$. If we set it equal to -1, the Birman–Schwinger principle concludes the proof. □

Lemma 3.6: Let $\lambda > 2$, i.e. $k \in (0,1)$. Then for all $m,n \in \mathbb{Z}$

$$\left|\left(\frac{\partial M(\lambda)}{\partial \lambda}\right)_{m,n}\right| \leq \sqrt{|v_n|}\frac{|m-n|^2}{k^{-1}-k}\sqrt{|v_m|}.$$

Proof. If we denote $N := |m-n|$, it suffices to show

$$\left|\frac{\partial}{\partial\lambda}\left(\frac{k^N-1}{k^{-1}-k}\right)\right| \leq \frac{N^2}{k^{-1}-k}.$$

Let us first evaluate the derivative above,

$$\frac{\partial}{\partial\lambda}\left(k+k^{-1}\right) = \frac{\partial}{\partial\lambda}\lambda, \qquad \frac{\partial k}{\partial\lambda} - \frac{1}{k^2}\frac{\partial k}{\partial\lambda} = 1, \qquad \frac{\partial k}{\partial\lambda} = \frac{k}{k-k^{-1}},$$

$$\begin{aligned}\frac{\partial}{\partial\lambda}\left(\frac{k^N-1}{k^{-1}-k}\right) &= \frac{\partial}{\partial k}\left(\frac{k^N-1}{k^{-1}-k}\right)\cdot\frac{\partial k}{\partial\lambda} = \frac{Nk^{N+2}-k^{N+2}-Nk^N-k^N+k^2+1}{(k^{-1}-k)^2}\cdot\frac{k}{k-k^{-1}}\\ &= \frac{Nk^N}{(k^{-1}-k)^2} - \frac{k+k^{-1}}{(k^{-1}-k)^2}\cdot\frac{1-k^N}{k^{-1}-k}.\end{aligned}$$

Using mathematical induction and the inequality $k+k^{-1}\geq 2$, one can show that

$$\frac{Nk^N}{(k^{-1}-k)^2} \leq \frac{k+k^{-1}}{(k^{-1}-k)^2}\cdot\frac{1-k^N}{k^{-1}-k}.$$

Below we verify that the following inequality holds true

$$\left|\frac{\partial}{\partial\lambda}\left(\frac{k^N-1}{k^{-1}-k}\right)\right| = \frac{k+k^{-1}}{(k^{-1}-k)^2}\cdot\frac{1-k^N}{k^{-1}-k} - \frac{Nk^N}{(k^{-1}-k)^2} \leq \frac{N^2}{k^{-1}-k}.$$

Multiplying the inequality by the positive number $(k^{-1}-k)^3$ we get

$$\left(k+k^{-1}\right)\left(1-k^N\right) - Nk^N\left(k^{-1}-k\right) \leq N^2\left(k^{-1}-k\right)^2. \tag{3.1.2}$$

Let us show the inequality

$$k+k^{-1}-2 \leq (k^{-1}-k)^2. \tag{3.1.3}$$

If we rearrange its term, we get an equivalent inequality $k^2-k-k^{-1}+k^{-2}\geq 0$. This inequality may be obtained by the following estimation

$$0 \leq (k-1)^2 = k(k-1)+1-k \leq k(k-1)+k^{-1}(k^{-1}-1) = k^2-k-k^{-1}+k^{-2}.$$

We shall prove the inequality (3.1.2) using mathematical induction

- $N = 0:\ \big(k+k^{-1}\big)\big(1-k^0\big) - 0k^0\big(k^{-1}-k\big) = 0 \le 0 = 0^2\big(k^{-1}-k\big)^2.$
- $N \mapsto N+1:$ We want to prove

$$\big(k+k^{-1}\big)\big(1-k^{N+1}\big) - (N+1)k^{N+1}\big(k^{-1}-k\big) \le (N+1)^2\big(k^{-1}-k\big)^2.$$

Expanding the terms we get

$$\begin{aligned}\underline{k+k^{-1}-2} + 2 - k^{N+2} - k^N - Nk^N + Nk^{N+2} - k^N + k^{N+2}\\ \le N^2\big(k^{-1}-k\big)^2 + 2N\big(k^{-1}-k\big)^2 + \underline{\big(k^{-1}-k\big)^2}.\end{aligned}$$

If we remove the underlined terms, then because of (3.1.3) it suffices to show that

$$2\big(1-k^N\big) - kNk^N\big(k^{-1}-k\big) \le N^2\big(k^{-1}-k\big)^2 + 2N\big(k^{-1}-k\big)^2.$$

If we use the inequality $k^{-1}+k \ge 2$ and add zero, we get the inequality below

$$\begin{aligned}\underline{\big(k^{-1}+k\big)\big(1-k^N\big) - Nk^N\big(k^{-1}-k\big)} + Nk^N\big(k^{-1}-k\big) - kNk^N\big(k^{-1}-k\big)\\ \le \underline{N^2\big(k^{-1}-k\big)^2} + 2N\big(k^{-1}-k\big)^2.\end{aligned}$$

Here, the underlined terms reduce because of the induction hypothesis (3.1.2); we get

$$\begin{aligned}Nk^N\big(k^{-1}-k\big)\big(1-k\big) &\le 2N\big(k^{-1}-k\big)^2\\ k^N\big(1-k\big) &\le 2\big(k^{-1}-k\big)\\ k^N &\le 2\frac{k^{-1}-k}{1-k} = 2(1+k^{-1}).\end{aligned}$$

The last inequality clearly holds.

This proves the inequality (3.1.2) and by extension the whole lemma.

□

Proposition 3.7: Let $v \in \ell^1(\mathbb{Z}, m^2)$ be a real-valued sequence and let $\varepsilon > 0$. If

$$\sum_{n\in\mathbb{Z}} v_n > 0,$$

then H_ε has an eigenvalue greater than 2. If there is such an eigenvalue, it follows

$$\sqrt{\lambda^2-4} = \varepsilon\sum_{n\in\mathbb{Z}} v_n + \mathcal{O}(\varepsilon^2) \qquad \text{as } \varepsilon \to 0.$$

Proof. Suppose that $\varepsilon\|v\|_{\ell^1(\mathbb{Z},m^2)} < 1$. Plugging into (3.1.1) the following identity

$$(I+\varepsilon M(\lambda))^{-1} = I - \varepsilon M(\lambda)(I+\varepsilon M(\lambda))^{-1} \tag{3.1.4}$$

we get

$$k^{-1} - k = \varepsilon\sum_{n\in\mathbb{Z}} v_n + \mathcal{O}(\varepsilon^2) \qquad \text{as } \varepsilon \to 0. \tag{3.1.5}$$

