Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Discrete Schrödinger Operator with a Complex Step Potential

Diskrétní Schrödingerův operátor s komplexním schodovitým potenciálem

Master's thesis

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Pokyny pro vypracování:

1) Shrňte dosavadní výsledky a doplňte analýzu spektrálních vlastností diskrétního Schrödingerova operátoru s komplexním schodovitým potenciálem pro obecnou vazebnou konstantu.
2) Najděte spodní a horní odhady normy rezolventy studovaného operátoru. Diskutujte a numericky ilustrujte důsledky pro pseudospektrum.
3) V závislosti na vazebné konstantě diskutujte existenci slabě vázaných stavů, nebo spektrální stabilitu vzhledem k malým poruchám potenciálu.
4) V závislosti na vazebné konstantě se pokuste najít explicitní podmínky na normu potenciálu garantující spektrální stabilitu.
5) Pokuste se o analýzu existence vlastních hodnot vložených v esenciálním spektru.

Seznam doporučené literatury:

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I declare that this Master's thesis is entirely my own work and I have listed all the used sources in the bibliography.

Prague, January 6, 2025 Bc. Vojtěch Bartoš

Název práce:

Diskrétní Schrödingerův operátor s komplexním schodovitým potenciálem

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Abstrakt: V této diplomové práci byly zkoumány spektrálních vlastnosti Schrödingerova operátoru H_α na $\ell^2(\mathbb{Z})$ s komplexním schodovitým potenciálem, kde vazebná konstanta α je volné komplexní číslo. Zkoumáme pseudospektrum tohoto operátoru. Po nalezení spodního a horního odhadu normy rezolventního operátoru jsme zkonstruovali nadmnožinu a podmnožinu pseudospektra, navíc jsme také odvodili asymptotické chování pseudospektra. Pomocí Birman–Schwingerova principu jsme studovali existenci a jednoznačnost slabě vázaných vlastních hodnot za jistých předpokladů na potenciál V. Operátor $H_0 + V$ má jednoznačné vlastní hodnoty. Naproti tomu, pokud $\alpha \neq 0$ a potenciál V je dostatečně malý, potom $H_\alpha + V$ nemá žádné vlastní hodnoty, neboli vykazuje spektralní stabilitu. Navíc spektrum zůstává čistě spojité.

Klíčová slova: Birmanův-Schwingerův princip, diskrétní Schrödingerův operátor, nesamosdruženost, pseudospektrum, schodovitý potenciál, slabé vazby, spektrální stabilita

Title:

Discrete Schrödinger Operator with a Complex Step Potential

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*Abstract:*We study spectral properties of a Schrödinger operator H_α on $\ell^2(\mathbb{Z})$ with a step-like potential where the coupling constant α is a free complex number. We investigate the pseudospectrum of this operator. After obtaining the lower and upper estimates of the resolvent operator's norm, we construct a superset and a subset of the pseudospectrum, in addition, we also derive the asymptotic behavior of the pseudospectrum. Utilizing the Birman–Schwinger principle, we study the existence and uniqueness of weak-coupled eigenvalues under certain assumptions on the potential V. The operator $H_0 + V$ has unique eigenvalues. On the other hand, if $\alpha \neq 0$ and the potential V is small enough, then $H_\alpha + V$ has no eigenvalues, i.e. H_α exhibits spectral stability. Moreover, we showed that the spectrum remains purely continuous.

Key words: Birman-Schwinger principle, discrete Schrödinger operator, non-self-adjointness, pseudospectrum, spectral stability, step-like potential, weak-coupling

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Introduction

Consider the continuous Laplace operator on the real line. Its domain may be defined as the space of twice differentiable functions, and its action is given by

$$\forall f \in C^2(\mathbb{R}) \qquad : \qquad \Delta f = -\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}.$$

Derivatives, by their nature, are defined by infinitesimal changes, but they can be approximated by finitely small changes. Consider a function f on the real line; we can discretize it simply by restricting its argument to whole numbers. While having at our disposal only the values of f at whole numbers, we can approximate the value of the Laplace operator applied on f at n by

$$-\Delta f(x)\big|_n \approx f(n+1) - 2f(n) + f(n-1).$$

It is important to note that the continuous Laplacian is highly valuable in theoretical analysis, mathematical modeling, and certain scientific disciplines, such as classical physics and differential geometry. It provides a foundation for understanding continuous systems. However, in many practical applications involving discrete data and numerical computations, the discrete Laplacian offers distinct advantages and is more directly applicable.

For our purposes, we will define the discrete Laplacian H_0 as the sum of the forward translation and the backward translation

$$\forall x = \{x_n\}_{n\in\mathbb{Z}} \subset \mathbb{C} \qquad : \qquad (H_0 x)_n := x_{n+1} + x_{n-1}.$$

As the title of this thesis suggests, we study the Schrödinger operator with a complex step potential defined as

$$\forall x = \{x_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{Z}) \ : \ (H_\alpha x)_n := \begin{cases} x_{n-1} + x_{n+1} & n < 0, \\ x_{n-1} + \alpha x_n + x_{n+1} & n \geq 0, \end{cases}$$

where α is a complex parameter. A Schrödinger operator is defined as the Laplace operator with some potential added, so in our case, the potential can be understood as the discrete Heaviside function multiplied by the complex coupling constant α. Such an operator is non-self-adjoint for any $\alpha \in \mathbb{C} \setminus \mathbb{R}$.

The study of self-adjoint operators has enjoyed great attention from researchers since the birth of quantum mechanics. However, the study of non-self-adjoint operators as toy models saw the light of day only in the past couple of decades. Although self-adjoint operators are indeed fundamental in quantum mechanics, where they guarantee real spectra, there are many compelling reasons to investigate non-self-adjoint operators. Moreover, from a purely mathematical perspective, non-self-adjoint operators exhibit intriguing spectral behavior such as non-trivial pseudospectra that can differ significantly from their self-adjoint counterparts.

The main motivation for the toy model we investigate is [8]. Therein the authors considered the operator

$$H := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}\,\mathrm{sgn}\,(x), \qquad \mathrm{Dom}(H) = \mathrm{W}^{2,2}(\mathbb{R}).$$

The operator H_α is a discrete counterpart of H with a simple generalization that the coupling constant is a free complex number. In many ways, we discovered similar behavior. A comprehensive comparison has been performed in [2]. Later, in [6], this continuous operator has been generalized in a similar manner with free coupling constants and further studied. In another relevant publication, [4], the authors consider a similar type of potential and add it to the Dirac operator.

In the first chapter, we introduce the notation used in this thesis. Then we state several standard results from functional analysis. Lastly, we introduce the Joukowsky transform which is a crucial part in the analysis of H_α and state properties of this transform needed for further analysis.

The aim of the second chapter is to describe spectral properties of unperturbed H_α. Firstly, we describe the spectrum of H_α and in doing so we obtain the resolvent operator. Next, in the case that $\mathrm{Im}\alpha \neq 0$, when H_α is non-self-adjoint, we investigate the pseudospectrum of H_α by obtaining upper and lower estimates of the resolvent's norm. We also provide the asymptotic formulas for the pseudospectrum. This chapter is concluded by describing the spectral measure of H_α for $\mathrm{Im}\alpha = 0$.

In the third chapter, we investigate how small perturbations of H_α change the spectrum. We replicate the results obtained by Simon in [16]. Therein the author showed the weak coupling property of the continuous Schrödinger operator. We do so in a more general manner and allow complex perturbations for H_0. Then we showe that for $\alpha \neq 0$ the operator H_α exhibits spectral stability under some assumptions on the potential.

Chapter 1

Preliminaries

1.1 Notation

Notation 1.1: Let $a, b \in \mathbb{C}$. We define the line segment between these two complex numbers as the set of all convex combinations:

$$[a, b] := \{at + b(1-t) \mid t \in [0, 1]\}.$$

Notation 1.2: The linear span of the set S is denoted as $\operatorname{span}(S)$.

Notation 1.3: For operators, I will always denote the identity operator. To simplify notation, we define

$$A + a := A + aI,$$

where $a \in \mathbb{C}$.

Notation 1.4: Let $v = \{v_n\}_{n\in\mathbb{Z}}$ be a complex-valued sequence. We define the diagonal matrix with generating sequence v as

$$\operatorname{diag}(v) := \{v_n \delta_{m,n}\}_{m,n\in\mathbb{Z}},$$

where $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ otherwise.

Notation 1.5: Let $v = \{v_n\}_{n\in\mathbb{Z}}$ be a complex-valued sequence. We define

$$\begin{aligned} & & V &:= \operatorname{diag}(v), \\ |v| &:= \big\{|v_n|\big\}_{n\in\mathbb{Z}}, & |V| &:= \operatorname{diag}(|v|), \\ |v|^{1/2} &:= \big\{\sqrt{|v_n|}\big\}_{n\in\mathbb{Z}}, & |V|^{1/2} &:= \operatorname{diag}(|v|^{1/2}), \\ v_{1/2} &:= \big\{\sqrt{|v_n|}\operatorname{sgn} v_n\big\}_{n\in\mathbb{Z}}, & V_{1/2} &:= \operatorname{diag}(v_{1/2}), \end{aligned}$$

where the signum function for complex arguments is given by

$$\operatorname{sgn} z := \begin{cases} \frac{z}{|z|} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

Notation 1.6: Let $v \in \ell^1(\mathbb{Z})$ and $k \in \mathbb{N}$. We define the weighted ℓ^1 norm of v by

$$\|v\|_{\ell^1(\mathbb{Z},m^k)} := |v_0| + \sum_{m\in\mathbb{Z}} |m|^k |v_m|.$$

We also define subspaces of $\ell^1(\mathbb{Z})$ by

$$\ell^1(\mathbb{Z}, m^k) := \{v \in \ell^1(\mathbb{Z}) \mid \|v\|_{\ell^1(\mathbb{Z},m^k)} < \infty\}.$$

One can verify that $\|\cdot\|_{\ell^1(\mathbb{Z},m^k)}$ satisfies the properties of a norm. Moreover, it is straightforward to observe that $\|v\|_{\ell^1(\mathbb{Z})} \leq \|v\|_{\ell^1(\mathbb{Z},m^k)} \leq \|v\|_{\ell^1(\mathbb{Z},m^l)}$ whenever $k \leq l$.

Notation 1.7: Let $\mathscr{H}$ be a hilbert space. We denote the space of bounded operators on $\mathscr{H}$ as $\mathscr{B}(\mathscr{H})$ and the space of compact operators on $\mathscr{H}$ as $\mathscr{K}(\mathscr{H})$.

Notation 1.8: Let $A \in \mathscr{B}(\mathscr{H})$. The *algebraic multiplicity* of an isolated eigenvalue of A is defined by

$$\nu_{\mathrm{a}}(\lambda) := \dim(\mathrm{Ran}(P_\lambda)),$$

where

$$P_\lambda := \frac{1}{2\pi\mathrm{i}} \oint_{\partial B(\lambda,\varepsilon)} (A - z)^{-1} \mathrm{d}z.$$

The radius ε is chosen such that $\overline{B(\lambda, \varepsilon)} \cap \sigma(A) = \emptyset$. The *discrete spectrum* and *essential spectrum* of A as defined by

$$\sigma_{\mathrm{d}}(A) := \{\lambda \in \sigma_{\mathrm{p}}(A) \mid \lambda \text{ is isolated and } \nu_{\mathrm{a}}(\lambda) < \infty\},$$
$$\sigma_{\mathrm{ess}}(A) := \sigma(A) \setminus \sigma_{\mathrm{d}}(A).$$

1.2 Standard results from functional analysis

Theorem 1.9 (Riesz–Schauder)**:** Let $A \in \mathscr{K}(\mathscr{H})$. Then

1. $0 \in \sigma(A)$,
2. every non-zero element of the spectrum is an eigenvalue,
3. the set of all eigenvalues is at most countable,
4. every non-zero eigenvalue has a finite multiplicity,
5. the spectrum has at most one accumulation point $\lambda = 0$.

Theorem 1.10: Let $A \in \mathscr{B}(\mathscr{H})$ and $V \in \mathscr{K}(\mathscr{H})$. Assume the following to hold.

- The interior of $\sigma(A)$ is empty in the topology of $\mathbb{C}$.
- Each connected component of $\rho(A)$ contains a point in $\rho(A + V)$.

Then $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(A + V)$.

The proof of Theorem 1.10 can be found in [14].

Lemma 1.11: Let $A, B \in \mathscr{B}(\mathscr{H})$. Then the spectra of AB and BA, excluding the point 0, are identical:

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}.$$

Proof. Choose any $\lambda \notin \sigma(AB) \cup \{0\}$, then there exists a bounded inverse operator to $(AB - \lambda)$. Then set

$$C := \frac{1}{\lambda}\big(- I + B(AB - \lambda)^{-1} A\big),$$

Clearly, C is a bounded operator. Let us show that C is the inverse to $(BA - \lambda)$

$$\begin{aligned}(BA - \lambda)C &= -\frac{1}{\lambda}BA + I + \frac{1}{\lambda}BAB(AB - \lambda)^{-1}A - B(AB - \lambda)^{-1}A \\ &= I + \frac{1}{\lambda}B\big(- I + AB(AB - \lambda)^{-1} - \lambda(AB - \lambda)^{-1}\big)A \\ &= I + \frac{1}{\lambda}B\big(- I + (AB - \lambda)(AB - \lambda)^{-1}\big)A = I + \frac{1}{\lambda}BOA = I,\end{aligned}$$

$$\begin{aligned}C(BA - \lambda) &= -\frac{1}{\lambda}BA + I + \frac{1}{\lambda}B(AB - \lambda)^{-1}ABA - B(AB - \lambda)^{-1}A \\ &= I + \frac{1}{\lambda}B\big(- I + (AB - \lambda)^{-1}AB - \lambda(AB - \lambda)^{-1}\big)A \\ &= I + \frac{1}{\lambda}B\big(- I + (AB - \lambda)^{-1}(AB - \lambda)\big)A = I + \frac{1}{\lambda}BOA = I,\end{aligned}$$

where O denotes the zero operator. We get that $\rho(AB) \setminus \{0\} \subset \rho(BA)$. This inclusion still holds if we remove zero from the right hand side $\rho(AB) \setminus \{0\} \subset \rho(BA) \setminus \{0\}$. By interchanging A and B we get the opposite inclusion. If we take the complement of this set equality we arrive at the assertion of the lemma. □

Theorem 1.12 (Birman–Schwinger principle)**:** Let $H, V \in \mathscr{B}(\mathscr{H})$, $\lambda \in \rho(H)$. Let $A, B \in \mathscr{B}(\mathscr{H})$ such that $V = AB$ and define the Birman–Schwinger operator by

$$K(\lambda) := B(H - \lambda)^{-1}A.$$

Then

1. $\lambda \in \sigma_{\mathrm{p}}(H + V) \quad \Longrightarrow \quad -1 \in \sigma_{\mathrm{p}}(K(\lambda))$,
2. $V \in \mathscr{K}(\mathscr{H}) \quad \& \quad -1 \in \sigma_{\mathrm{p}}(K(\lambda)) \quad \Longrightarrow \quad \lambda \in \sigma_{\mathrm{p}}(H + V)$.

Proof. For the first statement:

Suppose $\lambda \in \sigma_{\mathrm{p}}(H + V)$. Then, there exists a non-zero $\psi \in \mathscr{H}$ such that

$$(H + V)\psi = \lambda\psi \quad \Longleftrightarrow \quad (H - \lambda)\psi = -V\psi. \tag{1.2.1}$$

Since $\lambda \in \rho(H)$, the operator $(H - \lambda)$ has a bounded inverse. Applying $(H - \lambda)^{-1}$, we find

$$-\psi = (H - \lambda)^{-1}V\psi = (H - \lambda)^{-1}AB\psi.$$

Next, applying B to both sides and letting $\phi := B\psi$, we derive

$$-\phi = B(H - \lambda)^{-1}A\phi = K(\lambda)\phi. \tag{1.2.2}$$

Since $\phi \neq 0$ (as $\phi = 0$ leads to a contradiction with $\lambda \in \sigma_{\mathrm{p}}(H)$), it follows that -1 is an eigenvalue of $K(\lambda)$.

Next, we move to the second statement. If -1 is an eigenvalue of $K(\lambda)$ and in turn an element of the spectrum, then by Lemma 1.11 we have that $-1 \in \sigma\big((H-\lambda)^{-1}AB\big)$. Since V is a compact operator, $(H-\lambda)^{-1}V$ is also compact. Then by Theorem 1.9 follows that -1 is an eigenvalue of $(H-\lambda)^{-1}V$ and so

$$\exists \psi \neq 0 \quad : \quad (H-\lambda)^{-1}V\psi = -\psi.$$

If we apply $(H-\lambda)$ on the equation from the left, we arrive at

$$\exists \psi \neq 0 \quad : \quad (H+V)\psi = \lambda\psi,$$

that is $\lambda \in \sigma_{\mathrm{p}}(H+V)$. □

In this paper, we specialize Theorem 1.12 to the Hilbert space $\ell^2(\mathbb{Z})$ with additional assumptions on the potential V. Specifically, we assume that V is a diagonal operator, enabling the use of Notation 1.5.

A common choice for decomposing the diagonal operator V in the Birman–Schwinger principle is to set

$$A = |V|^{1/2}, \qquad B = V_{1/2}.$$

Additionally, we impose the assumption that the sequence v defining V is summable in absolute value, i.e., $v \in \ell^1(\mathbb{Z})$. This summability condition ensures that $\lim_{n\to\pm\infty} v_n = 0$. One easily checks that

$$V \in \mathscr{K}(\mathscr{H}) \qquad \Longleftrightarrow \qquad \lim_{n\to\pm\infty} v_n = 0.$$

Then it follows that V is a compact operator. Under these assumptions, the Birman–Schwinger principle establishes an equivalence formulated in the next theorem.

Theorem 1.13: Let $H \in \mathscr{B}(\ell^2(\mathbb{Z}))$, $\lambda \in \rho(H)$, $v \in \ell^1(\mathbb{Z})$. If we denote the Birman–Schwinger operator

$$K(\lambda) = V_{1/2}(H-\lambda)^{-1}|V|^{1/2},$$

then

$$\lambda \in \sigma_{\mathrm{p}}(H+V) \qquad \Longleftrightarrow \qquad -1 \in \sigma_{\mathrm{p}}(K(\lambda)).$$

Theorem 1.14 (Schur test)**:** Let $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ be a doubly infinite matrix of complex entries. If there exists a positive number p_j for all $j \in \mathbb{Z}$ such that

$$\sup_{j\in\mathbb{Z}} \frac{1}{p_j} \sum_{i\in\mathbb{Z}} |a_{i,j}| p_i =: \alpha < \infty \qquad \& \qquad \sup_{i\in\mathbb{Z}} \frac{1}{p_i} \sum_{j\in\mathbb{Z}} |a_{i,j}| p_j =: \beta < \infty,$$

then $A \in \mathscr{B}(\ell^2(\mathbb{Z}))$ and $\|A\| \leq \sqrt{\alpha\beta}$.

Proof. Choose an arbitrary $\psi \in \operatorname{span}\{e_n \mid n \in \mathbb{Z}\}$, then

$$\|A\psi\|^2 = \sum_{i\in\mathbb{Z}} \Big| \sum_{j\in\mathbb{Z}} a_{i,j}\psi_j \Big|^2 \leq \sum_{i\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} \sqrt{|a_{i,j}|p_j} \sqrt{\frac{|a_{i,j}|}{p_j}} |\psi_j| \right)^2$$
$$\overset{\text{C.–S.}}{\leq} \sum_{i\in\mathbb{Z}} \underbrace{\left(\sum_{j\in\mathbb{Z}} |a_{i,j}|p_j \right)}_{\leq \alpha p_i,\ \forall i} \left(\sum_{j\in\mathbb{Z}} \frac{|a_{i,j}|}{p_j} |\psi_j|^2 \right) \overset{\star}{\leq} \alpha \sum_{j\in\mathbb{Z}} \underbrace{\left(\frac{1}{p_j} \sum_{i\in\mathbb{Z}} |a_{i,j}|p_i \right)}_{\leq \beta} |\psi|_i^2 \leq \alpha\beta\|\psi\|^2.$$

C.–S. signifies the Cauchy–Schwarz inequality and after the starred inequality, the interchange of summation is justified by Tonneli's theorem. □

Remark. A few observations about this operator norm inequality:

1. If the given doubly infinite matrix A is symmetric in terms of the absolute value of its entries, then
$$\alpha = \beta \qquad \& \qquad \|A\| \leq \sup_{j\in\mathbb{Z}} \frac{1}{p_j} \sum_{i\in\mathbb{Z}} |a_{i,j}| p_i.$$
2. In some cases, we may obtain simpler estimates by setting $p_j = 1$ for all $j \in \mathbb{Z}$. While this approach might yield rougher bounds, it can simplify computations significantly in practical applications.

Weyl's criterion can be formulated in various ways. Here, we present the version best suited for the arguments that will follow. Typically, Weyl's criterion is stated for normal operators, where the criterion asserts an equivalence. However, since we intend to apply the theorem to a non-normal operator, we restrict our statement to a single implication that does not rely on this assumption.

Theorem 1.15 (Weyl's criterion)**:** Let $A \in \mathscr{B}(\mathscr{H})$ and $\lambda \in \mathbb{C}$. Then
$$\exists \{x_n\}_{n=1}^{\infty} \subset \mathscr{H} \setminus \{0\} \ : \ \lim_{n\to\infty} \frac{\|(A-\lambda)x_n\|}{\|x_n\|} = 0 \quad \Longrightarrow \quad \lambda \in \sigma(A).$$

Proof. We shall prove the contrapositive implication,
$$\lambda \in \rho(A) \quad \Longrightarrow \quad \exists K > 0, \quad \forall \{x_n\}_{n=1}^{\infty} \subset \mathscr{H} \setminus \{0\} \ : \ \lim_{n\to\infty} \frac{\|(A-\lambda)x_n\|}{\|x_n\|} \geq K.$$
Choose $\lambda \in \rho(A)$ and $\{x_n\}_{n=1}^{\infty} \subset \mathscr{H} \setminus \{0\}$ arbitrarily. Then for any $n \in \mathbb{N}$, we have
$$\|x_n\| = \|(A-\lambda)^{-1}(A-\lambda)x_n\| \leq \|(A-\lambda)^{-1}\| \|(A-\lambda)x_n\|.$$
Since $\lambda \in \rho(A)$, we have $\|(A-\lambda)^{-1}\| \in (0,\infty)$. This allows us to set
$$K := \frac{1}{\|(A-\lambda)^{-1}\|}.$$
And because $\forall n \in \mathbb{N} \ : \ x_n \neq 0$ we can estimate
$$K \leq \frac{\|(A-\lambda)x_n\|}{\|x_n\|}.$$
To conclude the proof, take the limit as $n \longrightarrow \infty$. ☐

Theorem 1.16 (Theorem XIII.106 in [14])**:** For any trace class operator A,
$$\det(I+A) = \prod_{j=1}^{N(A)} \big(1 + \lambda_j(A)\big)$$
where $\big\{\lambda_j(A)\big\}_{j=1}^{N(A)}$ are the eigenvalues of A counted with algebraic multiplicity.

1.3 The Joukowsky transform

A key mathematical tool in spectral analysis the operator H_α, which will be defined later, see (2.0.1), is the *Joukowsky transform.* It relates the spectral parameter λ to a new variable k through the equation

$$\lambda = k + k^{-1}. \tag{1.3.1}$$

Its inverse of this transform, $k = k(\lambda)$, will be derived later. When analyzing the operator H_0, we will denote the Joukowsky transform parameter by k to pay homage to [9] and to differentiate it from the case of H_α, where we use two instances of Joukowsky transform

$$\lambda = \xi + \xi^{-1}, \qquad \lambda - \alpha = \eta + \eta^{-1}. \tag{1.3.2}$$

Let us state some properties of this transform.

Proposition 1.17: Joukowsky transform given by (1.3.1) is a bijective map between the sets

$$\{k \in \mathbb{C} \mid 0 < |k| < 1\} \qquad \longleftrightarrow \qquad \mathbb{C} \setminus [-2, 2].$$

Furthermore, it maps $\{k \in \mathbb{C} \mid 0 < |k| \le 1\}$ onto the whole complex plane, though not injectively.

Proof. Let us parameterize the set

$$\{k \in \mathbb{C} \mid 0 < |k| < 1\} = \{k = re^{\mathrm{i}\phi} \mid r \in (0, 1),\ \phi \in (-\pi, \pi]\}.$$

The elements k of the punctured unit disk get transformed by the transform as

$$\lambda = re^{\mathrm{i}\phi} + \frac{1}{r}e^{-\mathrm{i}\phi} = \Big(r + \frac{1}{r}\Big)\cos\phi + \mathrm{i}\Big(r - \frac{1}{r}\Big)\sin\phi. \tag{1.3.3}$$

If we fix some value of r and consider λ as a function of ϕ, it traces ellipses in the complex plane with foci at -2 and 2. If we vary the value of r, we clearly see that we arrive at a parametrization of

$$\mathbb{C} \setminus [-2, 2] = \left\{\Big(r + \frac{1}{r}\Big)\cos\phi + \mathrm{i}\Big(r - \frac{1}{r}\Big)\sin\phi \;\Big|\; r \in (0, 1),\ \phi \in (-\pi, \pi)\right\}.$$

Furthermore, if we set $r = 1$ and vary ϕ we get a parametrization of the interval $[-2, 2]$. Clearly, in this case, the bijective nature is no longer present. □

Corollary 1.18: remove if not usedLet $A \subset \mathbb{C} \setminus [-2, 2]$ be a compact set, then there exist $0 < a < b < 1$ such that that the Joukowsky parameters of any $\lambda \in A$ satisfy $a \le |k| \le b$.

Equation (1.3.1) clearly shows how λ is dependent on k. Since the bijective nature of the Joukowsky transform is of great importance, it is useful also to have an explicit formula describing how k is dependent on λ.

Proposition 1.19: Let $\lambda \in \mathbb{C} \setminus [-2, 2]$. The inverse Joukowsky transform $k = k(\lambda)$ is given by

$$k(\lambda) = \begin{cases} \big(\lambda + \sqrt{\lambda^2 - 4}\big)/2 & \mathrm{Re}\lambda < 0 \text{ or } \mathrm{Re}\lambda = 0, \mathrm{Im}\lambda \le 0, \\ \big(\lambda - \sqrt{\lambda^2 - 4}\big)/2 & \mathrm{Re}\lambda > 0 \text{ or } \mathrm{Re}\lambda = 0, \mathrm{Im}\lambda \ge 0, \end{cases}$$

where $\sqrt{\cdot}$ is assumed to be its principal branch. Furthermore, the reciprocal value of k is given by

$$\big(k(\lambda)\big)^{-1} = \begin{cases} \big(\lambda - \sqrt{\lambda^2 - 4}\big)/2 & \mathrm{Re}\lambda < 0 \text{ or } \mathrm{Re}\lambda = 0, \mathrm{Im}\lambda \le 0, \\ \big(\lambda + \sqrt{\lambda^2 - 4}\big)/2 & \mathrm{Re}\lambda > 0 \text{ or } \mathrm{Re}\lambda = 0, \mathrm{Im}\lambda \ge 0. \end{cases}$$

Proof. Let us break down the proof of this proposition into three parts.

1. Form of k:

   If we multiply (1.3.1) by a non-zero k, we obtain a quadratic equation $k^2 - \lambda k + 1 = 0$, the solutions of which read
   $$k_\pm = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}.$$

2. Reciprocal value of k:

   Since the constant term in a quadratic equation is the product of its roots and the constant term in the studied equation is 1, we have $k_+ k_- = 1$; hence,
   $$k_\pm^{-1} = k_\mp = \frac{\lambda \mp \sqrt{\lambda^2 - 4}}{2}.$$
   From this follows the equivalence
   $$|k_+| \geq 1 \qquad \Longleftrightarrow \qquad |k_-| \leq 1.$$

3. Piece-wise nature of k:

   It suffices to show the following
   $$\mathrm{Re}\lambda > 0 \text{ or } (\mathrm{Re}\lambda = 0 \text{ and } \mathrm{Im}\lambda \geq 0) \quad \Longrightarrow \quad |k_-| \leq |k_+|. \tag{1.3.4}$$

   - Let $\mathrm{Re}\lambda \geq 0$, then one may write $\lambda = x + \mathrm{i}y$, where $x \geq 0$ and $y \in \mathbb{R}$. Considering the principal square root, we know that $\forall z \in \mathbb{C} \ : \ \mathrm{Re}\sqrt{z} \geq 0$. This allows us to write $\sqrt{\lambda^2 - 4} = u + \mathrm{i}v$, where also $u \geq 0$ and $v \in \mathbb{R}$. Let us rewrite the inequality in (1.3.4)
     $$\begin{aligned} |x + \mathrm{i}y - u - \mathrm{i}v| &\leq |x + \mathrm{i}y + u + \mathrm{i}v|, \\ (x-u)^2 + (y-v)^2 &\leq (x+u)^2 + (y+v)^2, \\ 0 &\leq 4xu + 4yv. \end{aligned}$$
     The assumption above and said property of complex square root implies that $4xu \geq 0$. To conclude this proof we need to show that $\mathrm{sgn}\, y = \mathrm{sgn}\, v$, i.e. $\mathrm{sgn}\,\mathrm{Im}\lambda = \mathrm{sgn}\,\mathrm{Im}\sqrt{\lambda^2 - 4}$. Applying the formula for the square root of a complex number
     $$\sqrt{a + \mathrm{i}b} = \sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}} + \mathrm{i}\,\mathrm{sgn}\,(b)\sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}}$$
     on $\sqrt{\lambda^2 - 4}$ we get
     $$\begin{aligned} \mathrm{Im}\sqrt{\lambda^2 - 4} &= \mathrm{Im}\sqrt{(x^2 - y^2 - 4) + \mathrm{i}(2xy)} \\ &= \mathrm{sgn}\,(2xy)\sqrt{\frac{\sqrt{(x^2 - y^2 - 4)^2 + (2xy)^2} - x^2 + y^2 + 4}{2}}. \end{aligned}$$
     From this one easily sees
     $$\mathrm{sgn}\left(\mathrm{Im}\sqrt{\lambda^2 - 4}\right) = \mathrm{sgn}\left(2\underbrace{\mathrm{Re}\lambda}_{\geq 0}\mathrm{Im}\lambda\right) = \mathrm{sgn}\left(\mathrm{Im}\lambda\right).$$

- Let $\mathrm{Re}\lambda = 0, \mathrm{Im}\lambda \geq 0$, then one may write $\lambda = \mathrm{i}c$, where $c \geq 0$. A simple estimation yields (1.3.4); we have

$$\left|\lambda - \sqrt{\lambda^2 - 4}\right| = \left|\mathrm{i}c - \sqrt{-c^2 - 4}\right| \leq c + \sqrt{c^2 + 4} = \left|\mathrm{i}c + \sqrt{-c^2 - 4}\right| = \left|\lambda + \sqrt{\lambda^2 - 4}\right|$$

Similarly, one would show the other implication

$$\mathrm{Re}\lambda < 0 \text{ or } \mathrm{Re}\lambda = 0, \mathrm{Im}\lambda \leq 0 \qquad \Longrightarrow \qquad |k_-| \geq |k_+|.$$

□

When deriving the Green's kernel in Section 2.1.3 a prominent expression arises, the Wronskian, denoted by $w = \xi - \eta^{-1}$. A property of the Wronskian of paramount importance is that for $\alpha \neq 0$ it never reaches 0 in absolute value. The following estimate will be used many times.

Lemma 1.20: Let $\alpha \neq 0$. Then for the Joukowsky parameters ξ and η of λ given by (1.3.2) we have

$$\exists K > 0 \ \forall \lambda \in \mathbb{C} \ : \ |\xi(\lambda) - \eta(\lambda)^{-1}| > K.$$

Proof. We will prove this lemma in two steps. First, we show that $\forall \lambda \in \mathbb{C} \ : \ \xi(\lambda) \neq \eta(\lambda)^{-1}$. After this, we show that $\lim_{\lambda \to \infty} |\xi(\lambda) - \eta(\lambda)^{-1}| \neq 0$. Combining these two statements one can clearly see that the expression is nowhere zero and also does not approach zero in the limit; hence, there has to exist a constant K with the desired property.

If we plug in the expressions of ξ and η^{-1} from Proposition 1.19 into $|\xi - \eta^{-1}|$ and multiply by 2, we arrive at

$$2(\xi - \eta^{-1}) = \begin{cases} \alpha + \sqrt{\lambda^2 - 4} + \sqrt{(\lambda - \alpha)^2 - 4}, \\ \alpha + \sqrt{\lambda^2 - 4} - \sqrt{(\lambda - \alpha)^2 - 4}, \\ \alpha - \sqrt{\lambda^2 - 4} + \sqrt{(\lambda - \alpha)^2 - 4}, \\ \alpha - \sqrt{\lambda^2 - 4} - \sqrt{(\lambda - \alpha)^2 - 4}, \end{cases}$$

where the piece-wise nature of the function depends on the real and imaginary parts of λ and $(\lambda - \alpha)$ but for finding the root of this function this distinction is inconsequential. As illustrated below, once we take squares, the piecewise details no longer affect the outcome, so we omit them. If we take one option, we get

$$\begin{aligned} \alpha + \sqrt{\lambda^2 - 4} &= \sqrt{(\lambda - \alpha)^2 - 4} \\ \alpha^2 + 2\alpha\sqrt{\lambda^2 - 4} + \alpha^2 - 4 &= \lambda^2 - 2\alpha\lambda + \alpha^2 - 4 \\ \sqrt{\lambda^2 - 4} &= \lambda \\ \lambda^2 - 4 &= \lambda^2 \\ -4 &= 0. \end{aligned}$$

There is clearly no solution to this equation.

In the limit $\lambda \to \infty$ both parameters ξ and η approach 0. Therefore, the term $\xi + \eta^{-1}$ not only does not vanish, it blows up. □

Remark. The aforementioned estimate works for every non-zero α. In later sections of the text, we will be mainly using the lower bound for the Wronskian. The aforementioned lemma allows us to define a constant dependent on α in the following way

$$\forall \alpha \neq 0 \ : \ C(\alpha) := \sup_{\lambda \in \mathbb{C}} \frac{1}{|\xi - \eta^{-1}|}. \tag{1.3.5}$$

Lemma 1.20 yields no quantitative information about the value of $C(\alpha)$. However, when α is real, Lemma 1.21 provides an exact value. For this case, a plot of $C(\alpha)$ is shown in Figure 1.1.

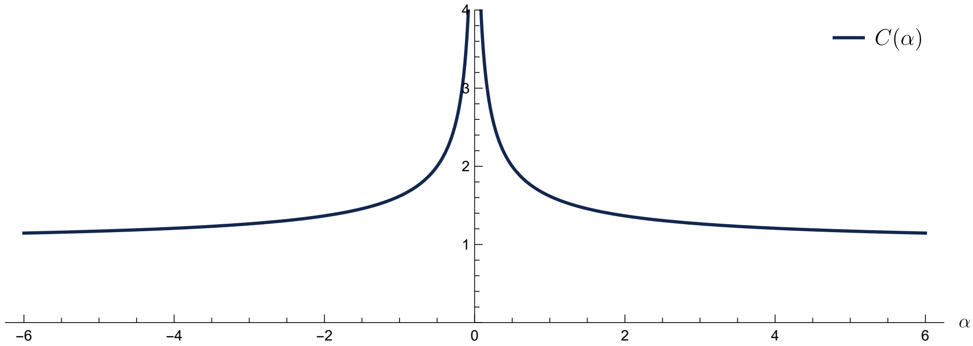


Figure 1.1: Plot of $C(\alpha)$ for real α.

Lemma 1.21: Let $\alpha \in \mathbb{R} \setminus \{0\}$, then

$$C(\alpha) = \left(|\alpha| \left(1 + \frac{|\alpha|}{2} - \sqrt{\frac{|\alpha|^2}{4} + |\alpha|} \right) \right)^{-\frac{1}{2}}.$$

Proof. Since the problem exhibits symmetry about $\alpha = 0$, we may assume without loss of generality that $\alpha > 0$. The objective is to determine

$$\inf_{\lambda \in \mathbb{C}} \left| \xi - \eta^{-1} \right|.$$

The function $\xi - \eta^{-1}$ is holomorphic on the domain $\mathbb{C} \setminus \big([-2, 2] \cup [-2 + \alpha, 2 + \alpha]\big)$. However, it is not continuous up to the boundary, as $\xi - \eta^{-1}$ may approach different values depending on the side from which the boundary segments are approached. This arises because the set excludes two, or in some cases only one, line segment(s) in the complex plane, depending on the parameter α. Consequently, care must be taken when applying the maximum modulus principle.

To address this, we decompose the complex plane into the upper and lower half-planes,

$$\mathbb{C}^{+\mathrm{i}} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}, \quad \mathbb{C}^{-\mathrm{i}} = \{z \in \mathbb{C} \mid \mathrm{Im} z < 0\}.$$

Although these regions are unbounded, we pass to the Joukowsky transform parameters ξ and η, which map them to the bounded regions

$$\mathbb{D}^{+\mathrm{i}} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0 \text{ and } |z| < 1\}, \quad \mathbb{D}^{-\mathrm{i}} = \{z \in \mathbb{C} \mid \mathrm{Im} z < 0 \text{ and } |z| < 1\}.$$

With this established, the maximum modulus principle ensures that the infimum of $|\xi - \eta^{-1}|$ is attained on the boundary of $\mathbb{D}^{+\mathrm{i}}$ or $\mathbb{D}^{-\mathrm{i}}$.

As the problem is symmetrical about $\alpha = 0$, we will assume $\alpha > 0$. For this proof, we will go back to the reciprocal value, i.e. our objective is to find

$$\inf_{\lambda \in \mathbb{C}} |\xi - \eta^{-1}|.$$

The function $\xi - \eta^{-1}$ is holomorphic on the set $\mathbb{C} \setminus \big([-2, 2] \ \cup \ [-2 + \alpha, 2 + \alpha]\big)$. But it is not continuous up to the boundary. We are removing two, possibly one, line segments from the complex plane. $\xi - \eta^{-1}$ can approach different values when approaching the line segment from either side. Because of this, we need to be careful when applying the maximum modulus principle. We avoid said problem by diving the complex plane into two half-planes, $\mathbb{C}^{+\mathrm{i}} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ and $\mathbb{C}^{-\mathrm{i}} = \{z \in \mathbb{C} \mid \mathrm{Im} z < 0\}$. These regions are clearly not bounded but if we transition to the perspective of the Joukowsky transform parameters ξ and η, we get $\mathbb{D}^{+\mathrm{i}} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0 \ \& \ |z| \leq 1\}$ and $\mathbb{D}^{-\mathrm{i}} = \{z \in \mathbb{C} \mid \mathrm{Im} z < 0 \ \& \ |z| \leq 1\}$. With this established, we can apply the maximum modulus principle and assert that the infimum is attained on the boundary of $\mathbb{D}^{+\mathrm{i}}$ or $\mathbb{D}^{-\mathrm{i}}$.

This naturally leads to three, not necessarily mutually exclusive cases, where A: $|\xi| = 1$, B: $|\eta| = 1$, or C: $\xi, \eta \in (-1, 1)$.

A: Let $|\eta| = 1$, then we can write $\eta = e^{\mathrm{i}\theta}$, where $\theta \in (-\pi, \pi]$ and we have

$$\xi + \xi^{-1} = \eta + \eta^{-1} + \alpha = 2\cos\theta + \alpha \begin{cases} \in [-2, 2] & \ldots \mathrm{A1} \\ \notin [-2, 2] & \ldots \mathrm{A2} \end{cases}$$

Now we discuss these sub-cases.

A1: Under the assumption $\alpha > 0$ the condition A1 reduces to $2\cos\theta + \alpha \leq 2$ and can occur only if $\alpha < 4$. Together with condition A both ξ and η are equal to one in absolute value; hence, there is a function $\theta \mapsto \phi(\theta)$ such that $\xi = e^{\mathrm{i}\phi(\theta)}$. Then the equation relating ξ and η reads

$$2\cos\phi(\theta) = 2\cos\theta + \alpha \tag{1.3.6}$$

and one can easily take the arc-cosine function to describe the function ϕ explicitly, but there is no need. Let us continue

$$\begin{aligned} \left|\xi - \eta^{-1}\right|^2 &= \left|e^{\mathrm{i}\phi(\theta)} - e^{-\mathrm{i}\theta}\right|^2 = \left|e^{\mathrm{i}(\phi(\theta)+\theta)} - 1\right|^2 \\ &= \big(\cos\big(\phi(\theta) + \theta\big) - 1\big)^2 + \sin^2\big(\phi(\theta) + \theta\big) \\ &= 2 - 2\cos\big(\phi(\theta) + \theta\big) \\ &= 2 - 2\cos\theta\cos\phi + 2\sin\theta\sin\phi. \end{aligned}$$

Next, we substitute $t = \cos\theta$, we then have $\cos\phi = t + \frac{\alpha}{2}$, we express the sine functions in terms of t, and we arrive at

$$\left|\xi - \eta^{-1}\right|^2 = 2 - 2t\Big(t + \frac{\alpha}{2}\Big) + 2\sqrt{1 - t^2}\sqrt{1 - \Big(t + \frac{\alpha}{2}\Big)^2} =: 2f(t).$$

The substitution we have utilized allows for $t \in [-1, 1]$. The condition A1 restricts $t \leq 1 - \frac{\alpha}{2}$. The function f is symmetric about $t = -\frac{\alpha}{4}$. This point is also the only critical point of f; furthermore, it is a global maximum of the function. Hence The minimum of f is attained at the endpoints. For case A1, we have

$$\min_{\mathrm{A1}} \left|\xi - \eta^{-1}\right|^2 = 2f(-1) = \alpha.$$

A2: Under the assumption $\alpha > 0$ the condition A2 reduces to $2\cos\theta + \alpha > 2$ and occurs for any α. In this case, ξ is a real number given by

$$\xi + \xi^{-1} = 2\cos\theta + \alpha. \tag{1.3.7}$$

The objective function then reads

$$\left|\xi - \eta^{-1}\right|^2 = \left|\xi - e^{-\mathrm{i}\theta}\right|^2 = (\xi - \cos\theta)^2 + \sin^2\theta = \xi^2 - 2\xi\cos\theta + 1 = \alpha\xi$$

Lemma 1.19 yields

$$\xi = \cos\theta + \frac{\alpha}{2} - \sqrt{\left(\cos\theta + \frac{\alpha}{2}\right)^2 - 1},$$

and so

$$\begin{aligned}\left|\xi - \eta^{-1}\right|^2 &= \alpha\left(\cos\theta + \frac{\alpha}{2} - \sqrt{\left(\cos\theta + \frac{\alpha}{2}\right)^2 - 1}\right)\\ &= \alpha\left(t + \frac{\alpha}{2} - \sqrt{\left(t + \frac{\alpha}{2}\right)^2 - 1}\right),\end{aligned}$$

where we again substituted $t = \cos\theta \in [-1, 1]$ and the condition A2 restricts $t > 1 - \frac{\alpha}{2}$. The function above is decreasing as a function of t, so the minimum is attained at $t = 1$. For case A2, we have

$$\min_{\mathrm{A2}}\left|\xi - \eta^{-1}\right|^2 = \alpha\left(1 + \frac{\alpha}{2} - \sqrt{\left(1 + \frac{\alpha}{2}\right)^2 - 1}\right).$$

Since $\left(1 + \frac{\alpha}{2} - \sqrt{\left(1 + \frac{\alpha}{2}\right)^2 - 1}\right)$ is some positive ξ which is less than one, we have

$$\min_{\mathrm{A}}\left|\xi - \eta^{-1}\right|^2 = \min_{\mathrm{A2}}\left|\xi - \eta^{-1}\right|^2$$

B: Let $|\xi| = 1$, then we can write $\xi = e^{\mathrm{i}\phi}$, where $\phi \in (-\pi, \pi]$ and we have

$$\eta + \eta^{-1} = 2\cos\phi - \alpha \begin{cases} \in [-2, 2] & \ldots \mathrm{B1} \\ \notin [-2, 2] & \ldots \mathrm{B2} \end{cases}$$

B1: This case is largely the same as A1. Here we have $2\cos\phi \geq \alpha - 2$. $|\eta| = 1$ and so there is a function $\phi \mapsto \theta(\phi)$ such that $\eta = e^{\mathrm{i}\theta(\phi)}$. With the substitution $t = \cos\phi$, we get

$$\left|\xi - \eta^{-1}\right|^2 = \quad \ldots \quad = 2 - 2t\left(t - \frac{\alpha}{2}\right) + 2\sqrt{1 - t^2}\sqrt{1 - \left(t - \frac{\alpha}{2}\right)^2} := 2f(t).$$

Condition B1 restricts $t \geq \frac{\alpha}{2} - 1$ and the result for this case reads

$$\min_{\mathrm{B1}}\left|\xi - \eta^{-1}\right|^2 = \alpha.$$

B2: For $\alpha > 0$ the condition B2 reads $2\cos\phi < \alpha - 2$. η is a real number given by

$$\eta + \eta^{-1} = 2\cos\phi - \alpha.$$

Then we have

$$\left|\xi-\eta^{-1}\right|^2=\left|e^{\mathrm{i}\phi}-\eta^{-1}\right|^2=(\cos\phi-\eta^{-1})^2+\sin^2\phi=\eta^2-2\eta\cos\phi+1=-\alpha\eta.$$

Lemma 1.19 yields

$$\eta^{-1}=\cos\phi-\frac{\alpha}{2}-\sqrt{\left(\cos\phi-\frac{\alpha}{2}\right)^2-1}.$$

After substituting $t=\cos\phi$, we arrive at

$$\left|\xi-\eta^{-1}\right|^2=-\alpha\left(t-\frac{\alpha}{2}-\sqrt{\left(t-\frac{\alpha}{2}\right)^2-1}\right)=:-\alpha g(t).$$

From the substitution $t\in[-1,1]$ but the condition B2 restricts $t\leq\frac{\alpha}{2}-1$ so we have

$$t\in\begin{cases}[-1,1] & \text{if}\quad \alpha\geq 4,\\ \left[-1,\frac{\alpha}{2}-1\right) & \text{if}\quad \alpha\in(0,4).\end{cases}$$

The function g is increasing and has no critical point, so g will attain its maximum or approach its supremum at the right endpoint. For this final case, we have

$$\inf_{\mathrm{B2}}\left|\xi-\eta^{-1}\right|^2=-\alpha\sup_t g(t)=\begin{cases}-\alpha g\left(\frac{\alpha}{2}-1\right)=\alpha & \alpha\in(0,4),\\ -\alpha g(1)=-\alpha\left(1-\frac{\alpha}{2}-\sqrt{\frac{\alpha^2}{4}-\alpha}\right) & \alpha\geq 4.\end{cases}$$

Since $\left(1-\frac{\alpha}{2}-\sqrt{\frac{\alpha^2}{4}-\alpha}\right)$ is some η^{-1} which is less than -1, we have

$$\min_{\mathrm{B}}\left|\xi-\eta^{-1}\right|^2=\alpha.$$

C: The last case for the discussion to be exhaustive is when both ξ and $\eta\in(-1,1)$, this corresponds to $\lambda\in\mathbb{R}\setminus\big([-2,2]\ \cup\ [-2+\alpha,2+\alpha]\big)$. Depending on the value of α, there are two or three connected components of this set. Let us discuss these three sub-cases separately.

C1: Let $\lambda<-2$. Then $\xi,\eta<0$ and we have

$$2|\xi-\eta^{-1}|=2(\xi-\eta^{-1})=\alpha+\sqrt{\lambda^2-4}+\sqrt{(\lambda-\alpha)^2-4}\,\bigg|_{\lambda=-2}=\alpha+\sqrt{(\alpha+2)^2-4}.$$

C2: Let $\lambda\in(2,\alpha-2)$, possible only for $\alpha>4$. Then $\xi>0$ and $\eta<0$. We have

$$\begin{aligned}2|\xi-\eta^{-1}|=2(\xi-\eta^{-1})&=\alpha-\sqrt{\lambda^2-4}+\sqrt{(\lambda-\alpha)^2-4}\\ &=\begin{cases}\alpha+\sqrt{(2-\alpha)^2-4} & \text{if}\ \ \lambda=2,\\ \alpha-\sqrt{(2-\alpha)^2-4} & \text{if}\ \ \lambda=\alpha-2.\end{cases}\end{aligned}$$

C3: Let $\lambda>\alpha+2$. Then $\xi,\eta>0$ and

$$2|\xi-\eta^{-1}|=2(\xi-\eta^{-1})=-\alpha+\sqrt{\lambda^2-4}+\sqrt{(\lambda-\alpha)^2-4}\,\bigg|_{\lambda=\alpha+2}=-\alpha+\sqrt{(\alpha+2)^2-4}.$$

For each of these sub-cases, the following argumentation applies. The Joukowsky inverse comes from Lemma 1.19. We used the fact that $\xi < \eta^{-1}$. None of these functions in terms of λ have any critical points; thus, we evaluate the functions at the endpoints. A simple calculation would show that the value for case C3 is minimal. Lastly, if we were to square this minimal value, divide by 4, and rearrange the terms, we would arrive at the expression for the minimal value for case A.

If we repeat the same argument that finalized case A, we get

$$\min \left|\xi - \eta^{-1}\right|^2 = \alpha \left(1 + \frac{\alpha}{2} - \sqrt{\left(1+\frac{\alpha}{2}\right)^2 - 1}\right).$$

Taking the reciprocal value of the square root of this expression we arrive at $C(\alpha)$. □

Lemma 1.22: Let $\lambda \in (-2, 2)$ and $\widetilde{\lambda} = \lambda + \mathrm{i}\varepsilon$. The Joukowsky parameter k and its reciprocal, associated with $\widetilde{\lambda}$, follow the asymptotic expansions

$$k = e^{\mathrm{i}\phi}\left(1 + \frac{\varepsilon}{2\sin\phi}\right) + \mathcal{O}(\varepsilon^2), \qquad k^{-1} = e^{-\mathrm{i}\phi}\left(1 - \frac{\varepsilon}{2\sin\phi}\right) + \mathcal{O}(\varepsilon^2) \qquad \text{as } \varepsilon \to 0_+,$$

where $\phi = -\arccos(\lambda/2) \in (-\pi, 0)$.

Proof. The unit circle $\{e^{\mathrm{i}\phi} \mid \phi \in (-\pi, \pi]\}$ is mapped non-injectively onto the interval $[-2, 2]$ by the Joukowsky transform. This is clear if we plug this parametrization into (1.3.1); moreover, we have $\lambda = 2\cos\phi$ and we can express ϕ in terms of λ. There are two options to do so, $\phi = \pm\arccos(\lambda/2)$; this follows from the non-injective nature of the mapping. We chose the negative sign because the lower half of the punctured unit disk is mapped onto the upper complex half-plane, where $\widetilde{\lambda}$ resides, by the Joukowsky transform. One can easily see this in (1.3.3). By this choice and the assumption $\lambda \in (-2, 2)$, we have $\phi \in (-\pi, 0)$. We obtain the exact expression for the Joukowsky parameter of $\widetilde{\lambda} = 2\cos\phi + \mathrm{i}\varepsilon$ from Lemma 1.19, which prescribes which of $\pm$ we need to choose

$$k = \frac{2\cos\phi + \mathrm{i}\varepsilon \pm \sqrt{(2\cos\phi + \mathrm{i}\varepsilon)^2 - 4}}{2}.$$

But the assumption of the lemma is that $\lambda \notin [-2, 2]$. Since this assumption is not met for $\varepsilon = 0$, choosing the negative branch ensures k at $\varepsilon = 0$ maps correctly under the transform:

$$k\Big|_{\varepsilon=0} = \frac{2\cos\phi - \sqrt{(2\cos\phi)^2 - 4}}{2} = \cos\phi - \sqrt{-\sin^2\phi} = \cos\phi - \mathrm{i}|\sin\phi| = e^{\mathrm{i}\phi},$$

where for $\phi \in (-\pi, 0)$ we have $\sin\phi < 0$. To capture the first-order behavior in ε, we differentiate k with respect to ε and then evaluate at $\varepsilon = 0$:

$$\frac{\mathrm{d}k}{\mathrm{d}\varepsilon} = \frac{\mathrm{i}}{2}\left(1 - \frac{\mathrm{i}\varepsilon + 2\cos\phi}{\sqrt{(2\cos\phi + \mathrm{i}\varepsilon)^2 - 4}}\right),$$

$$\frac{\mathrm{d}k}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} = \frac{\mathrm{i}}{2}\left(1 - \frac{2\cos\phi}{\sqrt{(2\cos\phi)^2 - 4}}\right) = \frac{\mathrm{i}}{2}\left(1 + \frac{\cos\phi}{\mathrm{i}\sin\phi}\right) = \frac{1}{2\sin\phi}e^{\mathrm{i}\phi}.$$

Combining these two calculations we get

$$k = e^{\mathrm{i}\phi}\left(1 + \frac{\varepsilon}{2\sin\phi}\right) + \mathcal{O}(\varepsilon^2).$$

To check the expansion for the reciprocal value we multiply the known expansion for k with the proposed expansion for k^{-1} and arrive at $kk^{-1} = 1 + \mathcal{O}(\varepsilon)$. $\square$

Remark. The asymptotic expansions of k when approaching λ from the other side is similar. One needs to take $\phi = \arccos(\lambda/2)$ and replace ε with $-\varepsilon$ in the asymptotic expression.