To prove the asymptotic formula it suffices to substitute for k and k^{-1} from Proposition 1.11. From the assumption $\sum_{n\in\mathbb{Z}} v_n > 0$ follows that for ε sufficiently small the right-hand side of (3.1.1) is positive; therefore there exists $k \in (0,1)$ which solves the equation since $k^{-1} - k$ is strictly positive. Since the relation between $k \in (0,1)$ and $x \in (0,\infty)$ given by $x = k^{-1} - k$ is bijective as is the relation between $k \in (0,1)$ and $\lambda \in (2,\infty)$ given by $\lambda = k^{-1} + k$, Proposition 3.5 states that there is a one-one correspondence between eigenvalues of H_ε greater than 2 and solutions of (3.1.1). So for all $\varepsilon < \varepsilon_0$ there is an eigenvalue of H_ε greater than 2. Let $\mu \geq \varepsilon$, then $H_0 + \mu V \geq H_0 + \varepsilon V$. We conclude this proof by showing that $H_0 + \mu V$ also has an eigenvalue greater than 2. By virtue of Corollary 1.9 we have

$$\lambda_1(H_0 + \mu V) \geq \lambda_1(H_0 + \varepsilon V) > 2.$$

□

Proposition 3.8: Let $v \in \ell^1(\mathbb{Z}, m^2)$ be a real-valued non-zero sequence and let $\varepsilon > 0$ be sufficiently small. Then H_ε has at most one eigenvalue greater than 2.

Proof. We are looking for eigenvalues $\lambda \in (2,\infty)$, i.e. $k \in (0,1)$. One can easily see that $H_0 + \varepsilon V \leq H_0 + \varepsilon|V|$ in the sense of quadratic forms. Suppose that $H_0 + \varepsilon|V|$ has at most one eigenvalue greater than 2, then from Theorem 1.8 we have

$$\forall l \geq 2 \ : \qquad \lambda_1(H_0 + \varepsilon|V|) \geq \lambda_l(H_0 + \varepsilon|V|) = \sup \sigma_{\text{ess}}(H_0 + \varepsilon|V|) = 2.$$

Then by virtue of the Corollary 1.9 we have

$$\forall l \geq 2 \ : \qquad 2 = \lambda_l(H_0 + \varepsilon|V|) \geq \lambda_l(H_0 + \varepsilon V) = \sup \sigma_{\text{ess}}(H_0 + \varepsilon V) = 2.$$

And so $H_0 + \varepsilon V$ has also at most one eigenvalue greater than 2. Without loss of generality, this allows us to restrict V to have only non-negative entries.

So that we can use Proposition 3.5 assume that $\varepsilon\|v\|_{\ell^1(\mathbb{Z},m^2)} < 1$. Let us define the function F as the right-hand side of (3.1.1) and take its derivative w.r.t. λ

$$F(\lambda,\varepsilon) := \varepsilon\big\langle v^{1/2}, (I + \varepsilon M(\lambda))^{-1} v^{1/2}\big\rangle,$$

$$\frac{\partial F}{\partial \lambda} = \varepsilon\Big\langle v^{1/2}, (I + \varepsilon M(\lambda))^{-1} \varepsilon \frac{\partial M(\lambda)}{\partial \lambda} (I + \varepsilon M(\lambda))^{-1} v^{1/2}\Big\rangle.$$

Next, we will plugging in (3.1.4) and estimate its absolute value:

$$\begin{aligned}
\left|\frac{\partial F}{\partial \lambda}\right| &= \varepsilon\Big|\Big\langle v^{1/2}, \varepsilon\frac{\partial M(\lambda)}{\partial \lambda} v^{1/2}\Big\rangle - \Big\langle v^{1/2}, \varepsilon^2 \frac{\partial M(\lambda)}{\partial \lambda} M(\lambda)(I + \varepsilon M(\lambda))^{-1} v^{1/2}\Big\rangle \\
&\quad - \Big\langle v^{1/2}, \varepsilon^2 M(\lambda)(I + \varepsilon M(\lambda))^{-1} \frac{\partial M(\lambda)}{\partial \lambda} v^{1/2}\Big\rangle \\
&\quad + \Big\langle v^{1/2}, \varepsilon^3 M(\lambda)(I + \varepsilon M(\lambda))^{-1} \frac{\partial M(\lambda)}{\partial \lambda} M(\lambda)(I + \varepsilon M(\lambda))^{-1} v^{1/2}\Big\rangle\Big| \\
&\leq \varepsilon^2 \Big|\Big\langle v^{1/2}, \frac{\partial M(\lambda)}{\partial \lambda} v^{1/2}\Big\rangle\Big| + 2\varepsilon^3 \|v^{1/2}\|^2 \Big\|\frac{\partial M(\lambda)}{\partial \lambda}\Big\| \|M(\lambda)\| \|(I + \varepsilon M(\lambda))^{-1}\| \\
&\quad + \varepsilon^4 \|v^{1/2}\|^2 \Big\|\frac{\partial M(\lambda)}{\partial \lambda}\Big\| \|M(\lambda)\|^2 \|(I + \varepsilon M(\lambda))^{-1}\|^2
\end{aligned}$$

Let us now estimate or in other ways investigate all the terms in this expression.