Corollary 1.23: Let $\lambda \in (-2, 2)$ and $\widetilde{\lambda} = \lambda + \mathrm{i}\varepsilon$. Then

$$|k|^2 = 1 + \frac{\varepsilon}{\sin\phi} + \mathcal{O}(\varepsilon^2), \qquad |k| = 1 + \frac{\varepsilon}{2\sin\phi} + \mathcal{O}(\varepsilon^2), \qquad \text{as } \varepsilon \to 0_+,$$
$$|k - k^{-1}|^2 = 4\sin^2\phi + \mathcal{O}(\varepsilon^2), \qquad |k - k^{-1}| = -2\sin\phi + \mathcal{O}(\varepsilon^2), \qquad \text{as } \varepsilon \to 0_+,$$

where $\phi = -\arccos(\lambda/2) \in (-\pi, 0)$.

Proof. If we take the asymptotic expansion for k and k^{-1} from the previous lemma, a simple calculation yields

$$|k|^2 = \left(1 + \frac{\varepsilon}{2\sin\phi} + \mathcal{O}(\varepsilon^2)\right)^2 = 1 + \frac{\varepsilon}{\sin\phi} + \mathcal{O}(\varepsilon^2),$$
$$|k - k^{-1}|^2 = \left| e^{\mathrm{i}\phi} - e^{-\mathrm{i}\phi} + \frac{\varepsilon}{2\sin\phi}\left(e^{\mathrm{i}\phi} + e^{-\mathrm{i}\phi}\right) + \mathcal{O}(\varepsilon^2) \right|^2 = 4\sin^2\phi + \mathcal{O}(\varepsilon^2).$$

as $\varepsilon \to 0_+$. If we apply the asymptotic expansion $\sqrt{1+x} = 1 + \frac{x}{2} + \mathcal{O}(\varepsilon^2)$ as $x \to 0_+$, we arrive at the asymptotic expansions for the absolute values of the desired expression. Note that since $\phi \in (-\pi, 0)$ we have $|\sin\phi| = -\sin\phi$. $\square$

Lemma 1.24: Let $\widetilde{\lambda} = 2 + \mathrm{i}\varepsilon$. Then the Joukowsky parameter k and its reciprocal, associated with $\widetilde{\lambda}$, follow the asymptotic expansions

$$k = 1 - \sqrt{\mathrm{i}\varepsilon} + \mathcal{O}(\varepsilon), \qquad k^{-1} = 1 + \sqrt{\mathrm{i}\varepsilon} + \mathcal{O}(\varepsilon), \qquad \text{as } \varepsilon \to 0_+.$$

Similarly, for $\lambda = -2 + \mathrm{i}\varepsilon$, we have

$$k = -1 + \sqrt{-\mathrm{i}\varepsilon} + \mathcal{O}(\varepsilon), \qquad k^{-1} = -1 - \sqrt{-\mathrm{i}\varepsilon} + \mathcal{O}(\varepsilon), \qquad \text{as } \varepsilon \to 0_+.$$

Proof. We will show proof only for $\widetilde{\lambda} = 2 + \mathrm{i}\varepsilon$ as the other case is analogous. Since $\forall\varepsilon > 0$: $\mathrm{Re}\widetilde{\lambda} > 0$, Lemma 1.19 states that

$$k = \frac{2 + \mathrm{i}\varepsilon - \sqrt{(2+\mathrm{i}\varepsilon)^2 - 4}}{2}.$$

Then after some manipulation of the expression we apply the Taylor expansion $\sqrt{1-x} = 1 + \mathcal{O}(x)$, as $x \to 0_+$, and arrive at the desired asymptotic formula

$$k = 1 + \frac{\mathrm{i}}{2}\varepsilon - \sqrt{\mathrm{i}\varepsilon}\sqrt{1 - \frac{\varepsilon}{4\mathrm{i}}} = 1 + \frac{\mathrm{i}}{2}\varepsilon - \sqrt{\mathrm{i}\varepsilon}\big(1 + \mathcal{O}(\varepsilon)\big) = 1 - \sqrt{\mathrm{i}\varepsilon} + \frac{\mathrm{i}}{2}\varepsilon + \mathcal{O}(\varepsilon^{3/2}) = 1 - \sqrt{\mathrm{i}\varepsilon} + \mathcal{O}(\varepsilon),$$

as $\varepsilon \to 0_+$. To check the expansion for the reciprocal value we multiply the known expansion for k with the proposed expansion for k^{-1} and arrive at $kk^{-1} = 1 + \mathcal{O}(\varepsilon)$, as $\varepsilon \to 0_+$. $\square$

Corollary 1.25: Let $\lambda \in \{-2, 2\}$ and $\widetilde{\lambda} = \lambda + \mathrm{i}\varepsilon$. Then

$$|k|^2 = 1 - \sqrt{2\varepsilon} + \mathcal{O}(\varepsilon), \qquad |k| = 1 - \frac{\sqrt{\varepsilon}}{\sqrt{2}} + \mathcal{O}(\varepsilon),$$
$$|k - k^{-1}|^2 = 4\varepsilon + \mathcal{O}\left(\varepsilon^{3/2}\right), \qquad |k - k^{-1}| = 2\sqrt{\varepsilon} + \mathcal{O}(\varepsilon), \qquad \text{as } \varepsilon \to 0_+.$$

Proof. Again, let us provide the calculations for $\lambda = 2$ as the other case is analogous. Take the asymptotic expansion for k from the previous lemma and calculate

$$|k|^2 = \left|1 - \sqrt{\mathrm{i}\varepsilon} + \mathcal{O}(\varepsilon)\right|^2 = \left(1 - \frac{1}{\sqrt{2}}\sqrt{\varepsilon}\right)^2 + \left(\frac{1}{\sqrt{2}}\sqrt{\varepsilon}\right)^2 + \mathcal{O}\left(\varepsilon^2\right) = 1 - \sqrt{2\varepsilon} + \mathcal{O}(\varepsilon),$$
$$|k - k^{-1}|^2 = \left|1 - \sqrt{\mathrm{i}\varepsilon} - 1 - \sqrt{\mathrm{i}\varepsilon} + \mathcal{O}(\varepsilon)\right|^2 = \left|-\sqrt{2\varepsilon} - \mathrm{i}\sqrt{2\varepsilon} + \mathcal{O}(\varepsilon)\right|^2 = 4\varepsilon + \mathcal{O}\left(\varepsilon^{3/2}\right),$$

as $\varepsilon \to 0_+$. As was done previously, if we apply the asymptotic expansion $\sqrt{1+x} = 1 + \frac{x}{2} + \mathcal{O}(\varepsilon^2)$ as $x \to 0_+$, we arrive at the asymptotic expansions for the absolute values of the desired expressions. □

Chapter 2

Spectral properties of H_α

The discrete nature of the operator arises from the underlying Hilbert space. It is the space of square summable doubly infinite sequences

$$\ell^2(\mathbb{Z}) = \Big\{ \{x_n\}_{n=-\infty}^{\infty} \mid \sum_{n=-\infty}^{\infty} |x_n|^2 < \infty \Big\}.$$

In this space we define an orthogonal basis

$$\forall n \in \mathbb{Z}: \qquad e_n := \{\delta_{k,n}\}_{k=-\infty}^{\infty}.$$

We now formally introduce the operator under study. The *discrete Schrödinger operator with complex step potential* will be denoted by H_α. It is dependent on one free complex parameter α. Thus, we are considering a single-parameter class of operators $\{H_\alpha\}_{\alpha \in \mathbb{C}}$. The operator is defined by its action on $\forall x \equiv \{x_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{Z})$ by

$$(H_\alpha x)_n = \begin{cases} x_{n-1} + x_{n+1} & n < 0, \\ x_{n-1} + \alpha x_n + x_{n+1} & n \geq 0. \end{cases} \tag{2.0.1}$$

The matrix representation of H_α reads

$$H_\alpha = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & \alpha & 1 & & \\ & & & 1 & \alpha & 1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Since $H_\alpha^* = H_{\overline{\alpha}}$, the operator H_α is not self-adjoint if $\operatorname{Im}\alpha \neq 0$. Furthermore, with this restriction on the parameter α H_α is not normal as is evident by

$$\big(H_\alpha H_\alpha^*\big)_{-1,0} = \overline{\alpha}, \qquad \big(H_\alpha^* H_\alpha\big)_{-1,0} = \alpha.$$

The operator H_α is bounded. To show this we decompose

$$H_\alpha = S + \alpha D + S^*, \tag{2.0.2}$$

where

$$\forall n \in \mathbb{Z}: \qquad S e_n := e_{n-1}, \quad S^* e_n = e_{n+1},$$

$$\forall n \geq 0: \quad De_n := e_n, \qquad \forall n < 0: \quad De_n := 0.$$

The following clearly holds,

$$\|S\| = \|D\| = \|S^*\| = 1 \qquad \Longrightarrow \qquad \|H_\alpha\| \leq \|S\| + |\alpha|\|D\| + \|S^*\| = 2 + |\alpha|.$$

2.1 Spectral analysis of H_α

The spectrum of H_α is purely continuous and coincides with $[-2, 2] \cup [-2+\alpha, 2+\alpha]$. While we originally proved this result in [1], we include a detailed derivation here for the sake of completeness. It is well known that the spectrum of H_0 is purely continuous and coincides with the interval $[-2, 2]$. For this reason, we assume $\alpha \neq 0$. The proof is organized into four distinct steps.

1. Prove that the point spectrum is empty.
2. Prove that the residual spectrum is empty.
3. Identify a suitable subset of the resolvent set of H_α, denoted by X.
4. Demonstrate that the complement of X is a subset of the spectrum.

2.1.1 Point spectrum

A complex number λ is an eigenvalue of H_α if and only if there exists a non-zero vector $x \in \ell^2(\mathbb{Z})$ such that it satisfies the eigenvalue equation $H_\alpha x = \lambda x$. In its expanded form the equation reads

$$\begin{aligned} \lambda x_n &= x_{n-1} + x_{n+1} \qquad n < 0, \\ (\lambda - \alpha) x_n &= x_{n-1} + x_{n+1} \qquad n \geq 0. \end{aligned}$$

If we define the sequence

$$\kappa_n := \begin{cases} \lambda & n < 0, \\ \lambda - \alpha & n \geq 0, \end{cases}$$

we can rewrite the equation above in a more concise form

$$\forall n \in \mathbb{Z} \qquad : \qquad \kappa_n x_n = x_{n+1} + x_{n-1}. \tag{2.1.1}$$

This is a difference equation of second order with a two-dimensional set of solutions. To find the solution, we apply the two Joukowsky transforms

$$\lambda = \xi + \xi^{-1}, \qquad \lambda - \alpha = \eta + \eta^{-1}.$$

This transformation of λ enables us to express the solution in a simple form. Assume that $\xi \neq \pm 1$ and $\eta \neq \pm 1$, as these special cases will be addressed later. For $n \leq -2$, the solutions to (2.1.1) are given by $x_n = \xi^n$ and $x_n = \xi^{-n}$, while for $n \geq 1$, the solutions are $x_n = \eta^{-n}$ and $x_n = \eta^n$. Indeed,

$$\left.\begin{aligned} \lambda x_n &= (\xi + \xi^{-1})\xi^n = \xi^{n+1} + \xi^{n-1} = x_{n+1} + x_{n-1} \\ \lambda x_n &= (\xi + \xi^{-1})\xi^{-n} = \xi^{-n+1} + \xi^{-n-1} = x_{n+1} + x_{n-1} \end{aligned}\right. \qquad n \leq -2,$$

$$\left.\begin{aligned} (\lambda - \alpha) x_n &= (\eta + \eta^{-1})\eta^n = \eta^{n+1} + \eta^{n-1} = x_{n+1} + x_{n-1} \\ (\lambda - \alpha) x_n &= (\eta + \eta^{-1})\eta^{-n} = \eta^{-n+1} + \eta^{-n-1} = x_{n+1} + x_{n-1} \end{aligned}\right. \qquad n \geq 1.$$

The general solution to (2.1.1) takes the form

$$\begin{aligned} x_n &= A\xi^n + B\xi^{-n} \qquad && n < 0, \\ x_n &= C\eta^n + D\eta^{-n} \qquad && n \geq 0. \end{aligned} \tag{2.1.2}$$

Since the space of solutions has dimension two, two out of the four constants A, B, C, D are dependent on the other two. We can determine them by plugging (2.1.2) into the difference equation (2.1.1) at the indices $n \in \{-1, 0\}$

$$\begin{aligned} n = -1 \quad &: \quad A\xi^{-2} + B\xi^2 + C + D = (\xi + \xi^{-1})(A\xi^{-1} + B\xi) \\ & \quad\;\; A + B = C + D, \end{aligned} \tag{2.1.3}$$

$$\begin{aligned} n = 0 \quad &: \quad A\xi^{-1} + B\xi + C\eta + D\eta^{-1} = (\eta + \eta^{-1})(C + D) \\ & \quad\;\; A\xi^{-1} + B\xi = C\eta^{-1} + D\eta. \end{aligned} \tag{2.1.4}$$

Under the standing assumption that $\xi \neq \pm 1$ and $\eta \neq \pm 1$ we can derive

$$\begin{aligned} A &= C\,\frac{\eta^{-1} - \xi}{\xi^{-1} - \xi} + D\,\frac{\eta - \xi}{\xi^{-1} - \xi}, & B &= C\,\frac{\eta^{-1} - \xi^{-1}}{\xi - \xi^{-1}} + D\,\frac{\xi - \eta^{-1}}{\xi - \xi^{-1}}, \\ C &= A\,\frac{\xi^{-1} - \eta}{\eta^{-1} - \eta} + B\,\frac{\xi - \eta}{\eta^{-1} - \eta}, & D &= A\,\frac{\xi^{-1} - \eta^{-1}}{\eta - \eta^{-1}} + B\,\frac{\xi - \eta^{-1}}{\eta - \eta^{-1}}. \end{aligned}$$

Let us define two auxiliary sets of doubly infinite sequences

$$\begin{aligned} \ell^2(+\infty) &:= \Big\{ \{x_n\}_{n \in \mathbb{Z}} \subset \mathbb{C} \;\Big|\; \sum_{n=0}^{+\infty} |x_n|^2 < \infty \Big\}, \\ \ell^2(-\infty) &:= \Big\{ \{x_n\}_{n \in \mathbb{Z}} \subset \mathbb{C} \;\Big|\; \sum_{n=-\infty}^{0} |x_n|^2 < \infty \Big\}. \end{aligned}$$

By choosing the values of two independent constants we get two specific solutions to the eigenvalue equation, let us denote them y and z

$$y_n = \begin{vmatrix} A = 0 \\ B = 1 \end{vmatrix} = \begin{cases} \xi^{-n} & n < 0, \\ \frac{\xi - \eta}{\eta^{-1} - \eta}\eta^n + \frac{\xi - \eta^{-1}}{\eta - \eta^{-1}}\eta^{-n} & n \geq 0, \end{cases} \tag{2.1.5}$$

$$z_n = \begin{vmatrix} C = 1 \\ D = 0 \end{vmatrix} = \begin{cases} \frac{\eta^{-1} - \xi}{\xi^{-1} - \xi}\xi^n + \frac{\eta^{-1} - \xi^{-1}}{\xi - \xi^{-1}}\xi^{-n} & n < 0, \\ \eta^n & n \geq 0. \end{cases} \tag{2.1.6}$$

One can clearly see that

$$\begin{aligned} |\xi| < 1 &\Longleftrightarrow y \in \ell^2(-\infty), \\ |\eta| < 1 &\Longleftrightarrow z \in \ell^2(+\infty). \end{aligned}$$

Let us verify that they are linearly independent,

$$\begin{vmatrix} y_0 & z_0 \\ y_1 & z_1 \end{vmatrix} = \begin{vmatrix} \frac{\xi-\eta}{\eta^{-1}-\eta} + \frac{\xi-\eta^{-1}}{\eta-\eta^{-1}} & 1 \\ \frac{\xi-\eta}{\eta^{-1}-\eta}\eta + \frac{\xi-\eta^{-1}}{\eta-\eta^{-1}}\eta^{-1} & \eta \end{vmatrix}$$

$$= \frac{\xi-\eta}{\eta^{-1}-\eta}\eta + \frac{\xi-\eta^{-1}}{\eta-\eta^{-1}}\eta - \frac{\xi-\eta}{\eta^{-1}-\eta}\eta - \frac{\xi-\eta^{-1}}{\eta-\eta^{-1}}\eta^{-1} = \xi - \eta^{-1} \neq 0.$$

Vectors y and z are linearly independent if and only if $\eta^{-1} \neq \xi$ which holds as long as $\alpha \neq 0$. One can easily verify this from (1.3.2). A general solution to the eigenvalue equation can be written as a linear combination of y and z. The last step is to show that no such linear combination is an element of $\ell^2(\mathbb{Z})$.

Choose any $a, b \in \mathbb{C}$ and set $x = ay + bz$. Then

$$x_n = \begin{cases} a\xi^{-n} + b\left(\frac{\eta^{-1}-\xi}{\xi^{-1}-\xi}\xi^n + \frac{\eta^{-1}-\xi^{-1}}{\xi-\xi^{-1}}\xi^{-n}\right) & n < 0, \\ a\left(\frac{\xi-\eta}{\eta^{-1}-\eta}\eta^n + \frac{\xi-\eta^{-1}}{\eta-\eta^{-1}}\eta^{-n}\right) + b\eta^n & n \geq 0, \end{cases}$$

Suppose $x \in \ell^2(\mathbb{Z})$ and from the definition of the Joukowsky transform we assert that $|\xi| \leq 1, |\eta| \leq 1$. Then

$$x \in \ell^2(\mathbb{Z}) \quad \Longrightarrow \quad x \in \ell^2(-\infty) \quad \& \quad x \in \ell^2(+\infty).$$

For $|\xi| = 1$ or $|\eta| = 1$ and $a, b \neq 0$, the necessary condition for series convergence is not met for

$$\sum_{n=0}^{\infty} |x_n|^2, \qquad \text{or} \qquad \sum_{n=-\infty}^{-1} |x_n|^2,$$

respectively. This leads to a contradiction with $x \in \ell^2(\mathbb{Z})$. Assuming $|\xi| < 1$ and $|\eta| < 1$, the term ξ^n diverges for negative indices. To satisfy the condition $x \in \ell^2(-\infty)$, this term must vanish. For this to hold, one of the following must be true:

$$b = 0, \qquad \text{or} \qquad \frac{\eta^{-1}-\xi}{\xi^{-1}-\xi} = 0.$$

The latter condition necessitates that $\eta^{-1} = \xi$, which is not possible under the assumption $\alpha \neq 0$. Thus, we conclude that $b = 0$.

In the sum spanning the positive indices, there is a divergent term η^{-n}. For this term to vanish, one of the following conditions must be satisfied:

$$a = 0, \qquad \text{or} \qquad \frac{\xi-\eta^{-1}}{\eta-\eta^{-1}} = 0.$$

Once again, we disregard the latter condition.

Consequently, we arrive at:

$$x \in \ell^2(\mathbb{Z}) \quad \Longleftrightarrow \quad a = 0 \;\; \& \;\; b = 0 \quad \Longleftrightarrow \quad \forall n \in \mathbb{Z} : x_n = 0.$$

Let us finally discuss the special cases $\xi = \pm 1$, $\eta = \pm 1$ and their combinations. First we choose $\xi = -1$ and $\eta \neq \pm 1$. Then for indices $n \leq -2$ the solutions to (2.1.1) read $x_n = (-1)^n$

and $x_n = n(-1)^n$ as is evident by

$$\begin{aligned} \lambda x_n &= -2(-1)^n = (-1)^{n+1} + (-1)^{n-1} = x_{n+1} + x_{n-1} \\ \lambda x_n &= -2n(-1)^{-n} = -(2n-1+1)(-1)^n \qquad\qquad n \le -2. \\ &= (n+1)(-1)^{n+1} + (n-1)(-1)^{n-1} = x_{n+1} + x_{n-1} \end{aligned}$$

If we consider $n \ge 1$ the solutions read $x_n = \eta^{-n}$ and $x_n = \eta^n$. The general solution is in the form

$$\begin{aligned} x_n &= A(-1)^n + Bn(-1)^{-n} \qquad n < 0, \\ x_n &= C\eta^n + D\eta^{-n} \qquad n \ge 0. \end{aligned} \tag{2.1.7}$$

To assert $x \in \ell^2(\mathbb{Z})$ we assume $A, B = 0$. If $|\eta| = 1$ and $C, D \neq 0$, then the necessary condition of series convergence is not met for positive indices. If $|\eta| < 1$ and x is a non-zero element of $\ell^2(\mathbb{Z})$, then $A, B, D = 0$ and $C \neq 0$. This results in a contradiction with a condition posed upon A, B, C, D, which we get by plugging (2.1.7) in (2.1.1) for $n = -1$

$$\begin{aligned} A - 2B + C\eta^0 + D\eta^0 &= -2(A - B) \\ A - 4B + C + D &= 0. \end{aligned}$$

Now we choose $\xi = -1$ and $\eta = 1$. Then the solution to the eigenvalue equation reads

$$\begin{aligned} x_n &= A(-1)^n + Bn(-1)^{-n} \qquad n < 0, \\ x_n &= C + Dn \qquad n \ge 0. \end{aligned}$$

If $x \in \ell^2(\mathbb{Z})$, the form of x necessitates that it is a trivial solution. The other special cases and their combination would be shown analogously.

Proposition 2.1: Let $\alpha \in \mathbb{C}$, then

$$\sigma_{\mathrm{p}}(H_\alpha) = \emptyset.$$

Proof. If $\alpha \neq 0$, the discussion above showed that there is no non-trivial solution to the eigenvalue equation that would lie in $\ell^2(\mathbb{Z})$. In the case of $\alpha = 0$, the lack of eigenvalues is well known. □

2.1.2 Residual spectrum

We first establish a general result that connects the absence of a point spectrum to the absence of a residual spectrum under certain assumptions. This foundational theorem, presented below, will subsequently enable us to demonstrate that the residual spectrum of H_α is indeed empty.

Theorem 2.2: Let $A \in \mathscr{B}(\ell^2(\mathbb{Z}))$ be self-adjoint and $B \in \mathscr{B}(\ell^2(\mathbb{Z}))$ a diagonal operator. Then

$$\sigma_{\mathrm{p}}(A + B) = \emptyset \qquad \Longrightarrow \qquad \sigma_{\mathrm{r}}(A + B) = \emptyset.$$

Proof. A general characterization of the residual spectrum states that

$$\sigma_{\mathrm{r}}(H) = \left\{\lambda \notin \sigma_{\mathrm{p}}(H) \mid \overline{\lambda} \in \sigma_{\mathrm{p}}(H^*)\right\},$$

where H is some general operator. A proof of this general characterization can be found in Chapter 5 of [12]. If substitute $H = A + B$ and apply the assumptions, the expression reduces to

$$\sigma_{\mathrm{r}}(A + B) = \left\{\lambda \in \mathbb{C} \mid \overline{\lambda} \in \sigma_{\mathrm{p}}(A + \overline{B})\right\}.$$

Now, let $\overline{\lambda} \in \sigma_{\mathrm{p}}(A+\overline{B})$. By definition, there exists a non-zero $\psi \in \ell^2(\mathbb{Z})$ such that $(A+\overline{B})\psi = \overline{\lambda}\psi$. Applying element-wise complex conjugation on this equation yields

$$(A + B)\overline{\psi} = \lambda\overline{\psi}.$$

This implies that $\lambda \in \sigma_{\mathrm{p}}(A+B)$. However, this contradicts the assumption that $\sigma_{\mathrm{p}}(A+B) = \emptyset$. Therefore, $\sigma_{\mathrm{p}}(A+\overline{B}) = \emptyset$. By the characterization of the residual spectrum, this directly implies that $\sigma_{\mathrm{r}}(A+B) = \emptyset$ as well. □

Proposition 2.3: Let $\alpha \in \mathbb{C}$, then

$$\sigma_{\mathrm{r}}(H_\alpha) = \emptyset.$$

Proof. In Theorem 2.2 we set $A = H_0$ and $B = \alpha D$, recall the decomposition (2.0.2). The operator H_0 is self-adjoint and its point spectrum is empty as was shown in Proposition 2.1. The operator αD is diagonal. Thus, by Theorem 2.2 we get that the residual spectrum of H_α is empty. □

2.1.3 Continuous spectrum

This section is more extensive than the previous one; hence, it is imperative to first discuss our approach. In those sections we proved that the point and residual parts of the spectrum are empty, i.e. the spectrum of H_α is purely continuous. The remaining question is what the spectrum is. First, we will define a suitable set, denoted by X. Then we will show that X is a subset of the resolvent set using Green's kernel approach. Lastly, we will show the opposite inclusion utilizing Weyl's criterion.