- From Proposition 3.4 we have $\|M(\lambda)\| \le \|v\|_{\ell^1(\mathbb{Z},m^2)} =: C_1$. In this notation, $\varepsilon < 1/C_1$.
- Having in mind the upper bound on ε we can independently on λ and ε estimate
$$\left\|\big(I+\varepsilon M(\lambda)\big)^{-1}\right\| = \Big\|\sum_{j=0}^{\infty}(-\varepsilon)^j M^j(\lambda)\Big\| \le \sum_{j=0}^{\infty}\varepsilon^j\|M(\lambda)\|^j \le \sum_{j=0}^{\infty}\varepsilon^j C_1^j = \frac{1}{1-\varepsilon C_1} =: C_2.$$
- There is a certain relation between ℓ^1 and ℓ^2 norms (note that we are restricting ourselves to non-negative entries in the sequence v):
$$\|v^{1/2}\|_{\ell^2}^2 = \langle v^{1/2}, v^{1/2}\rangle_{\ell^2} = \sum_{n\in\mathbb{Z}}\sqrt{v_n}\sqrt{v_n} = \sum_{n\in\mathbb{Z}} v_n = \|v\|_{\ell^1}.$$
- Because v is a non-zero sequence with non-negative entries, $\sum v_n > 0$; therefore, (2.14) allows us to estimate for ε sufficiently small $(k^{-1}-k)^{-1} \le C_3\varepsilon^{-1}$.
- We employed Lemma 3.6 and estimate
$$\begin{aligned}\Big|\Big\langle v^{1/2}, \frac{\partial M(\lambda)}{\partial\lambda}v^{1/2}\Big\rangle\Big| &\le \Big|\sum_{m\in\mathbb{Z}} v_m \sum_{m\in\mathbb{Z}} v_n \frac{|m-n|^2}{k^{-1}-k}\Big| = \frac{1}{k^{-1}-k}\sum_{m,n\in\mathbb{Z}}|v_n||v_m||m-n|^2 \\ &\le \frac{2}{k^{-1}-k}\sum_{m,n\in\mathbb{Z}}|v_n||v_m|(m^2+n^2) \le \frac{2\|v\|_{\ell^1(\mathbb{Z},m^2)}^2}{k^{-1}-k} \le \frac{2C_1^2C_3}{\varepsilon}.\end{aligned}$$
- There exist $C_4 > 0$ such that $\forall k \in [1/2, 1)\ :\ (\lambda-2)^{-1} \le C_4\varepsilon^{-2}$. Indeed, if we estimate for $k \in [1/2, 1)$
$$\big(k^{1/2}-k^{-1/2}\big)\big(\sqrt{2}+1/\sqrt{2}\big) \ge \big(k^{1/2}-k^{-1/2}\big)\big(k^{1/2}+k^{-1/2}\big) = k^{-1}-k \ge \frac{\varepsilon}{C_3},$$
we get
$$\lambda - 2 = k + k^{-1} - 2 = \big(k^{1/2}-k^{-1/2}\big)^2 \ge \frac{\varepsilon^2}{C_3^2(\sqrt{2}+1/\sqrt{2})^2} = \frac{\varepsilon^2}{C_4}$$
- The operator-valued function $z \mapsto M(z)$ is easily seen to be analytic in the region $\mathrm{Re}z > 2$. Moreover, the function $z \mapsto \|M(z)\|$ is continuous, and since any contour in the region $\mathrm{Re}z > 0$ is compact, $\|M(z)\|$ may be estimated from above by a constant K_1. The remark below Lemma 3.2 indicates that this constant may turn out to be C_1. The Cauchy formula states that
$$M(\lambda) = \frac{1}{2\pi\mathrm{i}}\oint_{|z-\lambda|=\delta}\frac{M(z)}{z-\lambda}\mathrm{d}z, \qquad \text{where } 0<\delta<\lambda-2.$$
The derivative of $M(\lambda)$ with respect to λ reads
$$\frac{\partial M(\lambda)}{\partial\lambda} = \frac{1}{2\pi\mathrm{i}}\oint_{|z-\lambda|=\delta}\frac{M(z)}{(z-\lambda)^2}\mathrm{d}z.$$
Taking its norm we obtain the estimate
$$\begin{aligned}\left\|\frac{\partial M(\lambda)}{\partial\lambda}\right\| &= \left\|\frac{1}{2\pi\mathrm{i}}\oint_{|z-\lambda|=\delta}\frac{M(z)}{(z-\lambda)^2}\mathrm{d}z\right\| \le \frac{1}{2\pi}\oint_{|z-\lambda|=\delta}\frac{\|M(z)\|}{\delta^2}|\mathrm{d}z| \\ &\le \frac{K_1}{2\pi\delta^2}\underbrace{\oint_{|z-\lambda|=\delta}|dz|}_{=2\pi\delta} = \frac{K_1}{\delta} \xrightarrow{\delta\to\lambda-2} \frac{K_1}{\lambda-2} \le \frac{K_1C_4}{\varepsilon^2} = \frac{C_5}{\varepsilon^2}\end{aligned}$$

Putting all these estimates together, we get

$$\left|\frac{\partial F}{\partial \lambda}\right| \leq \varepsilon^2 \frac{2C_1C_3}{\varepsilon} + 2\varepsilon^3 \|v\|_{\ell^1} \frac{C_5}{\varepsilon^2} C_1C_2 + \varepsilon^4 \|v\|_{\ell^1} \frac{C_5}{\varepsilon^2} C_1^2 C_2$$
$$\leq \varepsilon\Big(2C_1C_2 + 2\|v\|_{\ell^1} C_5C_1C_2 + \|v\|_{\ell^1} C_5C_1C_2\Big) = C\varepsilon \quad (3.1.6)$$

As stated before, the relation between $k \in (0,1)$ and $x \in (0,\infty)$ given by $x = k^{-1} - k$ is bijective and the relation between $k \in (0,1)$ and $\lambda \in (2,\infty)$ given by $\lambda = k^{-1} + k$ is also bijective. Hence, there is a one-one correspondence $\lambda \sim x$. Suppose there are two different eigenvalues greater than 2, denote them $\lambda_1 \sim x_1$ and $\lambda_2 \sim x_2$, taking their difference yields

$$x_1 - x_2 = F(\lambda_1, \varepsilon) - F(\lambda_2, \varepsilon) = \int_{x_1}^{x_2} \frac{\partial F}{\partial \lambda} \mathrm{d}\lambda$$

The estimation (3.1.6) allows us to find $\varepsilon > 0$ so small that $\left|\frac{\partial F}{\partial \lambda}\right| < \frac{1}{2}$; this yields the following argument

$$|x_2 - x_1| = \Big|\int_{x_1}^{x_2} \frac{\partial F}{\partial \lambda} \mathrm{d}\lambda\Big| \leq \frac{1}{2}|x_2 - x_1| \qquad \Longrightarrow \qquad x_2 = x_1.$$

And therefore, $\lambda_1 = \lambda_2$. □

Theorem 3.9: Let $v \in \ell^1(\mathbb{Z}, m^2)$. Then for all $\varepsilon > 0$

$$\sum v_n > 0 \qquad \Longrightarrow \qquad H_0 + \varepsilon V \text{ has an eigenvalue } > 2,$$
$$\sum v_n < 0 \qquad \Longrightarrow \qquad H_0 + \varepsilon V \text{ has an eigenvalue } < -2.$$

Moreover, if ε is small enough, the eigenvalue is unique and follows

$$\operatorname{sgn}\Big(\sum v_n\Big)\sqrt{\lambda^2 - 4} = \varepsilon \sum v_n + \mathcal{O}(\varepsilon^2) \qquad \text{as } \varepsilon \to 0.$$

Proof. This theorem follows clearly from Propositions 3.7, 3.8, and 3.1. □

In [11], they studied bound states of a large class of discrete Schrödinger-type operators in one and two dimensions. A special case of which is the operator H_0 in one dimension. In their paper, a different technique from ours was utilized; nevertheless, what we showed is in accordance with their results, though our result is not as general.

3.1.1 A comparison to the continuous setting

In [8] they study the bound states of weakly coupled continuous Schrödinger operator. In the previous section, we used similar steps that Simon used to give a sufficient and necessary condition for the existence of a bound state. Let us present two theorems from this paper that describe this behavior.

Theorem 3.10: Suppose that $V \in L^1(\mathbb{R}, (1 + x^2)\mathrm{d}x)$. Then, for $\varepsilon > 0$ sufficiently small, $-\mathrm{d}^2/\mathrm{d}x^2 + \varepsilon V$ has at most one negative eigenvalue. There is such an eigenvalue λ if and only if

$$\sqrt{-\lambda} = -\frac{\varepsilon}{2}\Big\langle V^{1/2}, (1 + \varepsilon M_\lambda)^{-1} |V|^{1/2}\Big\rangle.$$

The operator M_λ is the well-behaved part of the Birman–Schwinger operator, it is an analogue to the operator $M(\lambda)$ from the section above; M_λ is defined by its kernel

$$M_\lambda(x,y) := |V(x)|^{1/2}\frac{e^{-\sqrt{-\lambda}|x-y|}-1}{2\sqrt{-\lambda}}V^{1/2}(y).$$

Theorem 3.11: Let $V \in L^1(\mathbb{R},(1+x^2)\mathrm{d}x)$, V not a.e. zero. Then $-\mathrm{d}^2/\mathrm{d}x^2+\varepsilon V$ has a negative eigenvalue for all $\varepsilon > 0$ if and only if

$$\int V(x)\mathrm{d}x \le 0.$$

If it does have an eigenvalue λ, then it is unique and simple and obeys

$$\sqrt{-\lambda} = -\frac{\varepsilon}{2}\int V(x)\mathrm{d}x - \frac{\varepsilon^2}{4}\int V(x)|x-y|V(y)\mathrm{d}(x,y) + \mathcal{O}(\varepsilon^2) \qquad \text{as } \varepsilon \to 0.$$

3.2 Spectral stability of the discrete Schrödinger operator with complex step potential

In [10] it was shown that the continuous Schrödinger operator with complex step potential possesses some sort of spectral stability, though not complete. Later in this text, we will describe exactly what is meant by this when we compare the discrete and continuous settings. We show that the discrete version of the discrete Schrödinger operator with complex step potential exhibits similar behavior.

In this section, we consider only the case where $\mathrm{Im}\alpha \neq 0$. If we pose certain assumptions on the potential V, we can rule out the existence of bound states, i.e. we show that $H_\alpha+V$ possesses spectral stability. Note that our aim is to demonstrate spectral stability but some assumptions may be too restricting and some estimates may be rougher than needed.

The main tool we use in this section is the Birman–Schwinger principle, see Theorem 1.6. We will take the upper bound of the Hilbert–Schmidt norm of the Birman–Schwinger operator $K(\lambda)$ which scales proportionally to the norm of the potential V. Then we set the norm of V such that the norm of $K(\lambda)$ is strictly less than 1. From this follows the fact that λ is not an eigenvalue of $H_\alpha + V$.

The first step is to find the upper bound of the resolvent operator matrix entries uniform in $\lambda \in \rho(H_\alpha)$. In order to do so, let u first show some preliminary results.

Lemma 3.12: Let $m, n \ge 0$. A function given

$$f(z) := \frac{z^{m+n} - z^{|m-n|}}{z^{-1} - z}$$

can be continuously to the closed unit disc and it satisfies

$$\max_{|z|\le 1} |f(z)| = \frac{m+n-|m-n|}{2}.$$

Proof. Clearly, the definitory expression of the function f is not well defined at $z = 0, \pm 1$. Let us first show that in these points it can be continuously defined, i.e. there exist limits of f at these points. Let us consider the limit as $z \to 1$. For this purpose we parameterize $z = (1-\varepsilon)$, and analyzing the limit as $\varepsilon \to 0$ we get the asymptotic expression

$$\frac{z^{m+n} - z^{|m-n|}}{z^{-1} - z} = \frac{(1-\varepsilon)^{m+n} - (1-\varepsilon)^{|m-n|}}{(1-\varepsilon)^{-1} - (1-\varepsilon)} = \frac{1-(m+n)\varepsilon - 1 + |m-n|\varepsilon + \mathcal{O}(\varepsilon^2)}{1+\varepsilon+\mathcal{O}(\varepsilon^2) - 1 + \varepsilon}$$
$$= \frac{|m-n| - m - n}{2} + \mathcal{O}(\varepsilon),$$

which shows us that f has a limit at $z \to 1$. Similarly, let us show that there exists a limit of f as $z \to -1$, we parameterize $z = -(1-\varepsilon)$. Notice that $|m-n|$ and $m+n$ have the same parity; therefore, we can factor out $(-1)^{m+n} = (-1)^{|m-n|}$ in

$$\frac{z^{m+n} - z^{|m-n|}}{z^{-1} - z} = \frac{(-1)^{m+n}(1-\varepsilon)^{m+n} - (-1)^{|m-n|}(1-\varepsilon)^{|m-n|}}{(1-\varepsilon)^{-1} - (1-\varepsilon)}$$
$$= (-1)^{m+n}\frac{(1-\varepsilon)^{m+n} - (1-\varepsilon)^{|m-n|}}{(1-\varepsilon)^{-1} - (1-\varepsilon)} = (-1)^{m+n}\frac{|m-n| - m - n}{2} + \mathcal{O}(\varepsilon).$$

To show there exists a limit of f as $z \to 0$ let us rewrite

$$\frac{z^{m+n} - z^{|m-n|}}{z^{-1} - z} = -z^{|m-n|+1}\frac{1 - z^{m+n-|m-n|}}{1-z^2} \overset{\star}{=} -z^{|m-n|+1}\frac{1-z^{2k}}{1-z^2} = -z^{|m-n|+1}\sum_{j=0}^{k-1} z^{2j}.$$

The marked equality follows from the fact that $(m+n)$ and $|m-n|$ have the same parity; hence, their difference is necessarily even and one may write $2k = m+n-|m-n|$, where k is some natural number. By rewriting the function f in such a way we have also rid it of any issues when $z = 0$. Furthermore, if we take the absolute value of f written in such a way, we can estimate it from above by triangle inequality and get the assertion of this lemma. □

This lemma and Lemma 1.14 allow us to find a uniform upper bound of all matrix entries of $G(\lambda)$.

Lemma 3.13: Let us define a doubly infinite matrix M by setting

$$M_{m,n} := \begin{cases} \frac{m+n-|m-n|}{2} + \frac{1}{\widehat{w}} & m,n \geq 0, \\ \frac{1}{\widehat{w}} & m \geq 0, n < 0, \\ \frac{-m-n-|m-n|}{2} + \frac{1}{\widehat{w}} & m,n < 0, \\ \frac{1}{\widehat{w}} & m < 0, n \geq 0. \end{cases}$$

Then

$$\forall \lambda \in \mathbb{C},\ \forall m,n \in \mathbb{Z} \ : \quad |G_{m,n}(\lambda)| \leq M_{m,n}.$$

Proof. The Green Kernel theorem used to derive the resolvent operator does not define $G(\lambda)$ for $\lambda \in \sigma(H_\alpha)$. A byproduct of Lemma 3.12 is that we can continuously extend $G(\lambda)$ to the whole complex plane. If we estimate entries of $G(\lambda)$, see (1.2.8), with triangle inequality and further with Lemmas 3.12 and 1.14, we get the matrix M. □

Finally, let us estimate the norm of the Birman–Schwinger operator $K(\lambda)$.