Definition of the set X

Let us recall the solution of the eigenvalue difference equation y (2.1.5) and z (2.1.6), which depend on ξ and η. If both ξ and η are within absolute values less than 1, we can uniquely assign them to λ according to Proposition 1.17. Furthermore, we know that

$$|\xi| < 1 \iff y \in \ell^2(-\infty),$$
$$|\eta| < 1 \iff z \in \ell^2(+\infty).$$

If both conditions $|\xi| < 1$ and $|\eta| < 1$ are satisfied, we will have two vectors suitable for use in Green's kernel theorem. Ensuring these two conditions are met defines the set X, a candidate for the resolvent set of H_α. We have

$$|\xi| < 1 \implies \lambda = \xi + \xi^{-1} \in \mathbb{C} \setminus [-2, 2],$$
$$|\eta| < 1 \implies \lambda = \eta + \eta^{-1} + \alpha \in \mathbb{C} \setminus [-2 + \alpha, 2 + \alpha].$$

We define

$$X := \mathbb{C} \setminus \big([-2, 2] \cup [-2 + \alpha, 2 + \alpha]\big). \tag{2.1.8}$$

Green's kernel

In this section, we draw upon insights from [11]. This reference discusses the spectral properties of Jacobi operators, i.e., operators with tridiagonal matrix representations. Here, H_α

represents a Jacobi operator. The Green's kernel theorem as stated in [11] assumes the self-adjointness of the operator. However, it is evident that for a choice of parameter $\alpha \notin \mathbb{R}$, the operator H_α does not satisfy this assumption. We will proceed by adapting the construction of the Green's kernel and show that it indeed is the resolvent operator of H_α in our specific case. This requires demonstrating that it is a bounded operator and is the inverse of $(H_\alpha - \lambda)$. In doing so, we will establish that $X \subset \rho(H_\alpha)$. This construction depends on the vectors y and z, see (2.1.5). and (2.1.6), respectively. The Green's kernel $G(\lambda)$ takes values for indices $m, n \in \mathbb{Z}$ as follows

$$G_{m,n}(\lambda) = \frac{1}{w_k(y,z)} \begin{cases} z_n y_m & m \le n, \\ z_m y_n & m > n, \end{cases}$$

where $w \equiv w_k(y,z) = z_{k+1}y_k - z_k y_{k+1},\ k \in \mathbb{Z}$. Here, w is known as the Wronskian, and is non-zero if and only if y and z are linearly independent. This condition has already been verified. Notably, we denote G as dependent on λ; this is indeed the case, but this dependency is embedded within the vectors y and z, which are themselves defined in terms of ξ and η, uniquely determined by $\lambda \in X$. The Wronskian also depends on λ.

An important property of the Wronskian $w_k(y,z)$ is that it is independent of k. To see this, choose arbitrary $k \in \mathbb{Z}$ and recall the eigenvalue equation (2.1.1), from which we derive the relation $x_{n+1} = \kappa_n x_n - x_{n-1}$, then

$$\begin{aligned} w_k(y,z) = z_{k+1}y_k - z_k y_{k+1} &= (\kappa_k z_k - z_{k-1})y_k - z_k(\kappa_k y_k - y_{k-1}) \\ &= \kappa_k z_k y_k - z_{k-1}y_k - z_k\kappa_k y_k + z_k y_{k-1} = z_k y_{k-1} - z_{k-1}y_k = w_{k-1}(y,z). \end{aligned}$$

This allows us to omit the subscript on the Wronskian. For clarity in subsequent discussions, let us define

$$\widetilde{G} := wG. \tag{2.1.9}$$

We provide two equivalent matrix representations of $G(\lambda)$. One of these formulations may be less cumbersome to work with in certain scenarios than the other and vice versa. The matrix representations of $G(\lambda)$ read

$$G_{m,n}(\lambda) = \frac{1}{w} \begin{cases} \frac{\xi-\eta}{\eta^{-1}-\eta}\eta^{m+n} + \frac{\xi-\eta^{-1}}{\eta-\eta^{-1}}\eta^{|m-n|} & m,n \ge 0, \\ \eta^m \xi^{-n} & m \ge 0, n < 0, \\ \frac{\eta^{-1}-\xi}{\xi^{-1}-\xi}\xi^{|m-n|} + \frac{\eta^{-1}-\xi^{-1}}{\xi-\xi^{-1}}\xi^{-m-n} & m,n < 0, \\ \eta^n \xi^{-m} & m < 0, n \ge 0, \end{cases} \tag{2.1.10}$$

$$G_{m,n}(\lambda) = \begin{cases} \frac{\eta^{m+n}-\eta^{|m-n|}}{\eta^{-1}-\eta} + \frac{\eta^{m+n}}{w} & m,n \ge 0, \\ \frac{1}{w}\eta^m \xi^{-n} & m \ge 0, n < 0, \\ \frac{\xi^{|m-n|}-\xi^{-m-n}}{\xi-\xi^{-1}} + \frac{\xi^{-m-n}}{w} & m,n < 0, \\ \frac{1}{w}\eta^n \xi^{-m} & m < 0, n \ge 0. \end{cases} \tag{2.1.11}$$

Boundedness of $G(\lambda)$

Schur test, Theorem 1.14, will be used to obtain the upper bound. We set $p_j = 1$ for all $j \in \mathbb{Z}$. This way we get a readily available, albeit possibly rough, estimate. Let $\lambda \in X$ be arbitrary. In this setting, the Schur test implies

$$\|G(\lambda)\| \leq \sup_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \big|G_{m,n}(\lambda)\big|.$$

Finding the actual supremum proves to be quite difficult; therefore, we will be taking an estimate of $\sum_{n\in\mathbb{Z}} \big|G_{m,n}(\lambda)\big|$ from above while ridding it of the dependence on m. Recall the definition of $C(\alpha)$ from (1.3.5). Due to the form that $G(\lambda)$ takes, we need to investigate two branches:

- Let $m \geq 0$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} |G_{m,n}(\lambda)| &= \frac{1}{|w|} \sum_{n=0}^{\infty} \left| \frac{\xi - \eta}{\eta^{-1} - \eta} \eta^{m+n} + \frac{\xi - \eta^{-1}}{\eta - \eta^{-1}} \eta^{|m-n|} \right| + \frac{|\eta|^m}{|w|} \sum_{n=1}^{\infty} |\xi|^n \\
&\leq \frac{1}{|\eta^{-1} - \eta|} \left(\frac{|\xi - \eta|}{|w|} |\eta|^m \sum_{n=0}^{\infty} |\eta|^n + \sum_{n=-m}^{\infty} |\eta|^{|n|} \right) + \frac{|\eta|^m}{|w|} \sum_{n=1}^{\infty} |\xi|^n \\
&= \frac{1}{|\eta^{-1} - \eta|} \frac{1}{1 - |\eta|} \left(\frac{|\xi - \eta|}{|w|} |\eta|^m + 1 + |\eta| - |\eta|^{m+1} \right) + \frac{|\eta|^m |\xi|}{|w|(1 - |\xi|)} \\
&\leq \frac{1}{|\eta^{-1} - \eta|} \frac{1}{1 - |\eta|} \left(\frac{|\xi - \eta|}{|w|} + 1 + |\eta| \right) + \frac{|\xi|}{|w|(1 - |\xi|)} =: \widetilde{U}_1(\lambda) \\
&\leq \frac{1}{|\eta^{-1} - \eta|} \frac{1}{1 - |\eta|} \left(2C(\alpha) + 2 \right) + \frac{C(\alpha)}{1 - |\xi|} =: U_1(\lambda).
\end{aligned}$$

- Let $m < 0$. Then

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |G_{m,n}(\lambda)| &= \frac{1}{|w|} \sum_{n=1}^{\infty} \left| \frac{\eta^{-1} - \xi}{\xi^{-1} - \xi} \xi^{|m-n|} + \frac{\eta^{-1} - \xi^{-1}}{\xi - \xi^{-1}} \xi^{m+n} \right| + \frac{|\xi|^m}{|w|} \sum_{n=0}^{\infty} |\eta|^n \\
&\leq \frac{1}{|\xi^{-1} - \xi|} \left(\sum_{n=-m+1}^{\infty} |\xi|^{|n|} + \frac{|\xi^{-1} - \eta^{-1}|}{|w|} |\xi|^m \sum_{n=1}^{\infty} |\xi|^n \right) + \frac{|\xi|^m}{|w|} \sum_{n=0}^{\infty} |\eta|^n \\
&= \frac{1}{|\xi^{-1} - \xi|} \frac{1}{1 - |\xi|} \left(1 + |\xi| - |\xi|^m + \frac{|\xi^{-1} - \eta^{-1}|}{|w|} |\xi|^m \right) + \frac{|\eta|^m}{|w|(1 - |\eta|)} \\
&\leq \frac{1}{|\xi^{-1} - \xi|} \frac{1}{1 - |\xi|} \left(1 + |\xi| + \frac{|\xi^{-1} - \eta^{-1}|}{|w|} |\xi| \right) + \frac{|\eta|}{|w|(1 - |\eta|)} =: \widetilde{U}_2(\lambda) \\
&\leq \frac{1}{|\xi^{-1} - \xi|} \frac{1}{1 - |\xi|} \left((|\xi|^{-1} + |\eta|^{-1}) C(\alpha) + 2 \right) + \frac{C(\alpha)}{1 - |\eta|} =: U_2(\lambda).
\end{aligned}$$

In the analysis of both branches, the second-to-last estimate

$$\widetilde{U}(\lambda) := \max\{\widetilde{U}_1(\lambda), \widetilde{U}_2(\lambda)\} \tag{2.1.12}$$

is somewhat less crude and will be utilized in the study of the pseudospectra. To demonstrate the boundedness of the resolvent operator, we employ the estimate

$$U(\lambda) := \max\big\{U_1(\lambda), U_2(\lambda)\big\}.$$

While this estimate is rougher, it provides a clearer argument for boundedness.

Proposition 2.4: Let $\alpha \neq 0$. Then

$$\forall \lambda \in X \;:\quad \|(H_\alpha - \lambda)^{-1}\| \leq \max\left\{U_1(\lambda), U_2(\lambda)\right\} =: U(\lambda)$$

Proposition 2.5: Let $\alpha \in \mathbb{C}$. Then

$$\forall \lambda \in X : G(\lambda) \in \mathscr{B}(\ell^2(\mathbb{Z})).$$

Proof. For $\alpha = 0$, we have $X = \mathbb{C} \setminus [-2, 2]$, and the boundedness of the resolvent operator is well known. On the other hand, if $\alpha \neq 0$ and we utilize the upper bound from Proposition 2.5, it follows that for $\lambda \in X$, the upper bound $U(\lambda)$ is a finite number since $|\xi|, |\eta| \in (0, 1)$. □

$G(\lambda)$ is an inverse of $(H_\alpha - \lambda)$

As one may expect, here we need to show $(H_\alpha - \lambda)G = G(H_\alpha - \lambda) = I$. We will prove this by showing $\forall x \in \ell^2(\mathbb{Z}) : (H_\alpha - \lambda)Gx = x$. Recall that $wG = \widetilde{G}$ where w is the aforementioned Wronskian. If we use the notation from (2.1.1), we have

$$\begin{aligned}
w\big((H_\alpha - \lambda)Gx\big)_m &= \sum_{n=-\infty}^{\infty} \widetilde{G}_{m,n-1}x_n - \kappa_n \widetilde{G}_{m,n}x_n + \widetilde{G}_{m,n+1}x_n \\
&= \sum_{n=-\infty}^{m-1} z_m \underbrace{(y_{n-1} - \kappa_n y_n + y_{n+1})}_{=0} x_n + \sum_{n=m+1}^{\infty} y_m \underbrace{(z_{n-1} - \kappa_n z_n + z_{n+1})}_{=0} x_n \\
&\quad + (z_m y_{m-1} - \kappa_m z_m y_m + z_{m+1} y_m)x_m \overset{\star}{=} (z_m y_{m-1} - y_{m-1} z_m - z_m y_{m+1} + z_{m+1} y_m)x_m \\
&= (z_{m+1} y_m - z_m y_{m+1})x_m = w_m(y, z)x_m = w x_m.
\end{aligned}$$

Then we divide the equation by the non-zero w and arrive at the desired equation. When calculating, we used the fact that y and z are solutions of (2.1.1). If we were to interchange $(H_\alpha - \lambda)$ and G, we would arrive at the same result.

Proposition 2.6: Let $\alpha \in \mathbb{C}$. Then

$$\forall \lambda \in X \quad : \quad (H_\alpha - \lambda)G = G(H_\alpha - \lambda) = I.$$

Proposition 2.7: Let $\alpha \in \mathbb{C}$. Then

$$\sigma(H_\alpha) \subset [-2, 2] \,\cup\, [-2 + \alpha, 2 + \alpha]$$

Proof. By Proposition 2.5 and by Proposition 2.6 we have

$$\lambda \in X \quad \Longrightarrow \quad \lambda \in \rho(H_\alpha),$$

in other words

$$X \subset \rho(H_\alpha).$$

Transitioning to the complement we get the asserted inclusion. □

The inclusion $\rho(H_\alpha) \subset X$

In this section, we use one implication of Weyl's criterion as stated in Theorem 1.15. This particular formulation was chosen deliberately, as it addresses the needs of our analysis. The operator H_α is generally non-normal, and the specific implication that holds under these conditions suffices for our purposes.

For a given $\lambda \in \mathbb{C}$, suppose there exists a sequence $\{\psi^{(n)}\}_{n=1}^{\infty} \subset \ell^2(\mathbb{Z}) \setminus \{0\}$ such that

$$\lim_{n\to\infty} \frac{\|(H_\alpha - \lambda)\psi^{(n)}\|}{\|\psi^{(n)}\|} = 0.$$

Then $\lambda \in \sigma(H_\alpha)$. Since it is already established that $X \subset \rho(H_\alpha)$, we need not test this condition for all $\lambda \in \mathbb{C}$. Instead, we restrict λ to X^{c} and demonstrate that $\lambda \in \sigma(H_\alpha)$. This proves the inclusion $\rho(H_\alpha) \subset X$. Taking $\lambda \in X^{\mathrm{c}}$ we need to consider two cases $|\xi| = 1$ and $|\eta| = 1$, see (2.1.8).

For $\lambda \in X^{\mathrm{c}}$ such that $|\eta| = 1$ we define $\forall n \in \mathbb{N}$

$$(\zeta^{(n)})_k := \begin{cases} \eta^k & k \in \{0, 1, \dots, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, $\forall n \in \mathbb{N} \,:\, \zeta^{(n)} \in \ell^2(\mathbb{Z})$ since $\left\|\zeta^{(n)}\right\| = \sqrt{n}$. A simple calculation yields

$$\left((H_\alpha - \lambda)\zeta^{(n)}\right)_m = \begin{cases} 1 & m = -1, \\ -\eta^{-1} & m = 0, \\ -\eta^{n+1} & m = n, \\ \eta^n & m = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

This allows us to take the norm, $\left\|(H_\alpha - \lambda)\zeta^{(n)}\right\| = 2$, and evaluate the limit

$$\frac{\|(H_\alpha - \lambda)\zeta^{(n)}\|}{\|\zeta^{(n)}\|} = \frac{2}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0.$$

The set $\{\eta \mid |\eta| = 1\}$ corresponds with $[-2+\alpha, 2+\alpha]$ in the λ-plane. By applying Weyl's criterion we get $[-2+\alpha, 2+\alpha] \subset \sigma(H_\alpha)$

For $\lambda \in X^{\mathrm{c}}$ such that $|\xi| = 1$ we define $\forall n \in \mathbb{N}$

$$(\chi^{(n)})_k := \begin{cases} \xi^{-k} & k \in \{-n, \dots, 1, 0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Similar reasoning as above yields that $[-2, 2] \subset \sigma(H_\alpha)$.

Proposition 2.8: Let $\alpha \in \mathbb{C}$. Then

$$\sigma(H_\alpha) \subset [-2, 2] \,\cup\, [-2+\alpha, 2+\alpha].$$

Theorem 2.9: Let $\alpha \in \mathbb{C}$ and consider the operator H_α as given by (2.0.1). Then the spectrum of H_α is purely continuous and coincides with two line segments in the complex plane. More precisely

$$\sigma(H_\alpha) = \sigma_{\mathrm{c}}(H_\alpha) = [-2, 2] \cup [-2+\alpha, 2+\alpha] = [-2, 2] + \alpha\{0, 1\}.$$

Proof. In Proposition 2.1 we showed that the point spectrum is empty and similarly in Proposition 2.3 we showed that the residual spectrum is empty. Hence, the spectrum of H_α is purely continuous. In Propositions 2.7 and 2.8 were shown the two inclusions implying the set equality of spectrum in the theorem. □

The spectrum of H_α is depicted in Figure 2.1.

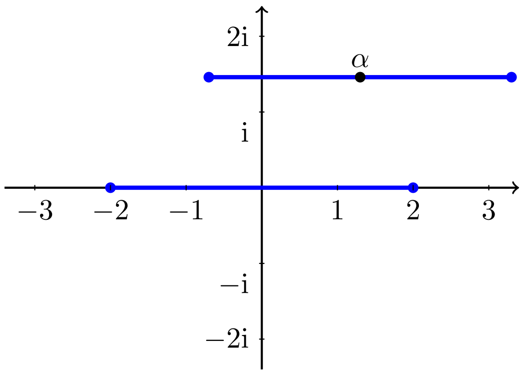


Figure 2.1: Spectrum of the operator H_α.

To conclude this section, we derive the numerical range of the operator H_α. Numerical range of an operator A is a subset of the complex plane $\mathbb{C}$ given by

$$\operatorname{Num}(A) = \left\{ \langle x, Ax \rangle \;\middle|\; x \in \ell^2(\mathbb{Z}),\ \|x\| = 1 \right\}.$$

The numerical range is always a convex subset of $\mathbb{C}$ and the spectrum of A is contained in its closure.

Theorem 2.10: Let $\alpha \in \mathbb{C}$. Then the numerical range of H_α is given by

$$\operatorname{Num}(H_\alpha) = (-2, 2) + \alpha[0, 1].$$

Proof. Let us define a two-parameter class of vectors from $\ell^2(\mathbb{Z})$ by

$$\widetilde{x}_n(q,t) := \begin{cases} \sqrt{t}\, q^n & n > 0, \\ 0 & n = 0, \\ \sqrt{1-t}\, q^{-n} & n < 0, \end{cases}$$

where $t \in [0, 1]$ and $q \in (-1, 1)$. Clearly, $\widetilde{x}(q, t)$ is an $\ell^2(\mathbb{Z})$ sequence. Its squared norm is given by

$$\|\widetilde{x}(q,t)\|^2 = \frac{q^2}{1-q^2}.$$

To construct a unit vector we define $x(q, t)$ as $\widetilde{x}(q, t)$ normalized by its norm. Recall the decomposition of H_α (2.0.2) and plug it into $\langle x, H_\alpha x\rangle$. A simple calculation yields

$$\langle x, H_\alpha x \rangle = 2\mathrm{Re}\langle x, Sx \rangle + \alpha \langle x, Dx \rangle = 2q + \alpha t. \tag{2.1.13}$$

Thus, it follows that

$$\left\{ 2q + \alpha t \mid q \in (-1, 1),\ t \in [0, 1] \right\} \subset \operatorname{Num}(H_\alpha). \tag{2.1.14}$$

In the remainder of the proof, we show that no other complex number may be in $\mathrm{Num}(H_\alpha)$.

Now, let $x \in \ell^2(\mathbb{Z})$ be an arbitrary unit vector. The inner product, $\langle x, Dx\rangle = \sum_{n=0}^{\infty} |x_n|^2$, represents the squared norm of $\ell^2(\mathbb{N}_0)$ part of x. Since x is a unit vector in $\ell^2(\mathbb{Z})$ and norms are non-negative, it follows that $\langle x, Dx\rangle \in [0, 1]$. Consequently, we have

$$\mathrm{Im}\langle x, H_\alpha x\rangle = \langle x, Dx\rangle \mathrm{Im}\alpha \in \mathrm{Im}\alpha[0, 1].$$

Thus, it follows that

$$\left\{2q + \alpha t \mid q \in \mathbb{R},\ t \notin [0, 1]\right\} \cap \mathrm{Num}(H_\alpha) = \emptyset. \tag{2.1.15}$$

Suppose there exists $x \in \ell^2(\mathbb{Z})$, $\|x\| = 1$, such that $\mathrm{Re}\langle x, Sx\rangle = \pm 1$. The Cauchy–Schwarz inequality yields

$$1 = |\mathrm{Re}\langle x, Sx\rangle| \leq |\langle x, Sx\rangle| \leq \|x\|\,\|Sx\| = \|x\|^2 = 1. \tag{2.1.16}$$

Since equality holds throughout, it follows that x and Sx are linearly dependent, i.e., $Sx = \beta x$ for some $\beta \in \mathbb{C}$. Furthermore, the equality $\mathrm{Re}\langle x, Sx\rangle = \pm 1$ implies $\langle x, Sx\rangle = \pm 1$. From this, we determine $\beta = \pm 1$, leading to $Sx = \pm x$. The operator S shifts the indices of x by 1. Therefore, all elements of x must be equal in absolute value. However, such a sequence cannot lie in $\ell^2(\mathbb{Z})$, as it would violate the square-summability condition unless it is a trivial sequence, in which case $\|x\| = 1$ does not hold. Thus, $\forall x \in \ell^2(\mathbb{Z}),\ \|x\| = 1\ :\ \mathrm{Re}\langle x, Sx\rangle \neq \pm 1$.

Moreover, from (2.1.16) immediately follows that $\forall x \in \ell^2(\mathbb{Z}),\ \|x\| = 1\ :\ \mathrm{Re}\langle x, Sx\rangle \in [-1, 1]$.Thus, from two arguments it follows that

$$\left\{2q + \alpha t \mid q \notin (-1, 1),\ t \in [0, 1]\right\} \cap \mathrm{Num}(H_\alpha) = \emptyset. \tag{2.1.17}$$

The union of the sets on the left-hand sides of (2.1.14), (2.1.15), and (2.1.17) covers the entire complex plane. Therefore, the subset relation in (2.1.14) is, in fact, an equality:

$$\mathrm{Num}(H_\alpha) = \left\{2q + \alpha t \mid q \in (-1, 1),\ t \in [0, 1]\right\} = (-2, 2) + \alpha[0, 1].$$

□

The numerical range of H_α is depicted in Figure 2.2.

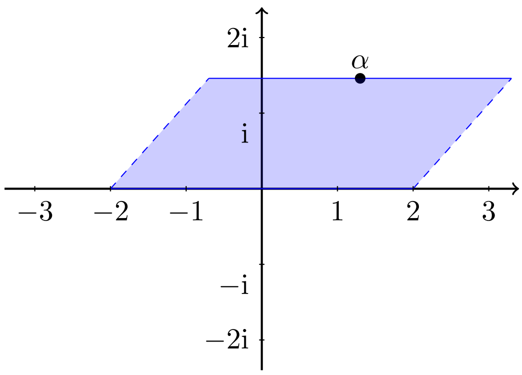


Figure 2.2: Numerical range of the operator H_α.

2.2 Pseudospectrum

In this chapter, we will almost solely rely on the book *Spectra and pseudospectra* by Lloyd Nicholas Trefethen and Mark Embree, see [17]. We will define what pseudospectra are and state their basic properties before we commence pseudospectral analysis of the studied operator H_α. Since these properties are not crucial for this paper and serve only to familiarize us with this new notion, we will not provide proofs; one may find them in [17].

Definition 2.11: Let $A \in \mathscr{B}(\mathscr{H})$ and $\varepsilon > 0$ be arbitrary. The *ε-pseudospectrum* of operator A is defined as the set

$$\sigma_\varepsilon(A) := \sigma(A) \cup \big\{\lambda \in \mathbb{C} \setminus \sigma(A) \;\big|\; \|(A-\lambda)^{-1}\| \geq \varepsilon^{-1}\big\}.$$

Theorem 2.12: Given $A \in \mathscr{B}(\mathscr{H})$, the pseudospectra $\big\{\sigma_\varepsilon(A)\big\}_{\varepsilon>0}$ have the following properties.

- Each $\sigma_\varepsilon(A)$ is a nonempty subset of $\mathbb{C}$.
- Any bounded connected component of $\sigma_\varepsilon(A)$ has a nonempty intersection with $\sigma(A)$.
- The pseudospectra are strictly nested supersets of the spectrum.

Describing the pseudospectra of self-adjoint operators is a relatively trivial endeavor because we can describe the norm of the resolvent in terms of the distance between its spectral parameter and the spectrum of the operator.

Theorem 2.13: Let $A \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator and $\lambda \in \varrho(A)$, then

$$\|(A-\lambda)^{-1}\| = \frac{1}{\operatorname{dist}(\sigma(A), \lambda)}.$$

Therefore, given a bounded self-adjoint operator with a known spectrum, one can explicitly describe the ε-pseudospectra as the ε-neighborhood of $\sigma(A)$.

Corollary 2.14: Let $A \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator, then

$$\varepsilon > 0: \qquad \sigma_\varepsilon(A) = \sigma(A) \cup \big\{\lambda \in \mathbb{C} \setminus \sigma(A) \;\big|\; \operatorname{dist}(\sigma(A), \lambda) \leq \varepsilon\big\} = \sigma(A) + D(\varepsilon),$$

where $D(\varepsilon)$ denotes a disc in the complex plane centered at the origin with radius ε.

Let us now apply these results to operator H_α. The following proposition immediately follows from the corollary above.

Proposition 2.15: Let α be a real number. Then

$$\varepsilon > 0: \qquad \sigma_\varepsilon(H_\alpha) = [-2, 2] \cup [-2+\alpha, 2+\alpha] \;+\; D(\varepsilon).$$

For the choice of $\alpha = 0$, the pseudospectrum of H_α is depicted in figure 2.3.

Setting aside the trivial case for a real parameter α, we will discuss the general case of α being any complex number. Describing the pseudospectra of non-self-adjoint operators is no more a trivial endeavor. Still, we can get an explicit expression for the pseudospectrum intersected with a certain set. This is justified by the following propositions, which is a standard result from functional analysis.

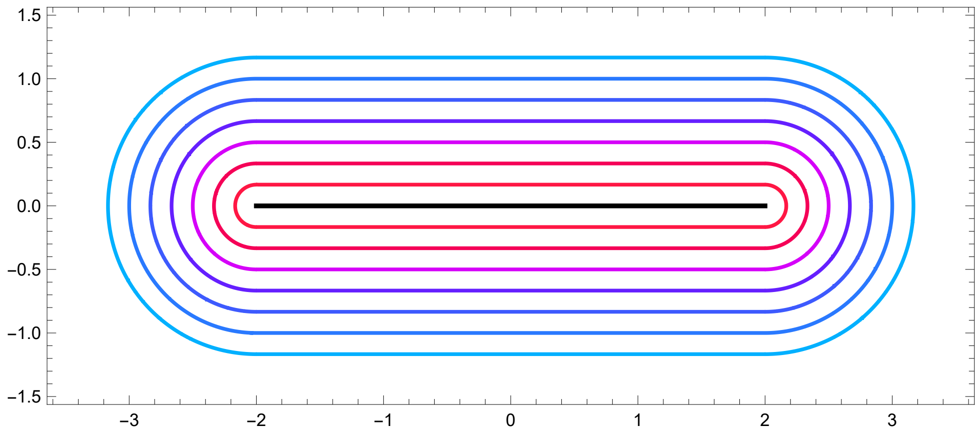


Figure 2.3: Pseudospectrum of H_0.