Proposition 3.14: Let $\mathrm{Im}\alpha \neq 0$ and $v \in \ell^1(\mathbb{Z}, m^2)$. Then the following estimate holds

$$\forall \lambda \in \rho(H_\alpha) \ : \qquad \|K(\lambda)\| \leq \frac{2}{\widehat{w}} \|v\|_{\ell^1(\mathbb{Z},m^2)}.$$

Proof. Before we move to the calculation itself, let us first show some estimates that we will be using. For $m, n \geq 0$ we have

$$m + n - |m-n| \leq m + n \qquad \text{and} \qquad \frac{(m+n-|m-n|)^2}{4} \leq \frac{(m+n)^2}{4} \leq \frac{m^2+n^2}{2}.$$

Similarly, for $m, n < 0$ we estimate

$$-m - n - |m-n| \leq -m - n \qquad \text{and} \qquad \frac{(-m-n-|m-n|)^2}{4} \leq \frac{(-m-n)^2}{4} \leq \frac{m^2+n^2}{2}.$$

In fact, we will be calculating the Hilbert–Schmidt norm as it dominates the standard operator norm and it is easier to calculate. Further on, we will use the matrix M from Lemma 3.13 to estimate the absolute value of matrix entries of $G(\lambda)$. With this established, we calculate

$$\|K(\lambda)\|^2 \leq \|K(\lambda)\|_{HS}^2 = \sum_{m,n\in\mathbb{Z}} \left| \sqrt{|v_m|}\; G_{m,n}(\lambda)\; \sqrt{|v_n|}\;\mathrm{sgn}\, v_n \right|^2$$
$$= \sum_{m,n\in\mathbb{Z}} |v_m| \left|G_{m,n}\right|^2 |v_n| \leq \sum_{m,n\in\mathbb{Z}} |v_m| M_{m,n}^2 |v_n| =: \sum_{m,n\in\mathbb{Z}} \widetilde{M}_{m,n}$$

$$\|K(\lambda)\|^2 \leq \underbrace{\sum_{\substack{m\geq 0\\ n\geq 0}} \widetilde{M}_{m,n}}_{A} + \underbrace{\sum_{\substack{m\geq 0\\ n< 0}} \widetilde{M}_{m,n}}_{B} + \underbrace{\sum_{\substack{m< 0\\ n< 0}} \widetilde{M}_{m,n}}_{C} + \underbrace{\sum_{\substack{m< 0\\ n\geq 0}} \widetilde{M}_{m,n}}_{D}.$$

We estimate each of the sums separately:

$$A = \sum_{\substack{m\geq 0\\ n\geq 0}} |v_m| \left(\frac{m+n-|m-n|}{2} + \frac{1}{\widehat{w}} \right)^2 |v_n|$$
$$= \underbrace{\sum_{\substack{m\geq 0\\ n\geq 0}} \frac{(m+n-|m-n|)^2}{4} |v_m||v_n|}_{A_1} + \underbrace{\sum_{\substack{m\geq 0\\ n\geq 0}} \frac{m+n-|m-n|}{\widehat{w}} |v_m||v_n|}_{A_2} + \underbrace{\frac{1}{\widehat{w}^2} \sum_{m\geq 0} |v_m| \sum_{n\geq 0} |v_n|}_{A_3}$$

$$A_1 \leq \frac{1}{2} \sum_{\substack{m\geq 0\\ n\geq 0}} \big(m^2+n^2\big)|v_m||v_n| = \frac{1}{2}\Big(\sum_{m\geq 0} m^2 |v_m| \sum_{n\geq 0} |v_n| + \sum_{m\geq 0} |v_m| \sum_{n\geq 0} n^2 |v_n| \Big)$$
$$= \sum_{m\geq 0} m^2 |v_m| \sum_{n\geq 0} |v_n|$$
$$A_2 \leq \frac{1}{\widehat{w}} \sum_{\substack{m\geq 0\\ n\geq 0}} \big(m+n\big)|v_m||v_n| = \frac{1}{\widehat{w}}\Big(\sum_{m\geq 0} m |v_m| \sum_{n\geq 0} |v_n| + \sum_{m\geq 0} |v_m| \sum_{n\geq 0} n |v_n| \Big)$$
$$= \frac{2}{\widehat{w}} \sum_{m\geq 0} m |v_m| \sum_{m\geq 0} |v_n|$$

$$C = \sum_{\substack{m<0\\n<0}} |v_m| \left(\frac{-m-n-|m-n|}{2} + \frac{1}{\widehat{w}} \right)^2 |v_m| = \underbrace{\sum_{\substack{m<0\\n<0}} \frac{(-m-n+|m-n|)^2}{4} |v_m||v_n|}_{C_1}$$

$$+ \underbrace{\sum_{\substack{m<0\\n<0}} \frac{-m-n-|m-n|}{\widehat{w}} |v_m||v_n|}_{C_2} + \underbrace{\frac{1}{\widehat{w}^2} \sum_{m<0} |v_m| \sum_{n<0} |v_n|}_{C_3}$$

$$\begin{aligned} C_1 &\leq \frac{1}{2} \sum_{\substack{m<0\\n<0}} \big(m^2+n^2\big)|v_m||v_n| = \frac{1}{2}\Big(\sum_{m<0} m^2|v_m| \sum_{n<0} |v_n| + \sum_{m<0} |v_m| \sum_{n<0} n^2|v_n| \Big) \\ &= \sum_{m<0} m^2|v_m| \sum_{n<0} |v_n| \\ C_2 &\leq \frac{1}{\widehat{w}} \sum_{\substack{m<0\\n<0}} \big(-m-n\big)|v_m||v_n| = \frac{1}{\widehat{w}}\Big(\sum_{m<0} |m||v_m| \sum_{n<0} |v_n| + \sum_{m<0} |v_m| \sum_{n<0} |n||v_n| \Big) \\ &= \frac{2}{\widehat{w}} \sum_{m<0} |m||v_m| \sum_{n<0} |v_n| \end{aligned}$$

$$B = \sum_{\substack{m\geq 0\\n<0}} |v_m| \frac{1}{\widehat{w}^2} |v_n| = \frac{1}{\widehat{w}^2} \sum_{m\geq 0} |v_m| \sum_{n<0} |v_n| \qquad D = \sum_{\substack{m<0\\n\geq 0}} |v_m| \frac{1}{\widehat{w}^2} |v_n| = \frac{1}{\widehat{w}^2} \sum_{m<0} |v_m| \sum_{n\geq 0} |v_n|.$$

$$\begin{aligned} &A_3 + B + C_3 + D \\ &\quad = \frac{1}{\widehat{w}^2}\Big(\sum_{m\geq 0} |v_m| \sum_{n\geq 0} |v_n| + \sum_{m\geq 0} |v_m| \sum_{n<0} |v_n| + \sum_{m<0} |v_m| \sum_{n<0} |v_n| + \sum_{m<0} |v_m| \sum_{n\geq 0} |v_n| \Big) \\ &\qquad = \frac{1}{\widehat{w}^2}\Big(\sum_{m\geq 0} |v_m| \sum_{n\in\mathbb{Z}} |v_n| + \sum_{m<0} |v_m| \sum_{n\in\mathbb{Z}} |v_n| \Big) = \frac{1}{\widehat{w}^2} \sum_{n\in\mathbb{Z}} |v_n| \sum_{m\in\mathbb{Z}} |v_m| = \frac{\|v\|^2_{\ell^1(\mathbb{Z})}}{\widehat{w}^2}, \end{aligned}$$

$$\begin{aligned} A_1 + C_1 &\leq \sum_{m\geq 0} m^2|v_m| \sum_{n\geq 0} |v_n| + \sum_{m<0} m^2|v_m| \sum_{n<0} |v_n| \\ &\leq \sum_{m\geq 0} m^2|v_m| \sum_{n\in\mathbb{Z}} |v_n| + \sum_{m<0} m^2|v_m| \sum_{n\in\mathbb{Z}} |v_n| = \sum_{m\in\mathbb{Z}} m^2|v_m| \sum_{n\in\mathbb{Z}} |v_n| = \|v\|_{\ell^1(\mathbb{Z},m^2)} \|v\|_{\ell^1(\mathbb{Z})}, \end{aligned}$$

$$\begin{aligned} A_2 + C_2 &\leq \frac{2}{\widehat{w}}\Big(\sum_{m\geq 0} m|v_m| \sum_{n\geq 0} |v_n| + \sum_{m<0} |m||v_m| \sum_{n<0} |v_n| \Big) \\ &\leq \frac{2}{\widehat{w}}\Big(\sum_{m\geq 0} m|v_m| \sum_{n\in\mathbb{Z}} |v_n| + \sum_{m<0} |m||v_m| \sum_{n\in\mathbb{Z}} |v_n| \Big) \\ &= \frac{2}{\widehat{w}} \sum_{m\in\mathbb{Z}} |m||v_m| \sum_{n\in\mathbb{Z}} |v_n| = \frac{2}{\widehat{w}} \|v\|_{\ell^1(\mathbb{Z},m^1)} \|v\|_{\ell^1(\mathbb{Z})}. \end{aligned}$$

Adding everything together we get the assertion of the proposition.

$$\begin{aligned}\|K(\lambda)\|^2 &\leq \frac{\|v\|^2_{\ell^1(\mathbb{Z})}}{\widehat{w}^2} + \|v\|_{\ell^1(\mathbb{Z},m^2)}\|v\|_{\ell^1(\mathbb{Z})} + \frac{2}{\widehat{w}}\|v\|_{\ell^1(\mathbb{Z},m^1)}\|v\|_{\ell^1(\mathbb{Z})} \\ &= \|v\|_{\ell^1(\mathbb{Z})}\Big(\frac{1}{\widehat{w}^2}\|v\|_{\ell^1(\mathbb{Z})} + \|v\|_{\ell^1(\mathbb{Z},m^2)} + \frac{2}{\widehat{w}}\|v\|_{\ell^1(\mathbb{Z},m^1)}\Big) \\ &\leq \frac{4}{\widehat{w}^2}\|v\|^2_{\ell^1(\mathbb{Z},m^2)}\end{aligned}$$

The last estimate is unnecessarily rough but as we have mentioned above our objective here is not optimality. At this point, we prefer simpler expressions to finer though more complicated upper bounds. ☐

Let us conclude this section with the culmination of these results.

Theorem 3.15: Let $\mathrm{Im}\alpha \neq 0$ and $v \in \ell^1(\mathbb{Z}, m^2)$. If we take $\varepsilon > 0$ so small that $2\varepsilon\|v\|_{\ell^1(\mathbb{Z},m^2)} < \widehat{w}$ then the potential εV does not change the spectrum, i.e.

$$\sigma(H_\alpha) = \sigma(H_\alpha + \varepsilon V).$$

Proof. If we estimate the norm of the Birman–Schwinger operator $\varepsilon K(\lambda)$ for the operator $H_\alpha + \varepsilon V$ according to Lemma 3.14 we get

$$\varepsilon\|K(\lambda)\| \leq \varepsilon\frac{2}{\widehat{w}}\|v\|_{\ell^1(\mathbb{Z},m^2)} < 1.$$

This rules out the possibility of -1 being an eigenvalue of $K^\varepsilon(\lambda)$. To finish the proof let us reiterate the Birman–Schwinger principle (Theorem 1.6):

$$\lambda \in \sigma_{\mathrm{p}}(H_\alpha + \varepsilon V) \qquad \Longleftrightarrow \qquad -1 \in \sigma_{\mathrm{p}}(K^\varepsilon(\lambda)).$$

Therefore, $\sigma_{\mathrm{p}}(H_\alpha + \varepsilon V) = \emptyset$. Since V is a compact operator, the perturbation does not change the essential spectrum, i.e. $\sigma_{\mathrm{ess}}(H_\alpha + \varepsilon V) = \sigma_{\mathrm{ess}}(H_\alpha)$, and so the assertion holds. ☐

Under similar assumptions on the potential V, the operator $H_\alpha + V$ exhibits spectral stability while the operator $H_0 + V$ has an eigenvalue no matter how small the potential is.

3.2.1 A comparison to the continuous setting

In this section we will compare our results to the result in [10]. The operator studied in this paper is the Schrödinger operator on $L^2(\mathbb{R})$ defined by

$$H := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}\,\mathrm{sgn}\,(x), \qquad \mathrm{Dom}(H) := W^{2,2}(\mathbb{R}).$$

The spectrum of H is shown to be purely essential and takes the form

$$\sigma(H) = \sigma_{\mathrm{ess}}(H) = [0, +\infty) + \mathrm{i}\{-1, 1\}.$$

In order to get a more precise analogue to the continuous operator H, we define the discrete operator $\widetilde{H}$ on $\ell^2(\mathbb{Z})$ by

$$\widetilde{H} := H_{2\mathrm{i}} - \mathrm{i}I = H_0 + \mathrm{i}\,\mathrm{sgn}\,(n).$$

In [14] we showed that the spectrum of $\widetilde{H}$ is also purely essential and coincides with the line segments $[-2, 2] \pm \mathrm{i}$, i.e.

$$\sigma(\widetilde{H}) = \sigma_{\mathrm{ess}}(\widetilde{H}) = [-2, 2] + \mathrm{i}\{-1, 1\}.$$

The main result of weak coupling analysis of H in [10] is the following theorem.