Proposition 2.16: Let $A \in \mathscr{B}(\mathscr{H})$ and $\lambda \in \rho(A)$. Then

$$\frac{1}{\operatorname{dist}\big(\lambda, \sigma(A)\big)} \leq \big\|(A-\lambda)^{-1}\big\|.$$

A lower bound can also be obtained simply from the definition of operator norm. Let $A \in \mathscr{B}(\mathscr{H})$, then

$$\forall y \in \mathscr{H},\ \|y\| = 1 : \qquad \|A\| = \sup_{\|x\|=1} \|Ax\| \geq \|Ay\|. \tag{2.2.1}$$

We shall refer to such a vector y as a *test vector*. However, in our case, Proposition 2.16 gives a more suitable lower bound.

Proposition 2.17 ([15, Proposition 2.8])**:** Let A be a closed operator on $\mathscr{H}$. Let U be a connected open subset of $\mathbb{C} \setminus \overline{\operatorname{Num}(A)}$. If there exists a number $\lambda_0 \in U$ which is contained in $\rho(A)$, then $U \subset \rho(A)$. Moreover,

$$\forall \lambda \in U \ : \ \big\|(A-\lambda)^{-1}\big\| \leq \frac{1}{\operatorname{dist}\big(\lambda, \operatorname{Num}(A)\big)}.$$

Lemma 2.18: Let $K \subset \mathbb{C}$ be bounded closed convex set, then its complement, $\mathbb{C} \setminus K$, is connected.

Proof. Let us establish notation for lines, rays, and line segments:

$$\langle a, b\rangle := \big\{(1-t)a + tb \ \big|\ t \in \mathbb{R}\big\} \qquad (a, b\rangle := \big\{(1-t)a + tb \ \big|\ t \in (0, +\infty)\big\}$$
$$[a, b] := \big\{(1-t)a + tb \ \big|\ t \in [0, 1]\big\} \qquad [a, b) := \big\{(1-t)a + tb \ \big|\ t \in [0, 1)\big\}$$

We will show that $\mathbb{C} \setminus K$ is path-connected; hence, connected. We need to construct a path connecting any two different points $v_1, v_2 \in \mathbb{C} \setminus K$. The set K is bounded; therefore, we can find $r > 0$ such that $K \subset B(0, r)$. If $\langle v_1, v_2\rangle \cap K = \emptyset$, we connect v_1 and v_2 with the line segment $[v_1, v_2]$. If the line $\langle v_1, v_2\rangle$ has a non-empty intersection with K, then we denote $\langle v_1, v_2\rangle \cap K = [k_1, k_2]$, where k_1 and k_2 may equal each other if the intersection is a singleton. Moreover, $\langle v_1, v_2\rangle \cap K$ is necessarily a convex set. Therefore, the line $\langle v_1, v_2\rangle$ decomposes into exactly three disjoint parts which read

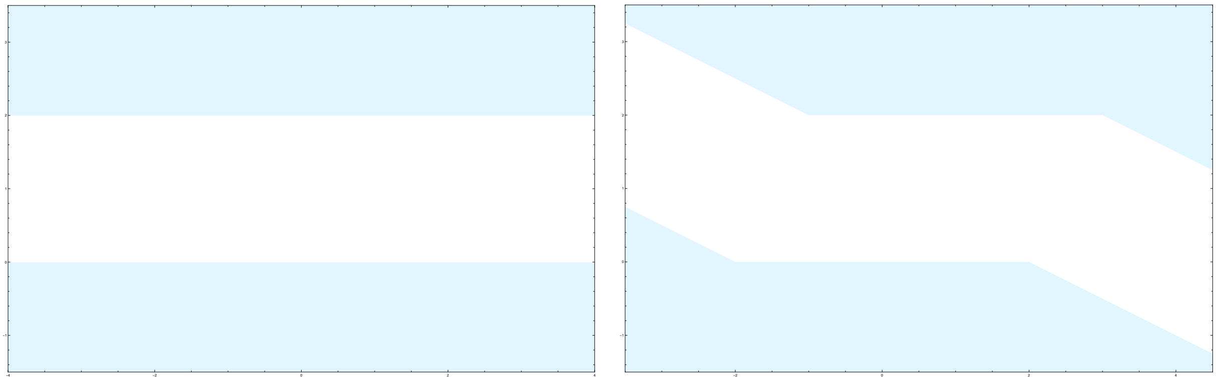
$$\langle v_1, v_2\rangle = \langle v_1, k_1) \cup [k_1, k_2] \cup (k_2, v_2\rangle.$$

Let us denote $\{b_1\} = \partial B(0,r) \cap [k_1, v_1\rangle$ and $\{b_2\} = \partial B(0,r) \cap [k_2, v_2\rangle$. Now we can connect $v_1, v_2 \in \mathbb{C} \setminus K$ with either one of the paths given by the set $[v_1, b_1) \cup \partial B(0,r) \cup (b_1, v_2]$. □

If an operator is bounded the closure of its numerical range is convex; therefore $\mathbb{C} \setminus \overline{\mathrm{Num}(A)}$ is connected. The restriction to bounded operators simplifies Proposition 2.17 in such a way that the inequality holds for all $\lambda \in \mathbb{C} \setminus \overline{\mathrm{Num}(A)}$.

Recall the numeric range of H_α is given by $\mathrm{Num}(H_\alpha) = (-2,2) + \alpha[0,1]$, see Theorem 2.10. Let us define

$$\Omega(\alpha) := \big\{ z \in \mathbb{C} \;\big|\; \mathrm{dist}\big(\lambda, \mathrm{Num}(H_\alpha)\big) = \mathrm{dist}\big(\lambda, \sigma(H_\alpha)\big) \big\}.$$



(a) $\alpha = 2\mathrm{i}$ (b) $\alpha = 2\mathrm{i} + 1$

Figure 2.4: Depictions of the set $\Omega(\alpha)$ for certain choices of α.

In Fig. 2.4, we have depicted the set $\Omega(\alpha)$ for certain values of α. For all $\lambda \in \Omega(\alpha)$ the upper and lower estimate of $\big\|(A-\lambda)^{-1}\big\|$ equal each other on; therefore, we can describe the ε-spectrum explicitly as

$$\Omega(\alpha) \cap \sigma_\varepsilon(H_\alpha) \;=\; \Omega(\alpha) \cap \big(\sigma(H_\alpha) + D(\varepsilon)\big).$$

To our knowledge, there is no way to describe the pseudospectra of H_α explicitly outside of $\Omega(\alpha)$; therefore, our task is to obtain a superset and a subset of each ε-pseudospectrum. This way we get a region where the boundary of the ε-pseudospectrum must reside. We will obtain these supersets and subsets by estimating the norm of the resolvent above and below. As stated above we obtain a lower bound by virtue of Proposition 2.16. These lower bounds are depicted in Figure 2.5. The tool we will use for obtaining upper estimates of an operator's norm is the Schur test, see Theorem 1.14. The Schur test has already been applied in Proposition 2.4, as already stated before, we shall be using the estimate as given by (2.1.12), i.e.

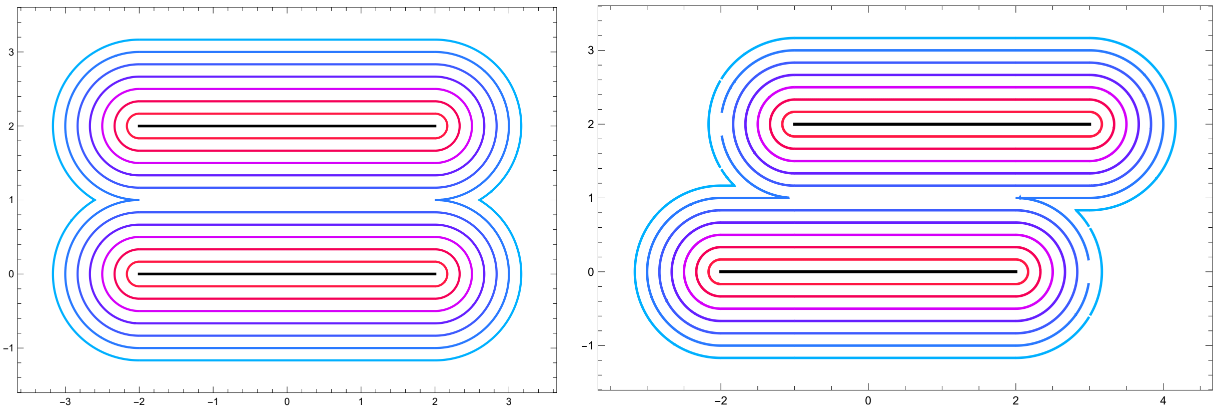
$$\forall \lambda \in \rho(H_\alpha) \;:\quad \|(H_\alpha - \lambda)^{-1}\| \le \widetilde{U}(\lambda).$$

These upper bounds are shown in Figure 2.6.

Theorem 2.19: Let $\mathrm{Im}\alpha \neq 0$. Then the ε-pseudospectra of H_α follow these inclusions:

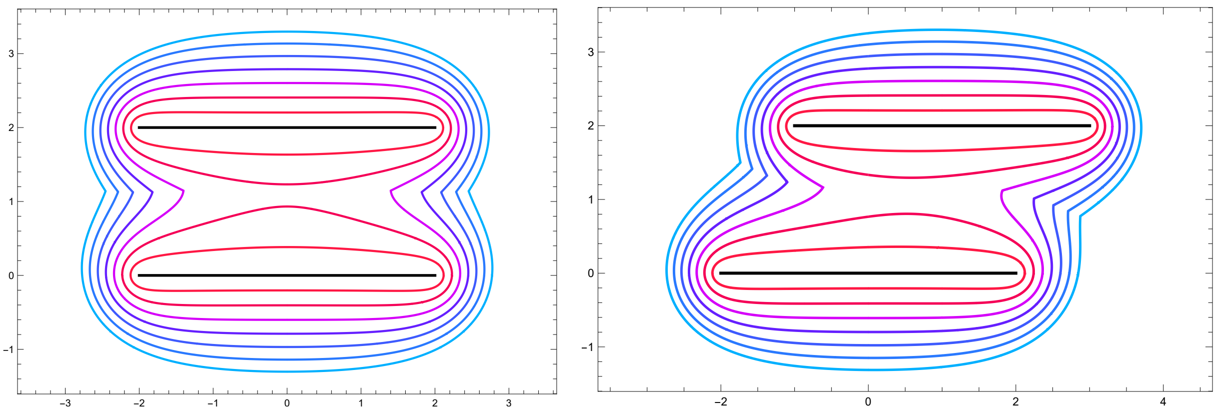
$$\Big\{ \lambda \in \mathbb{C} \;\Big|\; \frac{1}{\mathrm{dist}\big(\lambda, \sigma(H_\alpha)\big)} \ge \varepsilon^{-1} \Big\} \subset \sigma_\varepsilon(H_\alpha) \subset \big\{ \lambda \in \mathbb{C} \;\big|\; \widetilde{U}(\lambda) \ge \varepsilon^{-1} \big\}.$$

These estimates are used when we do not have an explicit expression for the resolvent operator's norm, i.e. on $\mathbb{C} \setminus \Omega(\alpha)$.



(a) $\alpha = 2\mathrm{i}$ (b) $\alpha = 2\mathrm{i} + 1$

Figure 2.5: Depictions of $\operatorname{dist}(\lambda, \sigma(H_\alpha)) = \varepsilon$ for certain values of ε.



(a) $\alpha = 2\mathrm{i}$ (b) $\alpha = 2\mathrm{i} + 1$

Figure 2.6: Depictions of $\widetilde{U}(\lambda) = \varepsilon$ for certain values of ε.

2.2.1 Asymptotic behavior

In this section we describe the behavior of pseudospectra near the spectrum of H_α. For $\lambda \in \sigma(H_\alpha)$ we derive asymptotic expansions of the upper and lower bounds of $\|(H_\alpha - \lambda - \mathrm{i}\varepsilon)^{-1}\|$ as $\varepsilon \to 0_+$.

As stated above, if $\operatorname{Im}\alpha = 0$ the pseudospectrum is trivially described. We assume the opposite, i.e. $\operatorname{Im}\alpha \neq 0$. In this case, the spectrum is comprised of two connected components. To ensure $\lambda + \mathrm{i}\varepsilon$ is not an element of the other connected component of the spectrum we assume $\varepsilon < |\operatorname{Im}\alpha|$.

A trick we can use while evaluating asymptotic pseudospectra is to separate the quadrants of the matrix representation of the resolvent operator

$$G(\lambda) = \begin{pmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{pmatrix} = \begin{pmatrix} G^{--} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & G^{-+} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ G^{+-} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & G^{++} \end{pmatrix}. \tag{2.2.2}$$

Taking the norm of the resolvent operator and estimating it with triangle inequality may be too rough of an estimate for global estimation but if we show that all but one term are asymptotically insignificant, we can make the asymptotic formula simpler. The quadrants are separated as in (2.1.10).

Since it is not ideal to work with matrices whose indices are on the negative half-line, we define matrices $\widehat{G}^{--}_{m,n} := G^{--}_{-m,-n}$, $\widehat{G}^{+-}_{m,n} := G^{+-}_{m,-n}$, and $\widehat{G}^{-+}_{m,n} := G^{-+}_{-m,n}$.

In [1] we showed that

$$\left\| \begin{pmatrix} G^{--} & 0 \\ 0 & 0 \end{pmatrix} \right\|_{\ell^2(\mathbb{Z})} = \|\widehat{G}^{--}\|_{\ell^2(\mathbb{N})}, \qquad \left\| \begin{pmatrix} 0 & G^{-+} \\ 0 & 0 \end{pmatrix} \right\|_{\ell^2(\mathbb{Z})} = \|\widehat{G}^{-+}\|_{\ell^2(\mathbb{N})},$$

$$\left\| \begin{pmatrix} 0 & 0 \\ G^{+-} & 0 \end{pmatrix} \right\|_{\ell^2(\mathbb{Z})} = \|\widehat{G}^{+-}\|_{\ell^2(\mathbb{N})}, \qquad \left\| \begin{pmatrix} 0 & 0 \\ 0 & G^{++} \end{pmatrix} \right\|_{\ell^2(\mathbb{Z})} = \|G^{++}\|_{\ell^2(\mathbb{N}_0)}.$$

With this established let us evaluate or estimate the norm of these operators.

The norms of G^{+-} and G^{-+} can be evaluated exactly by applying the following lemma.

Lemma 2.20: Let $p, q \in \mathbb{C}$, $|p|$, $|q| < 1$. We define an operator $A = A(p, q)$ on $\ell^2(\mathbb{N})$ whose matrix representation reads

$$\forall m, n \in \mathbb{N} \quad : \quad A_{m,n} := p^m q^n.$$

Then $A \in \mathscr{B}\big(\ell^2(\mathbb{N})\big)$. Moreover, we can calculate its norm

$$\|A\| = \sqrt{\frac{|p|^2}{1-|p|^2}\frac{|q|^2}{1-|q|^2}}.$$

Proof. Choose an arbitrary vector $\psi \in \ell^2(\mathbb{N})$, then the mth element of $A\psi$ reads

$$(A\psi)_m = \sum_{n=1}^{\infty} A_{m,n}\psi_n = \sum_{n=1}^{\infty} p^m q^n \psi_n = p^m \sum_{n=1}^{\infty} q^n \psi_n.$$

Next, we estimate the square of its norm

$$\|A\psi\|^2 = \sum_{m=1}^{\infty} |(A\psi)_m|^2 = \sum_{m=1}^{\infty} |p|^{2m} \Big| \sum_{n=1}^{\infty} q^n \psi_n \Big|^2 \leq \sum_{m=1}^{\infty} |p|^{2m} \Big(\sum_{n=1}^{\infty} |q|^n |\psi_n| \Big)^2$$
$$\overset{\text{C.–S.}}{\leq} \sum_{m=1}^{\infty} |p|^{2m} \sum_{n=1}^{\infty} |q|^{2n} \sum_{k=1}^{\infty} |\psi_k|^2 = \frac{|p|^2}{1-|p|^2}\frac{|q|^2}{1-|q|^2}\|\psi\|^2,$$

where the marked inequality denotes the Cauchy–Schwarz inequality. This calculation proves the assertion that A is a bounded operator on $\ell^2(\mathbb{N})$ and provides an upper bound for the norm of A. For the lower bound estimation let us choose a test vector $\psi_n := (\overline{q})^n$, then

$$\|A\psi\|^2 = \sum_{m=1}^{\infty} |(A\psi)_m|^2 = \sum_{m=1}^{\infty} |p|^{2m} \Big| \sum_{n=1}^{\infty} q^n (\overline{q})^n \Big|^2 = \sum_{m=1}^{\infty} |p|^{2m} \Big(\sum_{n=1}^{\infty} |q|^{2n} \Big)^2$$
$$= \sum_{m=1}^{\infty} |p|^{2m} \sum_{n=1}^{\infty} |q|^{2n} \sum_{k=1}^{\infty} |\overline{q}|^{2k} = \sum_{m=1}^{\infty} |p|^{2m} \sum_{n=1}^{\infty} |q|^{2n} \sum_{k=1}^{\infty} |\psi_k|^2 = \frac{|p|^2}{1-|p|^2}\frac{|q|^2}{1-|q|^2}\|\psi\|^2.$$

Since these two estimates are equal, we know it is the norm of A itself. $\square$

As a result of Lemma 2.20 we get

$$\|\widehat{G}^{+-}\| = \|\widehat{G}^{-+}\| = \frac{1}{|w|}\frac{1}{\sqrt{1-|\eta|^2}}\frac{|\xi|}{\sqrt{1-|\xi|^2}}. \tag{2.2.3}$$

The norms of the other quadrants are more complicated. We were not able to evaluate them exactly; hence, we give estimates. Multiple estimation techniques were used to take the upper bound, though not one yielded a better asymptotic formula than the others; therefore, we will show the most conspicuous one – the Schur test. We will get the lower bound by choosing suitable test vectors.

Proposition 2.21: Let $\lambda \in \rho(H_\alpha)$. If we denote

$$U^-(\lambda) := \frac{1}{|\xi^{-1}-\xi|}\frac{1}{1-|\xi|}\left((|\xi|^{-1}+|\eta|^{-1})C(\alpha)+2\right),$$

$$L^-(\lambda) := \frac{\sqrt{(1-|\xi|^2)^2|\xi| + 2\mathrm{Re}(-\xi^{-1}w)|\xi|(1-|\xi|^2) + |w|^2(1+|\xi|^2)}}{|w|\,|\xi^{-1}-\xi|\,(1-|\xi|^2)},$$

$$U^+(\lambda) := \frac{1}{|\eta^{-1}-\eta|}\frac{1}{1-|\eta|}\left(2C(\alpha)+2\right),$$

$$L^+(\lambda) := \frac{\sqrt{(1-|\eta|^2)^2 + 2\mathrm{Re}(-\eta w)|\eta|^2(1-|\eta|^2) + |\eta|^4|w|^2(1+|\eta|^2)}}{|w|\,|\eta|\,|\eta^{-1}-\eta|\,(1-|\eta|^2)}.$$

The upper and lower bounds of the norm of $\widehat{G}^{--}$ and G^{++} take the forms

$$U^- \geq \|\widehat{G}^{--}\| \geq L^-, \qquad U^+ \geq \|G^{++}\| \geq L^+,$$

where $C(\alpha)$ is the uniform upper bound of $|w|^{-1} = |\xi - \eta^{-1}|^{-1}$ from (1.3.5).

Proof. We utilize the formulation of G as given by (2.1.10). First, we establish the upper bounds for the norms using the Schur test. Let $m \geq 0$, then

$$\begin{aligned}
\sum_{n=0}^{\infty}|G^{++}_{m,n}(\lambda)| &= \frac{1}{|w|}\sum_{n=0}^{\infty}\left|\frac{\xi-\eta}{\eta^{-1}-\eta}\eta^{m+n} + \frac{\xi-\eta^{-1}}{\eta-\eta^{-1}}\eta^{|m-n|}\right| \\
&\leq \frac{1}{|\eta^{-1}-\eta|}\left(\frac{|\xi-\eta|}{|w|}|\eta|^m\sum_{n=0}^{\infty}|\eta|^n + \sum_{n=-m}^{\infty}|\eta|^{|n|}\right) \\
&= \frac{1}{|\eta^{-1}-\eta|}\frac{1}{1-|\eta|}\left(\frac{|\xi-\eta|}{|w|}|\eta|^m + 1 + |\eta| - |\eta|^{m+1}\right) \\
&\leq \frac{1}{|\eta^{-1}-\eta|}\frac{1}{1-|\eta|}\left(2C(\alpha)+2\right)
\end{aligned}$$

Since the dependence on m has been eliminated from the expression, the last estimate serves as an upper bound for $\|G^{++}(\lambda)\|$. A comparable method is applied to analyze G^{--}.

$$
\begin{aligned}
\sum_{n=1}^{\infty}|G^{--}_{m,n}(\lambda)| &= \frac{1}{|w|}\sum_{n=1}^{\infty}\left|\frac{\eta^{-1}-\xi}{\xi^{-1}-\xi}\xi^{|m-n|}+\frac{\eta^{-1}-\xi^{-1}}{\xi-\xi^{-1}}\xi^{m+n}\right| \\
&\leq \frac{1}{|\xi^{-1}-\xi|}\left(\sum_{n=-m+1}^{\infty}|\xi|^{|n|}+\frac{|\xi^{-1}-\eta^{-1}|}{|w|}|\xi|^{m}\sum_{n=1}^{\infty}|\xi|^{n}\right) \\
&= \frac{1}{|\xi^{-1}-\xi|}\frac{1}{1-|\xi|}\left(1+|\xi|-|\xi|^{m}+\frac{|\xi^{-1}-\eta^{-1}|}{|w|}|\xi|^{m}\right) \\
&\leq \frac{1}{|\xi^{-1}-\xi|}\frac{1}{1-|\xi|}\left((|\xi|^{-1}+|\eta|^{-1})C(\alpha)+2\right).
\end{aligned}
$$

For the lower bound we will be using estimation by a test vector, see (2.2.1). For $G^{++}(\lambda)$ and $\widehat{G}^{--}(\lambda)$ we define test vectors

$$
\psi=\{\eta^{n}\}_{n\in\mathbb{N}_0} \qquad \text{and} \qquad \varphi=\{\xi^{n}\}_{n\in\mathbb{N}},
$$

respectively. Their norms are

$$
\|\psi\|=\frac{1}{\sqrt{1-|\eta|^{2}}} \qquad \text{and} \qquad \|\varphi\|=\frac{|\xi|}{\sqrt{1-|\xi|^{2}}}.
$$

Let us calculate the m element of $G^{++}(\lambda)\psi$. Let $m\in\mathbb{N}_0$, then

$$
\begin{aligned}
\left(G^{++}(\lambda)\psi\right)_m &= \frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{\xi-\eta}{\eta^{-1}-\eta}\eta^{m+n}+\frac{\xi-\eta^{-1}}{\eta-\eta^{-1}}\eta^{|m-n|}\right)\eta^{n} \\
&= \frac{1}{\eta^{-1}-\eta}\left(\frac{\xi-\eta}{w}\eta^{m}\sum_{n=0}^{\infty}\eta^{2n}+\sum_{n=0}^{\infty}\eta^{|m-n|+n}\right) \\
&= \frac{1}{\eta^{-1}-\eta}\left(\left(\frac{\xi-\eta}{w}+1\right)\frac{1}{1-\eta^{2}}+m\right) \\
&= \frac{\eta^{m}}{w(1-\eta^{2})}\left(1-m\eta w\right).
\end{aligned}
$$

If we calculate

$$
|1-m\eta w|^{2}=\left(1-m\mathrm{Re}(\eta w)\right)^{2}+\left(m\mathrm{Im}(\eta w)\right)^{2}=1-2m\mathrm{Re}(\eta w)+m^{2}|\eta w|^{2},
$$

then the squared absolute value of $\left(G^{++}(\lambda)\psi\right)_m$ reads

$$
\left|\left(G^{++}(\lambda)\psi\right)_m\right|^{2}=\frac{1}{|w|^{2}|1-\eta^{2}|^{2}}\left(|\eta|^{2m}-m|\eta|^{2m}2\mathrm{Re}(\eta w)+m^{2}|\eta|^{2m}|\eta w|^{2}\right). \tag{2.2.4}
$$

In order to calculate the sum over indices m of the expression above, let us show special power series, so-called *low-order polylogarithms*, a special case of which is the geometric series. Let $z\in(0,1)$, then

$$
\sum_{n=0}^{\infty}z^{n}=\frac{1}{1-z}, \qquad \sum_{n=1}^{\infty}nz^{n}=\frac{z}{(1-z)^{2}}, \qquad \sum_{n=1}^{\infty}n^{2}z^{n}=\frac{z(1+z)}{(1-z)^{3}}.
$$

After we take the square root of the sum of (2.2.4) and divide it by the norm of ψ we get L^{+} after simple manipulations. The calculation of the lower bound of $\|\widehat{G}^{--}(\lambda)\|$ denoted by L^{-} is analogous and we also omit details. □

With the estimates or the exact formulas for the norms, we can evaluate their asymptotic expansions. The asymptotic formulas are properly derived in proof of Proposition 3.9 from Lemmas 1.22 and 1.24. There are less cases to consider in this setting than in the later part; therefore, we leave most of the calculations for later. Here, we need to differentiate two cases, whether λ is an endpoint of either connected component of the spectrum or not as the asymptotic formulas differ significantly.

- Let us denote $\lambda \in \mathrm{int}(\sigma(H_\alpha))$ if $\lambda \in (-2,2) \cup (-2+\alpha, 2+\alpha)$.
- Let us denote $\lambda \in \mathrm{cl}(\sigma(H_\alpha))$ if $\lambda \in \{-2, 2, -2+\alpha, 2+\alpha\}$.

We will calculate asymptotic behavior for each quadrant of $G(\lambda)$ individually.