Theorem 3.16: Let $V \in L^1(\mathbb{R}, (1+x^2)\mathrm{d}x)$ and denote the closed half-strip $\mathcal{S} := [0, +\infty) + \mathrm{i}[-1, 1]$. There exists a positive constant C (independent of V and ε) such that, whenever

$$\varepsilon \int_{\mathbb{R}} V(x)(1+x^2)\mathrm{d}x \leq \frac{1}{C},$$

we have

$$\sigma_{\mathrm{p}}(H + \varepsilon V) \subset \mathcal{S} \cap \left\{ z \in \mathbb{C} \;\middle|\; \mathrm{Re} z \geq \frac{C}{\varepsilon^2 \|V\|^2_{L^1(\mathbb{R})}} \right\}.$$

A more intuitive, albeit less precise, interpretation of this theorem is that the continuous operator H exhibits spectral stability outside the half-strip $\mathcal{S}$, while any eigenvalue inside it is further from the origin of the complex plane, the smaller the perturbation; while the assumptions from the theorem above are satisfied.

If we adjust Theorem 3.15 from the preceding section to resemble Theorem 3.16 more closely, we get the following theorem.

Theorem 3.17: Let $v \in \ell^1(\mathbb{Z}, m^2)$. Then the point spectrum of $\widetilde{H} + \varepsilon V$ is empty whenever

$$\varepsilon \|v\|_{\ell^1(\mathbb{Z}, m^2)} < \frac{1}{2\varphi^2},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Proof. In [14] we set the two Joukowsky transforms to be

$$\lambda = \xi + \xi^{-1}, \qquad \lambda - \alpha = \eta + \eta^{-1}.$$

This allowed us to find the matrix representation of $(H_\alpha - \lambda)^{-1}$ in the form (1.2.4). One can easily notice that if we set the two Joukowsky transforms to be

$$\lambda + \mathrm{i} = \xi + \xi^{-1}, \qquad \lambda - \mathrm{i} = \eta + \eta^{-1},$$

we get the matrix representation of $(\widetilde{H} - \lambda)^{-1}$ in the exact same form. Therefore, Theorem 3.15 still holds. Since the point spectrum of $\widetilde{H}$ is empty, the point spectrum of $\widetilde{H} + \varepsilon V$ is empty as well. In order to evaluate the constant $\widehat{w}$ from Lemma 1.14, we need to slightly modify the proof of the lemma. Let us define two sets, U^η and U^ξ, and state basic inequalities on these sets

$$U^\eta = \{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda > 0\} \qquad |\eta| \leq 1, \qquad |\mathrm{Im}(\lambda + \mathrm{i})| \geq 1,$$
$$U^\xi = \{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda \leq 0\} \qquad |\xi| \leq 1, \qquad |\mathrm{Im}(\lambda - \mathrm{i})| \geq 1.$$

From Proposition 1.12 we get

$$\mathrm{Im}(\lambda - \mathrm{i}) \neq 0 \quad \Longrightarrow \quad |\eta| \leq \frac{1}{2}\left(\sqrt{|\mathrm{Im}(\lambda - \mathrm{i})|^2 + 4} - |\mathrm{Im}(\lambda - \mathrm{i})|\right),$$
$$\mathrm{Im}(\lambda + \mathrm{i}) \neq 0 \quad \Longrightarrow \quad |\xi| \leq \frac{1}{2}\left(\sqrt{|\mathrm{Im}(\lambda + \mathrm{i})|^2 + 4} - |\mathrm{Im}(\lambda + \mathrm{i})|\right).$$

If we set $\delta := (\sqrt{5} - 1)/2$, we get $|\eta| \leq \delta$ on U^ξ and $|\xi| \leq \delta$ on U^η by the same argument as in Lemma 1.14. So, as before, we have $\forall \lambda \in \mathbb{C} \;:\; w(\lambda) \geq \widehat{w} := 1 - \delta$. One can notice that the constant δ is the reciprocal value of the golden ratio φ. A straightforward calculation shows that $1 - \delta = \varphi^{-2}$. □

So in contrast with the continuous operator H, the discrete operator $\widetilde{H}$ exhibits complete spectral stability under similar assumptions on the potential V.

Chapter 4

Dirac interaction

The Dirac interaction is a special case of diagonal perturbation, where we take $V = c\delta_0$. The coupling constant c of the Dirac interaction is an arbitrary complex number and δ_0 considered as an operator on $\ell^2(\mathbb{Z})$ is defined on $x \equiv \{x_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ as

$$(\delta_0 x)_n = x_0 \delta_{n,0}.$$

Our aim in this chapter is to find eigenvalues of $H_\alpha + c\delta_0$ given a coupling constant c. The tool we will be using for this is the Birman–Schwinger principle, see Theorem 1.6.

The operator δ_0 is a projection, i.e. $\delta_0^2 = \delta_0$. Therefore, for purposes of the Birman–Schwinger principle we may decompose $c\delta_0 = (c\delta_0)\cdot(\delta_0)$. The matrix entries of the Birman–Schwinger operator $K(\lambda)$ in this setting read

$$K_{m,n}(\lambda) = \left(c\delta_0(H_\alpha - \lambda)^{-1}\delta_0\right)_{m,n} \begin{cases} c(H_\alpha - \lambda)^{-1}_{0,0} = \frac{c}{\xi - \eta^{-1}} & m, n = 0, \\ 0 & m, n \neq 0. \end{cases}$$

Theorem 1.6 states that λ is an eigenvalue of $H_\alpha + c\delta_0$ if and only if -1 is an eigenvalue of $K(\lambda)$. Since the operator $K(\lambda)$ takes in this setting a rather simple form, spectral analysis of the operator $H_\alpha + c\delta_0$ reduces to solving the algebraic equation $\frac{c}{\xi-\eta^{-1}} = -1$ and if we rearrange it, it reads

$$c = \eta^{-1} - \xi. \tag{4.0.1}$$

In the previous chapter, we showed that H_α exhibits spectral stability if the perturbation is small enough. Let us illustrate that result in this more specific setting.

Proposition 4.1: Let α have a non-zero imaginary part. Then whenever $|c| < 1-\delta$, the spectra of H_α and $H_\alpha + c\delta_0$ are identical, where δ is a constant dependent only on the parameter α.

Proof. There is no solution to (4.0.1) if the absolute value of $c - \xi + \eta^{-1}$ is strictly above zero. Let us estimate

$$\left|c - \xi + \eta^{-1}\right| \geq \left|\xi - \eta^{-1}\right| - |c| \geq 1 - \delta - |c|,$$

where the last inequality follows from Lemma 1.14. If we set $1 - \delta - |c| > 0$, we get a sufficient condition for the absence of eigenvalues. □

For a more careful analysis of the equation (4.0.1), we need to express the parameters ξ and η in terms of λ. Proposition 1.11 states

$$\xi = \begin{cases} \frac{\lambda+\sqrt{\lambda^2-4}}{2} & \mathrm{Re}\lambda \leq 0, \\ \frac{\lambda-\sqrt{\lambda^2-4}}{2} & \mathrm{Re}\lambda \geq 0, \end{cases} \qquad \eta^{-1} = \begin{cases} \frac{\lambda-\alpha-\sqrt{(\lambda-\alpha)^2-4}}{2} & \mathrm{Re}(\lambda-\alpha) \leq 0, \\ \frac{\lambda-\alpha+\sqrt{(\lambda-\alpha)^2-4}}{2} & \mathrm{Re}(\lambda-\alpha) \geq 0. \end{cases}$$

If we plug this into the equation (4.0.1), we get a more explicit, albeit piece-wise formulation that is dependent on the real part of α

- $\mathrm{Re}\alpha \leq 0$:

$$\begin{aligned}
\mathrm{Re}\lambda \in (-\infty, \mathrm{Re}\alpha] &\quad\Longrightarrow\quad 2c + \alpha = -\sqrt{\lambda^2 - 4} - \sqrt{(\lambda - \alpha)^2 - 4} \\
\mathrm{Re}\lambda \in (\mathrm{Re}\alpha, 0] &\quad\Longrightarrow\quad 2c + \alpha = -\sqrt{\lambda^2 - 4} + \sqrt{(\lambda - \alpha)^2 - 4} \\
\mathrm{Re}\lambda \in (0, \infty) &\quad\Longrightarrow\quad 2c + \alpha = +\sqrt{\lambda^2 - 4} + \sqrt{(\lambda - \alpha)^2 - 4}
\end{aligned}$$

- $\mathrm{Re}\alpha = 0$:

$$\begin{aligned}
\mathrm{Re}\lambda \in (-\infty, 0] &\quad\Longrightarrow\quad 2c + \alpha = -\sqrt{\lambda^2 - 4} - \sqrt{(\lambda - \alpha)^2 - 4} \\
\mathrm{Re}\lambda \in (0, \infty) &\quad\Longrightarrow\quad 2c + \alpha = +\sqrt{\lambda^2 - 4} + \sqrt{(\lambda - \alpha)^2 - 4}
\end{aligned}$$

- $\mathrm{Re}\alpha \geq 0$:

$$\begin{aligned}
\mathrm{Re}\lambda \in (-\infty, 0] &\quad\Longrightarrow\quad 2c + \alpha = -\sqrt{\lambda^2 - 4} - \sqrt{(\lambda - \alpha)^2 - 4} \\
\mathrm{Re}\lambda \in (0, \mathrm{Re}\alpha] &\quad\Longrightarrow\quad 2c + \alpha = +\sqrt{\lambda^2 - 4} - \sqrt{(\lambda - \alpha)^2 - 4} \\
\mathrm{Re}\lambda \in (\mathrm{Re}\alpha, \infty) &\quad\Longrightarrow\quad 2c + \alpha = +\sqrt{\lambda^2 - 4} + \sqrt{(\lambda - \alpha)^2 - 4}
\end{aligned}$$

Any solution to these equations within the respective regions is an eigenvalue of $H_\alpha + c\delta_0$. Finding these solutions proved, however, to be quite difficult. They can surely be obtained analytically because by carefully squaring the equation, we get quartic polynomial equations in terms of λ which have solutions in radicals; however, these are difficult to work with. A more careful analysis of these equations may be the subject of further research projects.

Conclusion

Since this research project is a direct continuation of my Bachelor's degree project, see [14], we built upon the results obtained in that project. We incorporated elementary findings regarding the Joukowsky transform from the previous project and further extended and proved additional assertions. Moreover, we began this paper with the knowledge of the spectrum of H_α and its resolvent operator. The content of the first chapter consisted also of some standard results from functional analysis and spectral theory.

The second chapter was devoted to the pseudospectral analysis of H_α. The ε-pseudospectra are strictly nested supersets of the spectrum where the resolvent operator's norm is large. After we mentioned the trivial case of self-adjoint operators we showed several techniques for estimating $\|(H_\alpha - \lambda)^{-1}\|$ from above and below. The Schur test served as the primary tool for obtaining upper bounds, for which we provided a formulation in $\ell^2(\mathbb{Z})$. Using these estimates we constructed a subset and a superset of the ε-pseudospectrum on the region of the complex plane where we do not have general mathematical tools to describe it exactly. Asymptotic formulas for the estimates of the resolvent operator's norm were also given.

In [8], the existence and uniqueness of weakly-coupled bound states were described for the operator $-\mathrm{d}^2/\mathrm{d}x^2 + \varepsilon V$. The authors demonstrated that if V is integrable with the weight $(1 + x^2)\mathrm{d}x$ and ε is sufficiently small, the aforementioned operator has at most one eigenvalue. Furthermore, this eigenvalue corresponds to a solution of a specific algebraic equation. For all values of ε, an eigenvalue exists if and only if the mean value of V is non-positive. We demonstrated that similar assertions hold true in the discrete setting. Specifically, the operator $H_0 + \varepsilon V$ has at most one eigenvalue when the potential is summable with quadratic weight and ε is sufficiently small. However, the other assertion made in the continuous case is slightly more intricate for H_0 due to its bounded nature. We proved that if the mean value of the potential is positive, there exists an eigenvalue greater than 2. Similarly, if the mean value of the potential is negative, there exists an eigenvalue less than -2.

We showed a similar behavior regarding spectral stability between the continuous Schrödinger operator with a complex step potential, as described in [10], and its discrete counterpart that we studied. When the potential applied to H_α is summable with quadratic weight and sufficiently small, we provided proof that it does not generate any eigenvalues. In other words, the operator H_α exhibits complete spectral stability. They, under similar assumptions imposed on the potential, showed that the continuous operator H may not have eigenvalues outside the closure of the numerical range. Furthermore, the smaller the potential, the greater the distance any eigenvalues must be from the origin. The primary tool for these results in weak-coupling analysis and spectral stability was the Birman–Schwinger principle.

The last chapter introduced the Dirac interaction and formulated the problem. Moreover, we demonstrated the spectral stability property of H_α stated in the previous chapter.

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