Lemma 2.22: Assume $\alpha \neq 0$. Let $\lambda \in \sigma(H_\alpha)$. Then

$$\|G^{+-}(\lambda + \mathrm{i}\varepsilon)\| = \|G^{-+}(\lambda + \mathrm{i}\varepsilon)\| = \mathcal{O}\left(\varepsilon^{-1/2}\right) \qquad \text{as } \varepsilon \to 0_+.$$

Proof. The norms of G^{+-} and G^{-+} are equal, see (2.2.3); hence, we are to derive aymptotic expansion of

$$\frac{1}{|w|} \frac{1}{\sqrt{1-|\eta|^2}} \frac{|\xi|}{\sqrt{1-|\xi|^2}}.$$

As the approach for λ from either connected component of the spectrum is similar, we show only the case $\lambda \in [-2,2]$. Since we assume $\mathrm{Im}\alpha \neq 0$, we have $|\eta| \to c \in (0,1)$. We need to further differentiate two cases.

$\lambda \in \mathrm{cl}([-2,2])$, i.e. $\lambda \in \{-2,2\}$. In this case we obtain the asymptotic expansion of $|\xi|^2$ from Corollary 1.25. Then

$$\|G^{+-}(\lambda + \mathrm{i}\varepsilon)\| = \frac{1}{|w|_{\varepsilon=0}} \frac{1}{\sqrt{1-c^2}} \frac{1+\mathcal{O}\left(\varepsilon^{1/2}\right)}{\sqrt{\sqrt{2\varepsilon}+\mathcal{O}(\varepsilon)}} = \mathcal{O}\left(\varepsilon^{-1/4}\right) = \mathcal{O}\left(\varepsilon^{-1/2}\right), \qquad \text{as } \varepsilon \to 0_+$$

since $w = \mathcal{O}(1)$ as $\varepsilon \to 0_+$.

$\lambda \in \mathrm{int}([-2,2])$, i.e. $\lambda \in (-2,2)$. In this case we obtain the asymptotic expansion of $|\xi|^2$ from Corollary 1.23. Then

$$\|G^{+-}(\lambda + \mathrm{i}\varepsilon)\| = \frac{1}{|w|_{\varepsilon=0}} \frac{1}{\sqrt{1-c^2}} \frac{1+\mathcal{O}(\varepsilon)}{\sqrt{\frac{\varepsilon}{\sin\phi}+\mathcal{O}(\varepsilon^2)}} = \mathcal{O}\left(\varepsilon^{-1/2}\right), \qquad \text{as } \varepsilon \to 0_+$$

where $\phi = -\arccos(\lambda/2)$. □

Lemma 2.23: Assume $\alpha \neq 0$. Let $\lambda \in [-2,2]$. Then

$$U^+(\lambda+\mathrm{i}\varepsilon) = \frac{2}{\varepsilon}(C(\alpha)+1) + \mathcal{O}(1) \qquad U^-(\lambda+\mathrm{i}\varepsilon) = \mathcal{O}(1) \qquad \text{as } \varepsilon \to 0_+,$$
$$L^+(\lambda+\mathrm{i}\varepsilon) = \frac{1}{\sqrt{2}\,\varepsilon} + \mathcal{O}(1) \qquad L^-(\lambda+\mathrm{i}\varepsilon) = \mathcal{O}(1) \qquad \text{as } \varepsilon \to 0_+.$$

Let $\lambda \in [-2+\alpha, 2+\alpha]$. Then

$$U^+(\lambda+\mathrm{i}\varepsilon) = \mathcal{O}(1) \qquad U^-(\lambda+\mathrm{i}\varepsilon) = \frac{2}{\varepsilon}(C(\alpha)+1) + \mathcal{O}(1) \qquad \text{as } \varepsilon \to 0_+,$$
$$L^+(\lambda+\mathrm{i}\varepsilon) = \mathcal{O}(1) \qquad L^-(\lambda+\mathrm{i}\varepsilon) = \frac{1}{\sqrt{2}\,\varepsilon} + \mathcal{O}(1) \qquad \text{as } \varepsilon \to 0_+.$$

Proof. FINISH PROOF □

Theorem 2.24: Let $\alpha \neq 0$ and $\lambda \in \sigma(H_\alpha)$. Then the asymptotic behavior of the resolvent operator's norm reads

$$\frac{1}{\sqrt{2}\,\varepsilon} + \mathcal{O}\left(\varepsilon^{-1/2}\right) \leq \|(H_\alpha - \lambda)^{-1}\| \leq \frac{2}{\varepsilon}\left(C(\alpha)+1\right) + \mathcal{O}\left(\varepsilon^{-1/2}\right) \qquad \text{as } \varepsilon \to 0_+.$$

Proof. Taking the triangle inequality of the decomposed resolvent operator, see (2.2.2), we estimate $\|(H_\alpha - \lambda)^{-1}\|$ from above and below. If we then estimate further using Proposition 2.21, we get

$$\|(H_\alpha - \lambda)^{-1}\| \leq \|\widehat{G}^{--}\| + \|\widehat{G}^{+-}\| + \|\widehat{G}^{-+}\| + \|G^{++}\| \leq U^- + 2\|\widehat{G}^{-+}\| + U^+, \tag{2.2.5}$$

$$\|(H_\alpha - \lambda)^{-1}\| \geq \|\widehat{G}^{--}\| - \|\widehat{G}^{+-}\| - \|\widehat{G}^{-+}\| - \|G^{++}\| \geq L^- - 2\|\widehat{G}^{-+}\| - U^+, \tag{2.2.6}$$

$$\|(H_\alpha - \lambda)^{-1}\| \geq \|G^{++}\| - \|\widehat{G}^{+-}\| - \|\widehat{G}^{-+}\| - \|\widehat{G}^{--}\| \geq L^+ - 2\|\widehat{G}^{-+}\| - U^-. \tag{2.2.7}$$

These estimates allowed us to separate asymptotically insignificant terms. We get the assertion of the theorem simply by plugging in the asymptotic expressions from Lemmas 2.22 and 2.23. □

2.3 Spectral measure

In this section, we derive the spectral measure associated with the operator H_α, more precisely we derive the Radon–Nikodym derivative of the spectral measure with respect to the Lebesgue measure, the *spectral density*. To ensure the spectral measure is well-defined, we restrict our analysis to $\alpha \in \mathbb{R}$, in which case H_α is a self-adjoint operator. The spectral density of the discrete operator H_α is matrix-valued, and its entries are determined by the Stone's formula

$$\frac{\mathrm{d}\mu_{m,n}}{\mathrm{d}\lambda}(\lambda) = \frac{1}{2\pi\mathrm{i}} \lim_{\varepsilon\to 0_+} \left(G_{m,n}(\lambda + \mathrm{i}\varepsilon) - G_{m,n}(\lambda - \mathrm{i}\varepsilon)\right),$$

where $\lambda \in \sigma(H_\alpha)$. If need be, we denote the spectral density as $\mathrm{d}\mu_{m,n}$.

Let $z = \lambda + \mathrm{i}\varepsilon$. To calculate the spectral density one must take the limit as z approaches $\lambda \in \sigma(A)$ from each side. By using the Joukowsky transform, the analysis of the spectral density simplifies significantly. Instead of taking limits as $z \to \lambda \in (-2,2)$ from either side of the cut in the λ-plane, we exploit the fact that the interval $(-2,2)$ is the image of the unit circle $\partial\mathbb{D}$ under the transform. Specifically, each point $\lambda \in (-2,2)$ corresponds to two distinct pre-images on $\partial\mathbb{D}$:

$$k_1 = e^{i\theta}, \quad k_2 = e^{-i\theta} = \overline{k_1}, \quad \phi \in (-\pi, 0).$$

One can easily see this in Lemma 1.22. Thus, the discontinuity of the Green's kernel at $\lambda \in (-2,2)$ is captured by evaluating G at these two pre-images in the k-plane. This approach avoids the need to compute explicit limits in the λ-plane, providing a more direct and elegant analysis.

This applies directly to analyzing H_0. When using this argument on H_α one must take into account the to parameters ξ and η.

2.3.1 Spectral measure of H_0

If we set $\alpha = 0$, the analysis is quite straightforward. The resolvent operator for H_0 reads

$$G_{m,n}(\lambda) = \frac{k^{|m-n|}}{k - k^{-1}}.$$

For $\lambda \in (-2, 2)$ we have corresponding $\theta \in (0, \pi)$ by $\lambda = 2\cos\theta$. Due to the argument above, we have $k = e^{\mathrm{i}\theta}$ and

$$\lim_{\substack{z \to 2\cos\theta \\ \mathrm{Im} z > 0}} \left(G_{m,n}(z) - G_{m,n}(\overline{z})\right) \quad = G_{m,n}(k) - G_{m,n}(\overline{k}).$$

We abuse the notation by writing $G(\lambda) = G(k)$, where the dependence on λ is implicit through the Joukowsky transform.

We set $N = |m - n|$ and evaluate

$$\frac{k^n}{k - k^{-1}} - \frac{\overline{k}^n}{\overline{k} - \overline{k}^{-1}} = \frac{e^{-\mathrm{i}n\theta}}{e^{-\mathrm{i}\theta} - e^{\mathrm{i}\theta}} - \frac{e^{\mathrm{i}n\theta}}{e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}} = \frac{e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta}}{e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}} = \frac{\mathrm{i}\cos(n\theta)}{\sin\theta}.$$

This results in the spectral density of H_0 taking the form

$$\frac{\mathrm{d}\mu_{m,n}}{\mathrm{d}\lambda} = \frac{1}{2\pi}\frac{\cos\left(|m-n|\theta\right)}{\sin\theta}.$$

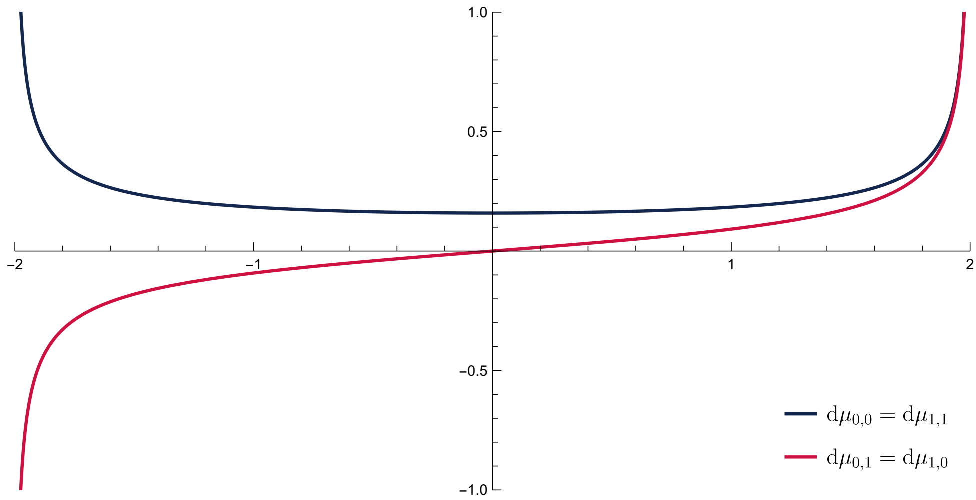


Figure 2.7: Spectral density of H_0.

2.3.2 Spectral measure of H_α

When considering $\alpha \neq 0$ real, the situation becomes quite tedious. There is one or two connected components of the spectrum. We need to differentiate which connected component, or line segment, we are approaching and whether we are approaching their intersection. In this section, we will be using the (2.1.11) formulation of the resolvent operator.

- Let $\lambda \in (2, 2) \ \cap \ (-2 + \alpha, 2 + \alpha)$. Of course, this set is non-empty if and only if $\lambda \in (-4, 4)$. Since λ resides in both line segments simultaneously, both Joukowsky parameters ξ and η are equal to 1 in absolute value; therefore, we can write $\xi = e^{\mathrm{i}\phi}$ and $\eta = e^{\mathrm{i}\theta}$, where $\phi(\theta) = \arccos\left(\cos\theta + \frac{\alpha}{2}\right)$ from (1.3.6). Once again we abuse the notation and write $G(\lambda) = G(\xi, \eta)$. For each quadrant of the matrix $G(\lambda)$ we calculate the spectral density separately.

– Let $m \geq 0$ and $n < 0$. Then

$$G_{m,n}(\xi,\eta) - G_{m,n}(\overline{\xi},\overline{\eta}) = \frac{\eta^m \xi^n}{\xi - \eta^{-1}} - \frac{\overline{\eta}^m \overline{\xi}^n}{\overline{\xi} - \overline{\eta}^{-1}} = \frac{e^{\mathrm{i}m\theta} e^{-\mathrm{i}n\phi}}{e^{\mathrm{i}\phi} - e^{-\mathrm{i}\theta}} - \frac{e^{-\mathrm{i}m\theta} e^{\mathrm{i}n\phi}}{e^{-\mathrm{i}\phi} - e^{\mathrm{i}\theta}}$$
$$= \mathrm{i}\frac{\sin\big(m\theta - n\phi - \phi\big) - \sin\big(m\theta - n\phi + \theta\big)}{1 - \cos\big(\phi + \theta\big)},$$

$$\frac{\mathrm{d}\mu_{m,n}}{\mathrm{d}\lambda} = \frac{1}{2\pi} \frac{\sin\big(m\theta - (n+1)\phi\big) - \sin\big((m+1)\theta - n\phi\big)}{1 - \cos\big(\phi + \theta\big)}.$$

– Let $m < 0$ and $n \geq 0$. Then similarly we have

$$\frac{\mathrm{d}\mu_{m,n}}{\mathrm{d}\lambda} = \frac{1}{2\pi} \frac{\sin\big(n\theta - (m+1)\phi\big) - \sin\big((n+1)\theta - m\phi\big)}{1 - \cos\big(\phi + \theta\big)}.$$

– Let $m \geq 0$ and $n \geq 0$. Then

$$\frac{\mathrm{d}\mu_{m,n}}{\mathrm{d}\lambda} = \frac{1}{2\pi} \frac{\cos\big((m+n)\theta\big) - \cos\big(|m-n|\theta\big)}{\sin\theta} + \frac{1}{2\pi} \frac{\sin\big((m+n)\theta - \phi\big) - \sin\big((m+n+1)\theta\big)}{1 - \cos(\phi + \theta)}.$$

– Let $m < 0$ and $n < 0$. Then

$$\frac{\mathrm{d}\mu_{m,n}}{\mathrm{d}\lambda} = \frac{1}{2\pi} \frac{\cos\big((m+n)\phi - \cos\big(|m-n|\phi\big)\big)}{\sin\phi} + \frac{1}{2\pi} \frac{\sin\big((m+n)\phi - \theta\big) - \sin\big((m+n+1)\phi\big)}{1 - \cos(\phi + \theta)}.$$

- Let $\lambda \in (-2,2)$ and $\lambda \notin (-2+\alpha, 2+\alpha)$. In this case, we again have $|\xi| = 1$ so $\xi = e^{\mathrm{i}\phi}$. The parameter η is no longer equal to 1 in absolute value and so it is uniquely determined by λ. Since $\alpha \in \mathbb{R}$, $\eta \in (-1,1)$.

  – Let $m \geq 0$ and $n < 0$. Then

$$G_{m,n}(\xi,\eta) - G_{m,n}(\overline{\xi},\eta) = \frac{\eta^m e^{-\mathrm{i}n\phi}}{e^{\mathrm{i}n\phi} - \eta^{-1}} - \frac{\eta^m e^{+\mathrm{i}n\phi}}{e^{-\mathrm{i}n\phi} - \eta^{-1}}$$
$$= 2i\eta^m \frac{\sin(n\phi)\eta^{-1} - \sin\big((n+1)\phi\big)}{1 - 2\eta^{-1}\cos\phi + \eta^{-2}},$$

$$\frac{\mathrm{d}\mu_{m,n}}{\mathrm{d}\lambda} = \frac{\eta^m}{\pi} \frac{\sin(n\phi)\eta^{-1} - \sin\big((n+1)\phi\big)}{1 - 2\eta^{-1}\cos\phi + \eta^{-2}}.$$

  – Let $m < 0$ and $n \geq 0$. Then

$$\frac{\mathrm{d}\mu_{m,n}}{\mathrm{d}\lambda} = \frac{\eta^n}{\pi} \frac{\sin(m\phi)\eta^{-1} - \sin\big((m+1)\phi\big)}{1 - 2\eta^{-1}\cos\phi + \eta^{-2}}.$$

  – Let $m \geq 0$ and $n \geq 0$. Then

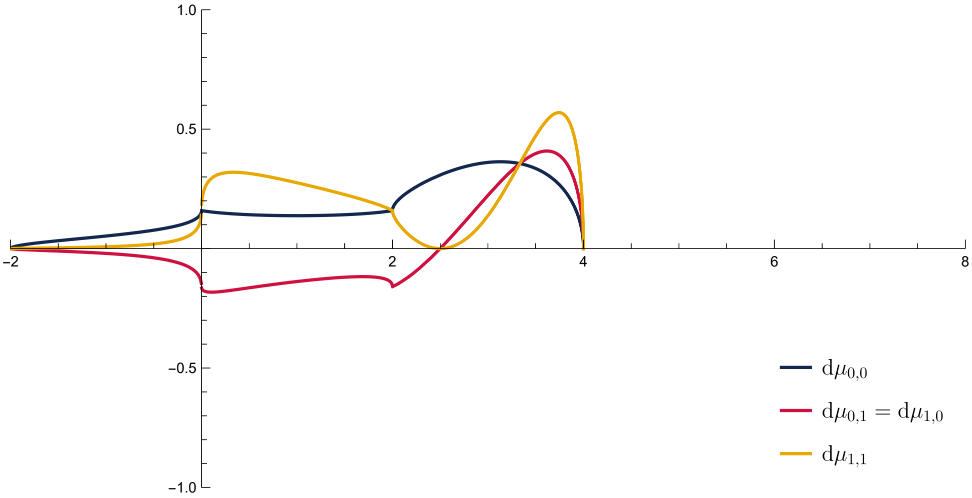
$$\frac{\mathrm{d}\mu_{m,n}}{\mathrm{d}\lambda} = -\frac{\eta^{m+n}}{\pi} \frac{\sin(\phi)}{1 - 2\eta^{-1}\cos\phi + \eta^{-2}}.$$

  – Let $m < 0$ and $n < 0$. Then

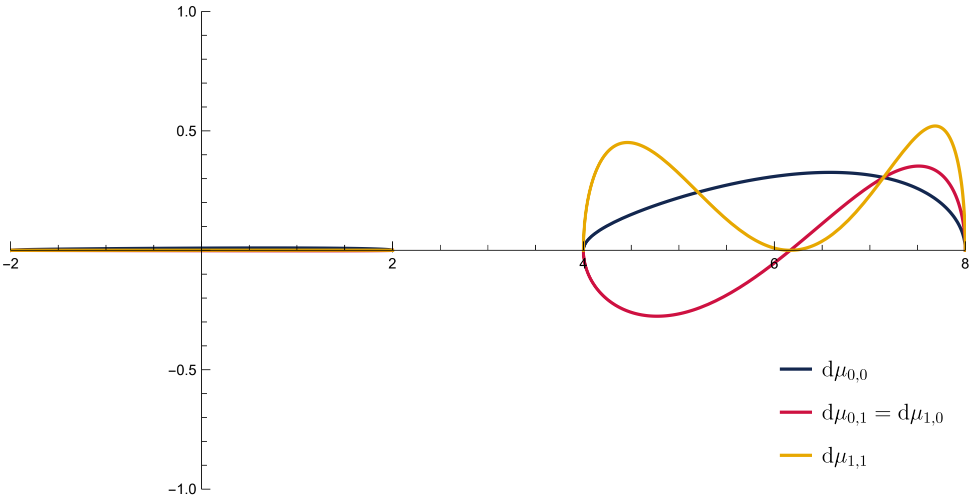
$$\frac{\mathrm{d}\mu_{m,n}}{\mathrm{d}\lambda} = \frac{1}{2\pi} \frac{\cos\big((m+n)\phi - \cos\big(|m-n|\phi\big)\big)}{\sin\phi} + \frac{1}{\pi} \frac{\eta^{-1}\sin\big((m+n)\phi\big) - \sin\big((m+n+1)\phi\big)}{1 - 2\eta^{-1}\cos\phi + \eta^{-2}}.$$

- Let $\lambda \notin (-2,2)$ and $\lambda \in (-2+\alpha, 2+\alpha)$. In this case, we again have $|\eta| = 1$ so $\eta = e^{\mathrm{i}\theta}$. This time the parameter ξ is no longer equal to 1 in absolute value and so it is uniquely determined by λ.

  - Let $m \geq 0$ and $n < 0$. Then
$$\frac{\mathrm{d}\mu_{m,n}}{\mathrm{d}\lambda} = \frac{\xi^{-n}}{\pi} \frac{\xi \sin(m\theta) - \sin\big((m+1)\theta\big)}{\xi^2 - 2\xi\cos\theta + 1}.$$
  - Let $m < 0$ and $n \geq 0$. Then
$$\frac{\mathrm{d}\mu_{m,n}}{\mathrm{d}\lambda} = \frac{\xi^{-m}}{\pi} \frac{\xi \sin(n\theta) - \sin\big((n+1)\theta\big)}{\xi^2 - 2\xi\cos\theta + 1}.$$
  - Let $m \geq 0$ and $n \geq 0$. Then
$$\frac{\mathrm{d}\mu_{m,n}}{\mathrm{d}\lambda} = \frac{1}{2\pi} \frac{\cos\big((m+n)\theta\big) - \cos\big(|m-n|\theta\big)}{\sin\theta} + \frac{1}{\pi} \frac{\xi \sin\big((m+n)\theta\big) - \sin\big((m+n+1)\theta\big)}{\xi^2 - 2\xi\cos\theta + 1}.$$
  - Let $m < 0$ and $n < 0$. Then
$$\frac{\mathrm{d}\mu_{m,n}}{\mathrm{d}\lambda} = -\frac{\xi^{-m-n}}{\pi} \frac{\sin(\theta)}{\xi^2 - 2\xi\cos\theta + 1}.$$



(a) $\alpha = 2$



(b) $\alpha = 6$

Figure 2.8: Spectral density of H_α for two values of α.

Chapter 3

Spectral properties of H_α under small complex perturbations

In this section, we demonstrate the existence, possibly uniqueness, and asymptotic behavior of eigenvalues, while a small potential V is added. If an operator emits eigenvalues under these small perturbations, we say that the operator exhibits *weak coupling*. Some operators do not emit eigenvalues when a small potential is added, the spectrum remains the same, when this is the case we say that the operator exhibits *spectral stability*.

3.1 Weak coupling of H_0

In this section, we work out a discrete analog to some results from [16], see also [13] for a pedagogical text. The article showed that under some assumptions posed upon the potential V, there exists a unique negative bound state of the Schrödinger operator $-\mathrm{d}^2/\mathrm{d}x^2 + V$ on $L^2(\mathbb{R})$.

Consider the discrete Laplace operator, i.e. discrete Schrödinger operator with zero potential, or H_0. Let us first describe some basic properties of this operator, since they do not immediately arise from the case of the Schrödinger operator with complex step potential. Though our notation H_α is consistent with this case, we denote the Laplace operator H_0. Its action on $x \equiv \{x_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ is given by

$$(H_0 x)_n = x_{n-1} + x_{n+1} \qquad n \in \mathbb{Z}.$$

As discussed in Section 1.3, the Joukowsky transform of the spectral parameter λ will be denoted by k, i.e. $\lambda = k + k^{-1}$. The operator H_0 is self-adjoint. As a well-known fact, we state that the spectrum of H_0 coincides with the interval $[-2, 2]$. The matrix representation of the resolvent operator which reads

$$(H_0 - \lambda)^{-1}_{m,n} = (H_0 - k - k^1)^{-1}_{m,n} = \frac{k^{|m-n|}}{k - k^{-1}}, \qquad m, n \in \mathbb{Z}, \quad 0 < |k| < 1,$$

is consistent with (2.1.11), if we set $\alpha = 0$.

Since the discrete version of the Laplacian is bounded, there arises a significant difference to the continuous version. The spectrum of the continuous Laplacian is $[0, \infty)$; therefore, one needs to consider only the left neighborhood of zero when looking for bound states. In the case of the discrete Laplace operator, we need to investigate both the neighborhood of 2 and the neighborhood of -2. However, the following proposition allows us to investigate, in fact, only one of these cases because the behavior is equivalent.

Proposition 3.1: Consider the Laplace operator H_0 and let $V = \operatorname{diag}\big(\{v_n\}_{n\in\mathbb{Z}}\big)$ be a bounded potential. Then the spectrum of the Schrödinger operator $H_V := H_0 + V$ satisfies

$$\sigma(H_V) = -\sigma(H_{-V}).$$

Proof. Let $x = \{x_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and consider the unitary operator U given by $(Ux)_n = (-1)^n x_n$. Then

$$\begin{aligned} U^{-1}H_V Ux &= UH_V Ux = UH_V Ux = UH_V U\{x_n\} = UH_V\{(-1)^n x_n\} \\ &= U\{(-1)^{n-1}x_{n-1} + (-1)^n v_n x_n + (-1)^{n-1}x_{n-1}\} = \{-x_{n-1} + v_n x_n - x_{n+1}\} = -H_{-V}x. \end{aligned}$$

Since this operation does not change the spectrum and scaling the operator by -1 also scales the spectrum as a subset of $\mathbb{C}$ by -1 we have

$$\sigma(H_V) = \sigma(-H_{-V}) = -\sigma(H_{-V}).$$

□

Before we move further, let us reiterate the Birman–Schwinger principle (Theorem 1.13) which states that, whenever $v \in \ell^1(\mathbb{Z})$, any $\lambda \in \rho(H_0)$ is an eigenvalue of $H_0 + V$ if and only if -1 is an eigenvalue of the Birman–Schwinger operator $K(\lambda)$. The entries of its matrix representation read

$$K_{m,n}(\lambda) = \sqrt{|v_m|}\,\frac{k^{|m-n|}}{k-k^{-1}}\,\sqrt{|v_n|}\,\operatorname{sgn} v_n, \qquad m,n \in \mathbb{Z}.$$

Later, we will be scaling the potential V by some small $\varepsilon > 0$, for this purpose, we will denote $H_\varepsilon := H_0 + \varepsilon V$. The Birman–Schwinger principle still holds in the following sense. Any $\lambda \in \rho(H_0)$ is an eigenvalue of H_ε if and only if -1 is an eigenvalue of $\varepsilon K(\lambda)$.

The trick is to decompose the Birman–Schwinger operator into the sum of a well-behaved operator and a rank-one operator as follows

$$K(\lambda) = L(\lambda) + M(\lambda),$$

where the matrix entries of these operators read

$$\begin{aligned} L_{m,n}(\lambda) &:= \sqrt{|v_m|}\,\frac{1}{k-k^{-1}}\,\sqrt{|v_n|}\,\operatorname{sgn} v_n, \\ M_{m,n}(\lambda) &:= \sqrt{|v_m|}\,\frac{k^{|m-n|}-1}{k-k^{-1}}\,\sqrt{|v_n|}\,\operatorname{sgn} v_n. \end{aligned}$$

The operator $L(\lambda)$ is rank-one. Indeed, if we take $\psi \in \ell^2(\mathbb{Z})$, we have for $m \in \mathbb{Z}$

$$\begin{aligned} \big(L(\lambda)\psi\big)_m &= \sum_{n\in\mathbb{Z}} L_{m,n}(\lambda)\psi_n = \sum_{n\in\mathbb{Z}} \sqrt{|v_m|}\,\frac{1}{k-k^{-1}}\,\sqrt{|v_n|}\,\operatorname{sgn} v_n\psi_n \\ &= \frac{1}{k-k^{-1}}\Big(\sum_{n\in\mathbb{Z}} \sqrt{|v_n|}\,\operatorname{sgn} v_n\psi_n\Big)\sqrt{|v_m|} = \frac{1}{k-k^{-1}}\langle v_{1/2}, \psi\rangle\sqrt{|v_m|}. \end{aligned}$$

Lemma 3.2: Let $N \geq 0$. The complex function f given by

$$f(z) := \frac{z^N - 1}{z^{-1} - z}$$

satisfies

$$\sup_{z \in \mathbb{D}^+} |f(z)| \leq \frac{N}{\sqrt{2}},$$

where $\mathbb{D}^+ := \{z \in \mathbb{C} \mid |z| \leq 1, \ \mathrm{Re} z > 0\}$.

Proof. Let N be an even number, then there is some k such that $N = 2k$. Then we rewrite

$$f(z) = -z\frac{1-(z^2)^k}{1-z^2} = -z\sum_{i=0}^{k-1} z^{2i}$$

and estimate its absolute value on the unit disk

$$|f(z)| \leq |z|\sum_{i=0}^{k-1} |z|^{2i} \leq k = \frac{N}{2} \leq \frac{N}{\sqrt{2}}.$$

Though the last estimate is unnecessarily crude, we still take it since it leads to a consistent result with N being odd.

If, on the other hand, N is an odd number the situation is more complex. We write $N = 2k+1$. The function f then takes the form

$$f(z) = -z\frac{1-z^{2k+1}}{1-z^2} = -\frac{z}{1+z}\frac{1-z^{2k+1}}{1-z} = -\frac{z}{1+z}\sum_{i=0}^{2k} z^i.$$

Let us define

$$g(z) := \frac{z}{1+z}.$$

Since g is holomorphic on $\mathbb{D}^+$ and continuous up to the boundary, we shall use the maximum modulus principle to determine

$$\sup_{z \in \mathbb{D}^+} |g(z)|.$$

We need only to evaluate the supremum on the boundary, which we decompose into a line segment and a half circle.

A Let $z = \mathrm{i}t$, where $t \in (-1, 1)$. Then we have

$$\begin{aligned} g(\mathrm{i}t) &= \frac{\mathrm{i}t}{1+\mathrm{i}t} = t\frac{\mathrm{i}+t}{1+t^2}, \\ |g(\mathrm{i}t)|^2 &= t^2\frac{1+t^2}{(1+t^2)^2} = \frac{t^2}{1+t^2}, \\ \frac{\mathrm{d}}{\mathrm{d}t}|g(\mathrm{i}t)|^2 &= \frac{2t}{(1+t^2)^2}. \end{aligned}$$

The only root of the derivative is at $t = t$ where a minimum is attained. Thus, this yields

$$\sup_{t \in (-1,1)} |g(\mathrm{i}t)| = |g(\mathrm{i})| = \frac{1}{\sqrt{2}}.$$

B Let $z = e^{\mathrm{i}\theta}$, where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then we have

$$g(e^{\mathrm{i}\theta}) = \frac{e^{\mathrm{i}\theta}}{1+e^{\mathrm{i}\theta}},$$
$$|g(e^{\mathrm{i}\theta})|^2 = \frac{(\cos^2\theta + \cos\theta - \sin^2\theta)^2 + (\sin\theta + 2\cos\theta\sin\theta)^2}{(2+2\cos\theta)^2} = \frac{1}{2+2\cos\theta},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}|g(e^{\mathrm{i}\theta})|^2 = \frac{\sin\theta}{2(1+\cos\theta)}.$$

Again the only root of the derivative is at $\theta = 0$ where the minimum is attained. Thus, the supremum is at the endpoint

$$\sup_{\theta\in(-\pi/2,\pi/2)} |g(e^{\mathrm{i}\theta})| = |g(e^{\frac{\mathrm{i}\pi}{2}})| = \frac{1}{\sqrt{2}}.$$

With this established, we can estimate the absolute value of f on the set $\mathbb{D}^+$

$$|f(z)| = |g(z)| \left|\sum_{i=0}^{2k} z^i\right| \le \frac{1}{\sqrt{2}} \sum_{i=0}^{2k} |z|^i \le \frac{1}{\sqrt{2}} \sum_{i=0}^{2k} 1 = \frac{1}{\sqrt{2}}(2k+1) = \frac{N}{\sqrt{2}}.$$

□

Remark. The matrix entries of $M(\lambda)$ as functions of λ are not defined at $\lambda = 0, 2$, corresponding to $k = 0, 1$, as was shown in the proof of Lemma 3.2, these are, in fact, removable singularities. By continuous extensions, we may consider M to be defined on $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \mathrm{Re} z \ge 0\}$.

Proposition 3.3: Let $v \in \ell^1(\mathbb{Z}, m^2)$, then

$$\forall \lambda \in \mathbb{C}, \mathrm{Re}\lambda \ge 0 \;:\; \|M(\lambda)\| \le \sqrt{2}\|v\|_{\ell^1(\mathbb{Z},m^2)}.$$

Proof. We estimate the standard operator norm by the Hilbert–Schmidt norm, then we estimate using Lemma 3.2:

$$\begin{aligned}
\|M(\lambda)\|^2 \le \|M(\lambda)\|_{HS}^2 &= \sum_{m,n\in\mathbb{Z}} |M_{m,n}(\lambda)|^2 = \sum_{m,n\in\mathbb{Z}} |v_m| \left(\frac{k^{|m-n|}-1}{k-k^{-1}}\right)^2 |v_n| \\
&\le \sum_{m,n\in\mathbb{Z}} |v_m| \frac{|m-n|^2}{2} |v_n| \le \sum_{m,n\in\mathbb{Z}} |v_m|(m^2+n^2)|v_n| \\
&= \sum_{m,n\in\mathbb{Z}} m^2|v_m||v_n| + \sum_{m,n\in\mathbb{Z}} n^2|v_m||v_n| = 2\sum_{m,n\in\mathbb{Z}} m^2|v_m||v_n| \\
&= 2\sum_{m\in\mathbb{Z}} m^2|v_m| \sum_{n\in\mathbb{Z}} |v_n| \le 2\left(|v_0| + \sum_{n\in\mathbb{Z}} |v_n|n^2\right)^2 = 2\|v\|_{\ell^1(\mathbb{Z},m^2)}^2.
\end{aligned}$$

□

Lemma 3.4: Let $v \in \ell^1(\mathbb{Z}, m^2)$. Then there exists ε_1 for which all $\varepsilon \in (0, \varepsilon_1)$ satisfy

$$\lambda = k + k^{-1} \in \sigma_p(H_\varepsilon) \qquad \Longleftrightarrow \qquad k - k^{-1} + \varepsilon\left\langle \overline{v_{1/2}},\ \big(I + \varepsilon M(\lambda)\big)^{-1} |v|^{1/2} \right\rangle = 0,$$

where $H_\varepsilon := H_0 + \varepsilon V$.

Proof. To ensure the invertibility of $\big(I+\varepsilon M(\lambda)\big)$ we use Lemma 3.3 and choose $\varepsilon_1 > 0$ such that any $\varepsilon < \varepsilon_1$ satisfies $\varepsilon\sqrt{2}\|v\|_{\ell^1(\mathbb{Z},m^2)} < 1$. Then we can write

$$I + \varepsilon K(\lambda) = \big(I + \varepsilon M(\lambda)\big)\big(I + \big(I + \varepsilon M(\lambda)\big)^{-1}\varepsilon L(\lambda)\big).$$

With all of this stated, we arrive at three equivalent statements

$$-1 \in \sigma_p\big(\varepsilon K(\lambda)\big) \iff \lambda \in \sigma_d(H_\varepsilon) \iff \det\big(I + \big(I + \varepsilon M(\lambda)\big)^{-1}\varepsilon L(\lambda)\big) = 0.$$

If we denote

$$\begin{aligned} X_m &:= \sum_{n\in\mathbb{Z}} \big(I + M(\lambda)\big)^{-1}_{m,n} |v_n|^{1/2} \\ Y_m &:= (v_{1/2})_m, \end{aligned}$$

we can write

$$\big(I + M(\lambda)\big)^{-1}\varepsilon L(\lambda) = \frac{\varepsilon}{k - k^{-1}} XY^T$$

and easily find the spectrum of this rank-one operator

$$\sigma\Big(\frac{\varepsilon}{k-k^{-1}}XY^T\Big) = \Big\{\frac{\varepsilon}{k-k^{-1}}X^TY, 0\Big\}.$$

Since a rank-one operator is a trace class operator, we can use Theorem 1.16,

$$\det(I + A) = \prod_{\lambda\in\sigma(A)} (1 + \lambda),$$

to get

$$\det\big(I + \big(I + \varepsilon M(\lambda)\big)^{-1}\varepsilon L(\lambda)\big) = 1 + \frac{\varepsilon}{k-k^{-1}}X^TY = 1 + \frac{\varepsilon}{k-k^{-1}}\big\langle\, \overline{v_{1/2}},\ \big(I + \varepsilon M(\lambda)\big)^{-1}|v|^{1/2}\,\big\rangle.$$

From this, the assertion of the lemma clearly follows. □

Theorem 3.5: Let $v \in \ell^1(\mathbb{Z}, m^2)$ and denote $U := \sum_{n\in\mathbb{Z}} v_n$. Then $\exists\varepsilon_0 > 0$ such that $\forall\varepsilon \in (0, \varepsilon_0)$ we have

$$\begin{aligned} \mathrm{Re}U > 0 &\ : \sigma_d(H_\varepsilon) \cap B_{\varepsilon_0}(2) = \{\lambda_\varepsilon\}, \\ \mathrm{Re}U < 0 &\ : \sigma_d(H_\varepsilon) \cap B_{\varepsilon_0}(-2) = \{\lambda_\varepsilon\}, \end{aligned}$$

and the following asymptotic formula holds

$$\lambda_\varepsilon = 2\mathrm{sgn}\,(\mathrm{Re}U) + \frac{\varepsilon^2U^2}{4} + o(\varepsilon^2) \qquad \text{as} \qquad \varepsilon \to 0_+.$$

Proof. Unless stated otherwise, the proof will show only the case $\mathrm{Re}U > 0$ as the other case is analogous. Choose $\varepsilon_1 > 0$ such that any $\varepsilon < \varepsilon_1$ satisfies $\varepsilon\sqrt{2}\|v\|_{\ell^1(\mathbb{Z},m^2)} < 1$ and apply Lemma 3.4. Then we need only show that the equation

$$k - k^{-1} + \varepsilon\big\langle\, \overline{v_{1/2}},\ \big(I + \varepsilon M(\lambda)\big)^{-1}|v|^{1/2}\,\big\rangle = 0 \tag{3.1.1}$$

has a solution. Let us investigate the inner product more closely

$$\begin{aligned}\left\langle\, \overline{v_{1/2}},\ \big(I + \varepsilon M(\lambda)\big)^{-1} |v|^{1/2}\,\right\rangle &= \sum_{n=0}^{\infty} (-1)^n \varepsilon^n \left\langle\, \overline{v_{1/2}},\ M^n(\lambda)|v|^{1/2}\,\right\rangle \\ &= U + \sum_{n=1}^{\infty} (-1)^n \varepsilon^n \left\langle\, \overline{v_{1/2}},\ M^n(\lambda)|v|^{1/2}\,\right\rangle \\ &= U + \widetilde{U},\end{aligned}$$

where $\widetilde{U}$ is dependent on λ and ε. Now let us estimate the absolute value of $\widetilde{U}$ and in doing so evaluate its asymptotic behavior as $\varepsilon \to 0_+$,

$$|\widetilde{U}| \leq \|v\|_{\ell^1} \sum_{n=1}^{\infty} \varepsilon^n \|M(\lambda)\|^n = \|v\|_{\ell^1} \frac{\varepsilon\sqrt{2}\|v\|_{\ell^1(\mathbb{Z},m^2)}}{1 - \varepsilon\sqrt{2}\|v\|_{\ell^1(\mathbb{Z},m^2)}} \leq C\varepsilon = \mathcal{O}(\varepsilon) \quad \text{as} \quad \varepsilon \to 0_+,$$

where $C > 0$ is a constant independent on λ and ε. Now let us reparametrize k in term of a new variable z, i.e. $z \mapsto k(z)$, like so

$$k - k^{-1} = -\varepsilon U(1 + z).$$

This reparametrization is a bijective map between

$$\mathbb{D} \setminus \{0\} \quad \longleftrightarrow \quad \mathbb{C} \setminus \Big[\frac{2\mathrm{i}}{\varepsilon U} - 1,\ -\frac{2\mathrm{i}}{\varepsilon U} - 1\Big].$$

By taking the convex combination of the two endpoints of the line segment we can easily show that 0 is not an element of the line segment if and only if $\mathrm{Re}U \neq 0$ and so z may be 0. This parametrization greatly simplifies equation (3.1.1) like so

$$\begin{aligned}-\varepsilon U(1 + z) + \varepsilon(U + \widetilde{U}) &= 0 \\ -zU + \widetilde{U} &= 0.\end{aligned}$$

For later use of Rouché's theorem, we define two function

$$\begin{aligned}f(z) &:= -zU + \widetilde{U}, \\ g(z) &:= -zU.\end{aligned}$$

These are clearly holomorphic functions on the domain of z. Let us estimate the absolute value of their difference

$$|f(z) - g(z)| = |\widetilde{U}| < \varepsilon C = \delta|U| = |g(z)|$$

for $z \in \partial B_\delta(0)$ where $\delta := \frac{\varepsilon C}{|U|}$. To ensure the holomorphic property of f and g, $B_\delta(0)$ needs to lie within the domain of z, i.e.

$$\Big[\frac{2\mathrm{i}}{\varepsilon U} - 1,\ -\frac{2\mathrm{i}}{\varepsilon U} - 1\Big] \cap B_\delta(0) = \emptyset.$$

To achieve this, we further need to restrict ε such that

$$\delta < \mathrm{dist}\Big(0,\ \Big[\frac{2\mathrm{i}}{\varepsilon U} - 1,\ -\frac{2\mathrm{i}}{\varepsilon U} - 1\Big]\Big).$$

This is possible since the interval does not intersect the origin whenever $\mathrm{Re}U \neq 0$. From Rouché's theorem then follows that $\forall \varepsilon \in (0, \varepsilon_1)$ the functions f and g have the same number of zeros including multiplicity in $B_\delta(0)$. Clearly the function g has exactly one zero in this set and that is the origin. Let us denote the unique simple zero of the function f as $z_0 = z_0(\varepsilon)$. Clearly as ε approaches zero, δ also approaches zero, and so $z_0(\varepsilon) = o(1)$ as $\varepsilon \to 0_+$. From the bijective nature of the reparametrization we get a unique k_0 such that

$$k_0 - k_0^{-1} = -\varepsilon U\big(1 + z_0(\varepsilon)\big).$$

The right-hand side of the equation tends to zero as $\varepsilon \to 0_+$ and so $k_0 = k_0(\varepsilon) \in \mathbb{D} \setminus \{0\}$ is either in some neighborhood of 1, H_1, or some neighborhood of -1, H_{-1}, depending on the sign of $\mathrm{Re}U$. Taking the real part of the equation

$$\mathrm{Re}\big(k_0 - k_0^{-1}\big) = -\varepsilon \mathrm{Re}U + o(\varepsilon) \qquad \text{as } \ \varepsilon \to 0_+,$$

we can clearly see the dependence on the sign of $\mathrm{Re}U$

$$\begin{aligned} \mathrm{Re}U > 0 \qquad &: \qquad k_0(\varepsilon) \in \mathbb{D} \cap H_1 \\ \mathrm{Re}U < 0 \qquad &: \qquad k_0(\varepsilon) \in \mathbb{D} \cap H_{-1}. \end{aligned}$$

Let us finish the proof by deriving the asymptotic formula. Again we assume that $\mathrm{Re}U > 0$ the other case is analogous. Under this assumption $k_0(\varepsilon)$ is in some neighborhood of 1 and so

$$\lambda_\varepsilon = k_0(\varepsilon) + k_0^{-1}(\varepsilon) \in H_2 \setminus [-2, 2].$$

The asymptotic formula then follows from

$$\begin{aligned} k_0(\varepsilon) - k_0^{-1}(\varepsilon) &= -\varepsilon U(1 + o(\varepsilon)) \\ \lambda_\varepsilon^2 &= 4 + \varepsilon^2 U^2 + o(\varepsilon^2) \\ \lambda_\varepsilon &= 2 + \frac{\varepsilon^2 U^2}{4} + o(\varepsilon^2), \end{aligned}$$

where we squared the first equation and expressed λ_ε from $k_0(\varepsilon)$ as $\lambda^2 - 4 = (k - k^{-1})^2$, then we took the square root of the second equation and expanded the square root to the first degree. □

Remark. The approach using the Rouché's theorem allows us to assert only the local uniqueness of the eigenvalue. For a given potential with a positive sum, there exists a neighborhood of 2 within which there is exactly one eigenvalue. This approach is weaker in its assertion when compared to [16] but relaxes the assumption on the potential V which is no longer restricted to real values. The reason is the lack of variational min-max principle for non-self-adjoint operators.

3.2 Spectral stability of H_α

In this section, we prove that the operator H_α exhibits spectral stability for all $\alpha \neq 0$. This means that if we apply a V, under certain assumptions the spectrum remains the same. Moreover, there are no eigenvalues embedded in the essential spectrum, i.e. the purely continuous spectrum of H_α remains purely continuous for $H_\alpha + V$.

First, we need to establish some auxiliary results.

Lemma 3.6: Let $m, n \geq 0$. The complex function f given by

$$f(z) := \frac{z^{m+n} - z^{|m-n|}}{z^{-1} - z}$$

satisfies

$$\sup_{|z| \leq 1} |f(z)| = \frac{m+n-|m-n|}{2}.$$

Proof. The expression defining the function f is not well defined at $z \in \{0, \pm 1\}$, where removable singularities exist. We can rewrite f as

$$\frac{z^{m+n} - z^{|m-n|}}{z^{-1} - z} = -z^{|m-n|+1} \frac{1 - z^{m+n-|m-n|}}{1 - z^2} \stackrel{\star}{=} -z^{|m-n|+1} \frac{1 - z^{2k}}{1 - z^2} = -z^{|m-n|+1} \sum_{j=0}^{k-1} z^{2j}.$$

The marked equality holds because $(m + n)$ and $|m - n|$ have the same parity, meaning their difference is even. Thus, we can write $2k = m + n - |m - n|$ for some integer k. Taking the absolute value of f in this form and applying the triangle inequality, we obtain $|f(z)| \leq k$, for $|z| \leq 1$. Finally, we confirm that this bound is achieved at $z = 1$. Substituting $z = 1$ into the expression for $f(z)$ yields

$$f(1) = \frac{m+n-|m-n|}{2}.$$

□

Next, we use this lemma to derive a global estimate for the absolute values of the Green's kernel of H_α. Recall the global estimate $|w|^{-1} \leq C(\alpha)$, see (1.3.5).

Lemma 3.7: Let us define a doubly infinite matrix M by setting

$$M_{m,n} := \begin{cases} \frac{m+n-|m-n|}{2} + C(\alpha) & \text{if } m, n \geq 0, \\ C(\alpha) & \text{if } m \geq 0, n < 0, \\ \frac{-m-n-|m-n|}{2} + C(\alpha) & \text{if } m, n < 0, \\ C(\alpha) & \text{if } m < 0, n \geq 0. \end{cases}$$

Then

$$\forall \lambda \in \rho(H_\alpha),\ \forall m, n \in \mathbb{Z} \ : \quad |G_{m,n}(\lambda)| \leq M_{m,n}.$$

Proof. If we estimate the absolute value of the entries of $G(\lambda)$ as expressed in (2.1.11) by applying triangle inequality, using Lemmas 3.6 and 1.20, we obtain the matrix M as an upper bound. □

With the estimate of the Green's kernel established, we can estimate the norm of the Birman–Schwinger operator. Recall the notation for weighted ℓ^1-spaces, see Notation 1.6.

Proposition 3.8: Let $\alpha \neq 0$ and $v \in \ell^1(\mathbb{Z}, m^2)$. Then the following estimate holds

$$\forall \lambda \in \rho(H_\alpha) \ : \qquad \|K(\lambda)\| \leq (1 + C(\alpha)) \, \|v\|_{\ell^1(\mathbb{Z}, m^2)}.$$

Proof. Before we move to the calculation itself, let us first show some estimates that we will be using. For $m, n \geq 0$ we have

$$m + n - |m-n| \leq m+n \qquad \text{and} \qquad \frac{(m+n-|m-n|)^2}{4} \leq \frac{(m+n)^2}{4} \leq \frac{m^2+n^2}{2}.$$

Similarly, for $m, n < 0$ we estimate

$$-m - n - |m-n| \leq -m-n \qquad \text{and} \qquad \frac{(-m-n-|m-n|)^2}{4} \leq \frac{(-m-n)^2}{4} \leq \frac{m^2+n^2}{2}.$$

We will be calculating the Hilbert–Schmidt norm as it dominates the operator norm and it is easier to calculate. Further on, we will use the matrix M from Lemma 3.7 to estimate the absolute value of matrix entries of $G(\lambda)$. With this established, we calculate

$$\begin{aligned}\|K(\lambda)\|^2 \leq \|K(\lambda)\|_{HS}^2 &= \sum_{m,n\in\mathbb{Z}} \left| \sqrt{|v_m|}\, G_{m,n}(\lambda)\, \sqrt{|v_n|} \operatorname{sgn} v_n \right|^2 \\ &= \sum_{m,n\in\mathbb{Z}} |v_m| \left| G_{m,n} \right|^2 |v_n| \leq \sum_{m,n\in\mathbb{Z}} |v_m| M_{m,n}^2 |v_n| =: \sum_{m,n\in\mathbb{Z}} \widetilde{M}_{m,n}\end{aligned}$$

$$\|K(\lambda)\|^2 \leq \underbrace{\sum_{\substack{m\geq 0\\ n\geq 0}} \widetilde{M}_{m,n}}_{A} + \underbrace{\sum_{\substack{m\geq 0\\ n< 0}} \widetilde{M}_{m,n}}_{B} + \underbrace{\sum_{\substack{m< 0\\ n< 0}} \widetilde{M}_{m,n}}_{C} + \underbrace{\sum_{\substack{m< 0\\ n\geq 0}} \widetilde{M}_{m,n}}_{D}.$$

We estimate each of the sums separately:

$$\begin{aligned}A &= \sum_{\substack{m\geq 0\\ n\geq 0}} |v_m| \left(\frac{m+n-|m-n|}{2} + C(\alpha) \right)^2 |v_n| \\ &= \underbrace{\sum_{\substack{m\geq 0\\ n\geq 0}} \frac{(m+n-|m-n|)^2}{4} |v_m||v_n|}_{A_1} + \underbrace{\sum_{\substack{m\geq 0\\ n\geq 0}} C(\alpha)\big(m+n-|m-n|\big)|v_m||v_n|}_{A_2} + \underbrace{C(\alpha)^2 \sum_{m\geq 0} |v_m| \sum_{n\geq 0} |v_n|}_{A_3}\end{aligned}$$

$$\begin{aligned}A_1 &\leq \frac{1}{2} \sum_{\substack{m\geq 0\\ n\geq 0}} \big(m^2+n^2\big)|v_m||v_n| = \frac{1}{2}\Big(\sum_{m\geq 0} m^2 |v_m| \sum_{n\geq 0} |v_n| + \sum_{m\geq 0} |v_m| \sum_{n\geq 0} n^2 |v_n| \Big) \\ &= \sum_{m\geq 0} m^2|v_m| \sum_{n\geq 0} |v_n| \\ A_2 &\leq C(\alpha) \sum_{\substack{m\geq 0\\ n\geq 0}} \big(m+n\big)|v_m||v_n| = C(\alpha)\Big(\sum_{m\geq 0} m|v_m| \sum_{n\geq 0} |v_n| + \sum_{m\geq 0} |v_m| \sum_{n\geq 0} n|v_n| \Big) \\ &= 2C(\alpha) \sum_{m\geq 0} m|v_m| \sum_{m\geq 0} |v_n|\end{aligned}$$

$$C=\sum_{\substack{m<0\\n<0}}|v_m|\left(\frac{-m-n-|m-n|}{2}+C(\alpha)\right)^2|v_m|=\underbrace{\sum_{\substack{m<0\\n<0}}\frac{(-m-n+|m-n|)^2}{4}|v_m||v_n|}_{C_1}$$

$$+\underbrace{\sum_{\substack{m<0\\n<0}}C(\alpha)\big(-m-n-|m-n|\big)|v_m||v_n|}_{C_2}+\underbrace{C(\alpha)^2\sum_{m<0}|v_m|\sum_{n<0}|v_n|}_{C_3}$$

$$\begin{aligned}C_1&\le\frac{1}{2}\sum_{\substack{m<0\\n<0}}\big(m^2+n^2\big)|v_m||v_n|=\frac{1}{2}\Big(\sum_{m<0}m^2|v_m|\sum_{n<0}|v_n|+\sum_{m<0}|v_m|\sum_{n<0}n^2|v_n|\Big)\\&=\sum_{m<0}m^2|v_m|\sum_{n<0}|v_n|\\C_2&\le C(\alpha)\sum_{\substack{m<0\\n<0}}\big(-m-n\big)|v_m||v_n|=C(\alpha)\Big(\sum_{m<0}|m||v_m|\sum_{n<0}|v_n|+\sum_{m<0}|v_m|\sum_{n<0}|n||v_n|\Big)\\&=2C(\alpha)\sum_{m<0}|m||v_m|\sum_{n<0}|v_n|\end{aligned}$$

$$B=\sum_{\substack{m\ge0\\n<0}}|v_m|C(\alpha)^2|v_n|=C(\alpha)^2\sum_{m\ge0}|v_m|\sum_{n<0}|v_n|\qquad D=\sum_{\substack{m<0\\n\ge0}}|v_m|C(\alpha)^2|v_n|=C(\alpha)^2\sum_{m<0}|v_m|\sum_{n\ge0}|v_n|.$$

$$\begin{aligned}&A_3+B+C_3+D\\&\quad=C(\alpha)^2\Big(\sum_{m\ge0}|v_m|\sum_{n\ge0}|v_n|+\sum_{m\ge0}|v_m|\sum_{n<0}|v_n|+\sum_{m<0}|v_m|\sum_{n<0}|v_n|+\sum_{m<0}|v_m|\sum_{n\ge0}|v_n|\Big)\\&\quad=C(\alpha)^2\Big(\sum_{m\ge0}|v_m|\sum_{n\in\mathbb{Z}}|v_n|+\sum_{m<0}|v_m|\sum_{n\in\mathbb{Z}}|v_n|\Big)=C(\alpha)^2\sum_{n\in\mathbb{Z}}|v_n|\sum_{m\in\mathbb{Z}}|v_m|=C(\alpha)^2\|v\|^2_{\ell^1(\mathbb{Z})},\end{aligned}$$

$$\begin{aligned}A_1+C_1&\le\sum_{m\ge0}m^2|v_m|\sum_{n\ge0}|v_n|+\sum_{m<0}m^2|v_m|\sum_{n<0}|v_n|\\&\le\sum_{m\ge0}m^2|v_m|\sum_{n\in\mathbb{Z}}|v_n|+\sum_{m<0}m^2|v_m|\sum_{n\in\mathbb{Z}}|v_n|=\sum_{m\in\mathbb{Z}}m^2|v_m|\sum_{n\in\mathbb{Z}}|v_n|=\|v\|_{\ell^1(\mathbb{Z},m^2)}\|v\|_{\ell^1(\mathbb{Z})},\end{aligned}$$

$$\begin{aligned}A_2+C_2&\le2C(\alpha)\Big(\sum_{m\ge0}m|v_m|\sum_{n\ge0}|v_n|+\sum_{m<0}|m||v_m|\sum_{n<0}|v_n|\Big)\\&\le2C(\alpha)\Big(\sum_{m\ge0}m|v_m|\sum_{n\in\mathbb{Z}}|v_n|+\sum_{m<0}|m||v_m|\sum_{n\in\mathbb{Z}}|v_n|\Big)\\&=2C(\alpha)\sum_{m\in\mathbb{Z}}|m||v_m|\sum_{n\in\mathbb{Z}}|v_n|=2C(\alpha)\|v\|_{\ell^1(\mathbb{Z},m^1)}\|v\|_{\ell^1(\mathbb{Z})}.\end{aligned}$$

Adding everything together we get the assertion of the proposition.

$$\begin{aligned}\|K(\lambda)\|^2 &\leq C(\alpha)^2\|v\|^2_{\ell^1(\mathbb{Z})} + \|v\|_{\ell^1(\mathbb{Z},m^2)}\|v\|_{\ell^1(\mathbb{Z})} + 2C(\alpha)\|v\|_{\ell^1(\mathbb{Z},m^1)}\|v\|_{\ell^1(\mathbb{Z})} \\ &= \|v\|_{\ell^1(\mathbb{Z})}\Big(C(\alpha)^2\|v\|_{\ell^1(\mathbb{Z})} + \|v\|_{\ell^1(\mathbb{Z},m^2)} + 2C(\alpha)\|v\|_{\ell^1(\mathbb{Z},m^1)}\Big) \\ &\leq \|v\|_{\ell^1(\mathbb{Z},m^2)}\Big(C(\alpha)^2\|v\|_{\ell^1(\mathbb{Z},m^2)} + \|v\|_{\ell^1(\mathbb{Z},m^2)} + 2C(\alpha)\|v\|_{\ell^1(\mathbb{Z},m^1)}\Big) \\ &= (1 + C(\alpha))^2\,\|v\|^2_{\ell^1(\mathbb{Z},m^2)}\end{aligned}$$

The last estimate is unnecessarily rough but as we have mentioned above our objective here is not optimality. At this point, we prefer simpler expressions to finer though more complicated upper bounds. □

The following result will be used to prove that there are no embedded eigenvalues in the essential spectrum.

Proposition 3.9: Assume $\alpha \neq 0$ and $v \in \ell^1(\mathbb{Z})$. Let $\lambda \in \sigma(H_\alpha)$ be an eigenvalue of $H_\alpha + V$, i.e. there exists a non-zero vector $u \in \ell^2(\mathbb{Z})$ satisfying $(H_\alpha + V)u = \lambda u$. Then

$$\forall m \in \mathbb{Z} \quad : \quad \big(\big(I + K(\lambda + \mathrm{i}\varepsilon)\big)V_{1/2}u\big)_m \longrightarrow 0 \qquad \text{as } \varepsilon \to 0_+.$$

Proof. Choose an $m \in \mathbb{Z}$ fixed. The eigenvalue equation implies $v_m u_m = \lambda u_m - (H_\alpha u)_m$. Then we have

$$\begin{aligned}\big(K(\lambda + \mathrm{i}\varepsilon)V_{1/2}u\big)_m = \big(V_{1/2}(H_\alpha - \lambda - \mathrm{i}\varepsilon)^{-1}Vu\big)_m &= \operatorname{sgn} v_m\sqrt{|v_m|}\sum_{n\in\mathbb{Z}} G_{m,n}(\lambda + \mathrm{i}\varepsilon)v_n u_n \\ &= \operatorname{sgn} v_m\sqrt{|v_m|}\sum_{n\in\mathbb{Z}} G_{m,n}(\lambda + \mathrm{i}\varepsilon)\big(\lambda u_m - (H_\alpha u)_m\big).\end{aligned}$$

Let us investigate only a part of the previous expression

$$\begin{aligned}\sum_{n\in\mathbb{Z}} G_{m,n}(\lambda + \mathrm{i}\varepsilon)v_n u_n &= \sum_{n\in\mathbb{Z}} G_{m,n}(\lambda + \mathrm{i}\varepsilon)\big(\lambda u_m + \mathrm{i}\varepsilon u_m - (H_\alpha u)_m\big) - \mathrm{i}\varepsilon\sum_{n\in\mathbb{Z}} G_{m,n}(\lambda + \mathrm{i}\varepsilon)u_m \\ &= \Big\langle \overline{G_{m,\cdot}(\lambda + \mathrm{i}\varepsilon)}, (\lambda + \mathrm{i}\varepsilon - H_\alpha)u\Big\rangle - \mathrm{i}\varepsilon\sum_{n\in\mathbb{Z}} G_{m,n}(\lambda + \mathrm{i}\varepsilon)u_m \\ &= \Big\langle \overline{(H_\alpha - \lambda - \mathrm{i}\varepsilon)G_{m,\cdot}(\lambda + \mathrm{i}\varepsilon)}, u\Big\rangle - \mathrm{i}\varepsilon\sum_{n\in\mathbb{Z}} G_{m,n}(\lambda + \mathrm{i}\varepsilon)u_m \\ &= -\langle e_m, u\rangle - \mathrm{i}\varepsilon\sum_{n\in\mathbb{Z}} G_{m,n}(\lambda + \mathrm{i}\varepsilon)u_m \\ &= -u_m - \mathrm{i}\varepsilon\Big\langle \overline{G_{m,\cdot}(\lambda + \mathrm{i}\varepsilon)},\ u\Big\rangle.\end{aligned}$$

Now we plug this result back into the previous expression and obtain

$$\big(\big(I + K(\lambda + \mathrm{i}\varepsilon)\big)V_{1/2}u\big)_m = -\mathrm{i}\varepsilon\operatorname{sgn} v_m\sqrt{|v_m|}\Big\langle \overline{G_{m,\cdot}(\lambda + \mathrm{i}\varepsilon)},\ u\Big\rangle.$$

If we show that the right hand side approaches vanishes as $\varepsilon \to 0_+$, the proof will be complete. To this end we estimate the absolute value of the right hand side of the expression above using the Cauchy–Schwarz inequality

$$\Big|-\mathrm{i}\varepsilon\operatorname{sgn} v_m\sqrt{|v_m|}\,\langle G_{m,\cdot}(\lambda + \mathrm{i}\varepsilon),\ u\rangle\Big| \leq \varepsilon\sqrt{|v_m|}\|G_{m,\cdot}(\lambda + \mathrm{i}\varepsilon)\|\|u\|.$$

The proof is complete if we show that

$$\varepsilon\|G_{m,\cdot}(\lambda+\mathrm{i}\varepsilon)\| \to 0 \qquad \text{as } \varepsilon \to 0_+. \tag{3.2.1}$$

Recall that we denoted $\widetilde{G} = wG$ and that from (1.3.5) we have $|w|^{-1} \leq C(\alpha)$. For the calculation of the norm $\|G_{m,\cdot}(\lambda+\mathrm{i}\varepsilon)\|$ the formulation (2.1.10) is the most suitable. ξ and η are the Joukowsky parameters of $\lambda+\mathrm{i}\varepsilon$. To ensure that $\lambda+\mathrm{i}\varepsilon$ is always in the resolvent set of H_α we assume that $\varepsilon < |\mathrm{Im}\alpha|$. This assumption necessitates that ξ and η are unique and less than 1 in absolute value. We have

$$\begin{aligned}\|\widetilde{G}_{m,\cdot}(\lambda+\mathrm{i}\varepsilon)\|^2 &= \sum_{n\in\mathbb{Z}} \left|\widetilde{G}_{m,n}(\lambda+\mathrm{i}\varepsilon)\right|^2 \\ &= \begin{cases} \sum_{n=0}^{\infty} \left|\frac{\xi-\eta}{\eta^{-1}-\eta}\eta^{m+n} - \frac{\xi-\eta^{-1}}{\eta-\eta^{-1}}\eta^{|m-n|}\right|^2 + \sum_{n=1}^{\infty} |\eta^m\xi^n|^2 & m \geq 0 \\ \sum_{n=1}^{\infty} |\eta^n\xi^{-m}|^2 + \sum_{n=1}^{\infty} \left|\frac{\eta^{-1}-\xi}{\xi^{-1}-\xi}\xi^{|m+n|} - \frac{\eta^{-1}-\xi^{-1}}{\xi-\xi^{-1}}\eta^{n-m}\right|^2 & m < 0. \end{cases}\end{aligned}$$

Let us estimate both expressions from above. For $m \geq 0$ we have

$$\begin{aligned}\|\widetilde{G}_{m,\cdot}(\lambda+\mathrm{i}\varepsilon)\|^2 &\leq 2|\eta|^{2m}\frac{|\xi-\eta|^2}{|\eta^{-1}-\eta|^2}\sum_{n=0}^{\infty}|\eta|^{2n} + 2\frac{|\xi-\eta^{-1}|^2}{|\eta-\eta^{-1}|^2}\sum_{n=-m}^{\infty}|\eta|^{2|n|} + |\eta|^{2m}\sum_{n=1}^{\infty}|\xi|^{2n} \\ &\leq 2\frac{4}{|\eta^{-1}-\eta|^2}\frac{1}{1-|\eta|^2} + 2\frac{2+2|\eta|^{-2}}{|\eta-\eta^{-1}|^2}\frac{2}{1-|\eta|^2} + \frac{1}{1-|\xi|^2}.\end{aligned} \tag{3.2.2}$$

And for $m < 0$ we estimate

$$\begin{aligned}\|\widetilde{G}_{m,\cdot}(\lambda+\mathrm{i}\varepsilon)\|^2 &\leq |\xi|^{-2m}\sum_{n=1}^{\infty}|\eta|^{2n} + 2\frac{|\eta^{-1}-\xi|^2}{|\xi^{-1}-\xi|^2}\sum_{n=1+m}^{\infty}\xi^{2|n|} + 2|\eta|^{-2m}\frac{|\eta^{-1}-\xi^{-1}|^2}{|\xi-\xi^{-1}|^2}\sum_{n=1}^{\infty}|\eta|^{2n} \\ &\leq \frac{1}{1-|\eta|^2} + 2\frac{2|\eta|^{-2}+2}{|\xi^{-1}-\xi|^2}\frac{2}{1-|\xi|^2} + 2\frac{2|\eta|^{-2}+2|\xi|^{-2}}{|\xi-\xi^{-1}|^2}\frac{1}{1-|\xi|^2}.\end{aligned} \tag{3.2.3}$$

For the remaining expression we need to derive asymptotic formulas as $\varepsilon \to 0_+$.

Recall the asymptotic expansion of the Joukowsky parameter from Lemmas 1.22 and 1.24. Corollaries 1.23 and 1.25 that followed yielded the asymptotic expansions for $|k|^2$ and $|k-k^{-1}|^2$. We apply these corollaries separately on ξ and η.

Let $\lambda \in (-2,2)$, then

$$|\xi|^{-2} = \mathcal{O}(1), \qquad \frac{1}{1-|\xi|^2} = \mathcal{O}\left(\varepsilon^{-1}\right), \qquad \frac{1}{|\xi-\xi^{-1}|^2} = \mathcal{O}(1) \qquad \text{as } \varepsilon \to 0_+.$$

Let $\lambda \in (-2+\alpha, 2+\alpha)$, then

$$|\eta|^{-2} = \mathcal{O}(1), \qquad \frac{1}{1-|\eta|^2} = \mathcal{O}\left(\varepsilon^{-1}\right), \qquad \frac{1}{|\eta-\eta^{-1}|^2} = \mathcal{O}(1) \qquad \text{as } \varepsilon \to 0_+.$$

For the endpoints we have

$$\begin{aligned}\lambda \in \{2,-2\} \quad &: \quad |\xi|^{-2} = \mathcal{O}(1), \quad \frac{1}{1-|\xi|^2} = \mathcal{O}\left(\varepsilon^{-1/2}\right), \qquad \frac{1}{|\xi-\xi^{-1}|^2} = \mathcal{O}\left(\varepsilon^{-1}\right), \\ \lambda \in \{2+\alpha, 2+\alpha\} \quad &: \quad |\eta|^{-2} = \mathcal{O}(1), \quad \frac{1}{1-|\eta|^2} = \mathcal{O}\left(\varepsilon^{-1/2}\right), \qquad \frac{1}{|\eta-\eta^{-1}|^2} = \mathcal{O}\left(\varepsilon^{-1}\right).\end{aligned}$$

as $\varepsilon \to 0_+$. The remaining cases obey

$$\begin{aligned} \lambda \notin [-2,2] \quad &: \quad \xi \to c_1 \quad \text{as } \varepsilon \to 0_+, \quad \text{where } |c_1| \in (0,1), \\ \lambda \notin [-2+\alpha, 2+\alpha] \quad &: \quad \eta \to c_2 \quad \text{as } \varepsilon \to 0_+, \quad \text{where } |c_2| \in (0,1). \end{aligned}$$

The desired expressions follow

$$\begin{aligned} \lambda \notin [-2,2] \quad &: \quad |\xi|^{-2} = \mathcal{O}(1), \qquad \frac{1}{1-|\xi|^2} = \mathcal{O}(1), \qquad \frac{1}{|\xi - \xi^{-1}|^2} = \mathcal{O}(1), \\ \lambda \notin [-2+\alpha, 2+\alpha] \quad &: \quad |\eta|^{-2} = \mathcal{O}(1), \qquad \frac{1}{1-|\eta|^2} = \mathcal{O}(1), \qquad \frac{1}{|\eta - \eta^{-1}|^2} = \mathcal{O}(1) \end{aligned}$$

as $\varepsilon \to 0_+$.

Recall that $wG = \widetilde{G}$, and so $\|G(\lambda + \mathrm{i}\varepsilon)\| \leq C(\alpha)\|\widetilde{G}(\lambda + \mathrm{i}\varepsilon)\| = \mathcal{O}(1)\,\|\widetilde{G}(\lambda + \mathrm{i}\varepsilon)\|$ as $\varepsilon \to 0_+$. By line segments we mean the two line segments $[-2,2]$ and $[-2+\alpha, 2+\alpha]$. Even though that in the case that $\alpha \in [-4,4]$ they combine into one large line segment, it is useful to view them as two separate line segments. In the remaining part of the proof, we break down every case we need to properly apply the asymptotic expansions and evaluate the asymptotic behavior of $\|G_{m,\cdot}(\lambda + \mathrm{i}\varepsilon)\|$. For $m \in \mathbb{Z}$ we have

$$\begin{aligned} A \quad &: \quad \|G_{m,\cdot}(\lambda + \mathrm{i}\varepsilon)\| = \mathcal{O}\left(\varepsilon^{-1/2}\right), \\ B \quad &: \quad \|G_{m,\cdot}(\lambda + \mathrm{i}\varepsilon)\| = \mathcal{O}\left(\varepsilon^{-3/4}\right), \end{aligned} \tag{3.2.4}$$

where

A: λ is not an endpoint of either line segment. We need to identify three distinct sub-cases

$$\begin{aligned} &\lambda \in (-2,2) \cup (-2+\alpha, 2+\alpha), \\ &\lambda \in (-2,2) \setminus [-2+\alpha, 2+\alpha], \\ &\lambda \in (-2+\alpha, 2+\alpha) \setminus [-2,2]. \end{aligned}$$

B: λ is an endpoint of at least on of the line segments. Again we identify certain sub-cases

$$\begin{aligned} &\lambda \in \{2, -2+\alpha\} && \text{for } \alpha \in (0,4), \\ &\lambda \in \{-2, +2+\alpha\} && \text{for } \alpha \in (-4,0), \\ &\lambda = \pm 4 && \text{for } \alpha = \pm 4, \\ &\lambda \in \{-2, 2, -2+\alpha, 2+\alpha\} && \text{for } \alpha \notin [-4,4]. \end{aligned}$$

Each of these sub-cases corresponds with a certain set of asymptotic formulas for the expressions. We plug them into (3.2.3) and (3.2.2) and obtain (3.2.4) regardless of m. This concludes the proof because it implies (3.2.1). □

Now we have everything necessary established to formulate and prove the main result of this section.

Theorem 3.10: Let $\alpha \neq 0$ and $v \in \ell^1(\mathbb{Z}, m^2)$. If we take the sequence v such that

$$(1 + C(\alpha))\,\|v\|_{\ell^1(\mathbb{Z},m^2)} < 1,$$

then the potential V does not change the spectrum and there are no embedded eigenvalues, i.e.

$$\sigma(H_\alpha) = \sigma(H_\alpha + V) = \sigma_{\mathrm{c}}(H_\alpha + V).$$

Proof. The interior of $\sigma(H_\alpha)$ is clearly empty in the topology of $\mathbb{C}$ and $\rho(H_\alpha)$ has one connected component which has a non-empty intersection with $\rho(H_\alpha + V)$ since $H_\alpha + V$ is bounded. The operator V is compact. This satisfies the assumptions of Theorem 1.10 and is states that

$$\sigma_{\mathrm{ess}}(H_\alpha) = \sigma_{\mathrm{ess}}(H_\alpha + V). \tag{3.2.5}$$

Denote $\big(1 + C(\alpha)\big)\|v\|_{\ell^1(\mathbb{Z},m^2)} =: q < 1$. If we estimate the norm of the Birman–Schwinger operator $K(\lambda)$ for $\lambda \in \rho(H_\alpha)$ according to Lemma 3.8, we get

$$\|K(\lambda)\| \leq q < 1. \tag{3.2.6}$$

This rules out the possibility of -1 being an eigenvalue of $K(\lambda)$. Let us reiterate the Birman–Schwinger principle (Theorem 1.13): For $\lambda \in \rho(H_\alpha)$ we have

$$\lambda \in \sigma_{\mathrm{p}}(H_\alpha + V) \qquad \Longleftrightarrow \qquad -1 \in \sigma_{\mathrm{p}}(K(\lambda)).$$

Therefore, $\lambda \in \rho(H_\alpha) \implies \lambda \notin \sigma_{\mathrm{p}}(H_\alpha + V)$. From (3.2.5) follows that in $\rho(H_\alpha)$ the operator $H_\alpha + V$ can have only eigenvalues. We then conclude that

$$\sigma(H_\alpha + V) \cap \rho(H_\alpha) = \emptyset.$$

Thus $\sigma(H_\alpha) = \sigma(H_\alpha + V)$ since the essential spectra are the same.

To show that the spectrum of $H_\alpha + V$ remain purely continuous, we use Proposition 3.9. Let $\lambda \in \sigma(H_\alpha)$. Assume $V \neq 0$ as the theorem clearly holds for $V = 0$. For a contradiction, suppose $\lambda \in \sigma_{\mathrm{p}}(H_\alpha + V)$. By the definition of an eigenvalue, there exists a non-zero vector $u \in \ell^2(\mathbb{Z})$ such that $(H_\alpha + V)u = \lambda u$. Then Proposition 3.9 states that

$$\forall m \in \mathbb{Z} \qquad : \qquad \big\langle e_n,\ (I + K(\lambda + \mathrm{i}\varepsilon))V_{1/2}u\big\rangle \longrightarrow 0 \qquad \text{as } \varepsilon \to 0_+.$$

Hence,

$$\forall x \in \ell^2(\mathbb{Z}) \qquad : \qquad \big\langle x,\ K(\lambda + \mathrm{i}\varepsilon)V_{1/2}u\big\rangle \longrightarrow -\big\langle x,\ V_{1/2}u\big\rangle \qquad \text{as } \varepsilon \to 0_+.$$

Notice that $V_{1/2}u \neq 0$ otherwise $\lambda \in \sigma_{\mathrm{p}}(H_\alpha) = \emptyset$. If we plug in $x = V_{1/2}u$ and take the absolute value, we get

$$\lim_{\varepsilon \to 0_+} \big|\big\langle V_{1/2}u,\ K(\lambda + \mathrm{i}\varepsilon)V_{1/2}u\big\rangle\big| = \big\|V_{1/2}u\big\|^2.$$

On the other hand, if we take the Cauchy–Schwarz inequality first and then plug in the estimate (3.2.6), we arrive at

$$\lim_{\varepsilon \to 0_+} \big|\big\langle V_{1/2}u,\ K(\lambda + \mathrm{i}\varepsilon)V_{1/2}u\big\rangle\big| \leq \lim_{\varepsilon \to 0_+} \|K(\lambda + \mathrm{i}\varepsilon)\| \ \big\|V_{1/2}u\big\|^2 \leq q \, \big\|V_{1/2}u\big\|^2.$$

This is a contradiction since $q < 1$. Therefore, $\lambda \in \sigma(H_\alpha) \implies \lambda \notin \sigma_{\mathrm{p}}(H_\alpha + V)$. Hence, the point spectrum of $H_\alpha + V$ is empty. From (3.2.5) it follows that $\lambda \in \sigma(H_\alpha) \implies \lambda \in \sigma_{\mathrm{c}}(H_\alpha + V) \cup \sigma_{\mathrm{r}}(H_\alpha + V)$.

Recall the decomposition (2.0.2). Since the point spectrum of $H_\alpha + V$ is empty, Theorem 2.2 asserts that the residual spectrum of $H_\alpha + V$ is empty. In Theorem 2.2, we set $A = H_0$ and $B = \alpha D + V$ which satisfy the assumptions of the theorem.

In conclusion, the potential V introduces no eigenvalues and the essential spectrum remains the same and purely continuous. $\square$

Conclusion

This master's thesis is a direct continuation of the author's Bachelor's degree project, see [1], and Research project, see [2]. We built upon the results obtained in those works. We introduced the Joukowsky transform and further extended and proved additional assertions. The content of the first chapter consisted also of some standard results from functional analysis and spectral theory.

In the second chapter, we reiterated the basic spectral analysis of H_α found in [1]. The bulk of this chapter was devoted to the pseudospectral analysis of H_α. The ε-pseudospectra are strictly nested supersets of the spectrum where the resolvent operator's norm is large. After we mentioned the trivial case of self-adjoint operators, we showed estimates for $\|(H_\alpha - \lambda)^{-1}\|$ from above and below. The Schur test served as the primary tool for obtaining upper bounds, for which we provided a formulation in $\ell^2(\mathbb{Z})$. The lower estimate was obtained by a general estimation technique utilizing the distance from the numerical range. Using these estimates we constructed a subset and a superset of the ε-pseudospectrum on the region of the complex plane where we do not have general mathematical tools to describe it exactly. Asymptotic formulas for the estimates of the resolvent operator's norm were also given. We have also described the spectral measure of H_α for $\mathrm{Im}\alpha = 0$.

The last chapter contains the main result. In [16], the existence and uniqueness of weakly-coupled bound states were described for the operator $-\mathrm{d}^2/\mathrm{d}x^2 + \varepsilon V$. The authors demonstrated that if V is integrable with the weight $(1 + x^2)\mathrm{d}x$ and ε is sufficiently small, the aforementioned operator has at most one eigenvalue. Furthermore, this eigenvalue corresponds to a solution of a specific algebraic equation. For all values of ε, an eigenvalue exists if and only if the mean value of V is non-positive. We demonstrated that similar assertions hold true in the discrete setting. Specifically, the operator $H_0 + \varepsilon V$ has at most one eigenvalue in small neighborhoods of 2 or -2 depending on the sign of the sum of V when the potential is summable with quadratic weight and ε is sufficiently small. However, the approach used, which asserts only local uniqueness of the eigenvalue, allowed us to generalize the result to complex potentials.

We showed a similar behavior regarding spectral stability between the continuous Schrödinger operator with a complex step potential, as described in [8], and its discrete counterpart that we studied. When the complex step potential added to H_α is summable with quadratic weight and sufficiently small, we provided proof that it does not generate any eigenvalues. Moreover, the spectrum remains purely continuous. In other words, the operator H_α exhibits complete spectral stability. The authors of [8], under similar assumptions imposed on the potential, showed that the continuous operator H may not have eigenvalues outside the closure of the numerical range. Furthermore, the smaller the potential, the greater the distance any eigenvalues must be from the origin. The primary tool for these results in weak-coupling analysis and spectral stability was the Birman–Schwinger principle.

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