Spectral bounds for 1D discrete Schrödinger and Dirac operators with complex potentials

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- The discrete Schrödinger operator: $\boxed{H_V = H_0 + V,}$

$$H_V = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & 1 & v_{-1} & 1 & & & \\ & & 1 & v_0 & 1 & & \\ & & & 1 & v_1 & 1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

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- The resolvent of H_0:
$$(H_0 - \lambda)^{-1}_{m,n} = \frac{k^{|m-n|}}{k - k^{-1}}, \quad \forall m, n \in \mathbb{Z},$$
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- The Joukowski map:
$$\lambda(k) = k^{-1} + k$$
is 1–1 mapping of the punctured unit disk $0 < |k| < 1$ onto $\mathbb{C} \setminus [-2, 2]$.

ℓ^1-potentials

Theorem (ℓ^1-potential)

Let $v \in \ell^1(\mathbb{Z})$. Then

$$\sigma_{\mathrm{p}}(H_V) \subset \left\{ \lambda \in \mathbb{C} \setminus (-2,2) \;\middle|\; |\lambda^2 - 4| \leq \|v\|^2_{\ell^1(\mathbb{Z})} \right\}.$$

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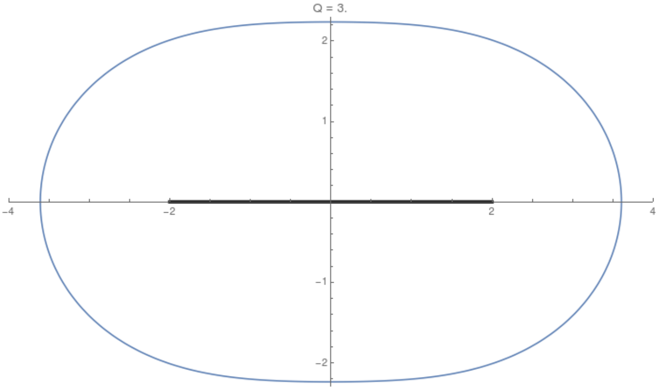
In addition, the estimate is optimal in the following sense:

To any boundary point of the spectral enclosure which does not belong to $(-2,2)$, there exists an ℓ^1–potential V so that this boundary point is an eigenvalue of the corresponding discrete Schrödinger operator H_V.

Geometry of the boundary curve

The boundary curve for $Q := \|v\|_{\ell^1(\mathbb{Z})}$: $$|\lambda^2 - 4| = Q^2.$$



Proof

- The goal is to prove:

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- For $\lambda \notin [-2,2] \equiv \sigma(H_0)$, the proof relies on the Birman–Schwinger principle (one implication):

$$\lambda \in \sigma_p(H_V) \implies -1 \in \sigma_p(K(\lambda)),$$

for

$$K(\lambda) := |V|^{1/2} (H_0 - \lambda)^{-1} V_{1/2},$$

and

$$|V|^{1/2} e_n = \sqrt{|v_n|} e_n \quad \text{and} \quad V_{1/2} e_n = \operatorname{sgn}(v_n) \sqrt{|v_n|} e_n$$

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with the complex signum function $\operatorname{sgn} z = z/|z|$, if $z \neq 0$, and $\operatorname{sgn} 0 = 0$.
- In particular,

$$\lambda \in \sigma_p(H_V) \implies \|K(\lambda)\| \geq 1.$$

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$$\left|(H_0 - \lambda)^{-1}_{m,n}\right| = \frac{|k|^{|m-n|}}{|k - k^{-1}|} \leq \frac{1}{|k - k^{-1}|} = \frac{1}{\sqrt{|\lambda^2 - 4|}}, \quad \forall m, n \in \mathbb{Z}.$$

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- For any $\psi \in \ell^2(\mathbb{Z})$, we estimate

$$\|K(\lambda)\psi\|^2_{\ell^2(\mathbb{Z})} \leq \sum_{m\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} \sqrt{|\upsilon_m|} \left|(H_0 - \lambda)^{-1}_{m,n}\right| \sqrt{|\upsilon_n|}|\psi_n| \right)^2$$
$$\leq \frac{\|\upsilon\|_{\ell^1(\mathbb{Z})}}{|\lambda^2 - 4|} \left(\sum_{m\in\mathbb{Z}} \sqrt{|\upsilon_n|}|\psi_n| \right)^2 \leq \frac{\|\upsilon\|^2_{\ell^1(\mathbb{Z})}}{|\lambda^2 - 4|} \|\psi\|^2_{\ell^2(\mathbb{Z})}.$$

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$$\|K(\lambda)\|^2 \leq \frac{\|\upsilon\|^2_{\ell^1(\mathbb{Z})}}{|\lambda^2 - 4|}$$

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- In other words,

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- Thus, if $\lambda \in \sigma_p(H_V)$, then $\boxed{|\lambda^2 - 4| \leq \|\upsilon\|^2_{\ell^1(\mathbb{Z})}.}$

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An incomplete list of authors:

Abramov, Aslanyan, Behrndt, Cuenin, Davies, Enblom, Frank, Fanelli, Ibrogimov, Krej–čiřík, Langer, Laptev, Lee, Lieb, Lotoreichik, Rohleder, Safronov, Seiringer, Seo, Tretter, Vega,...

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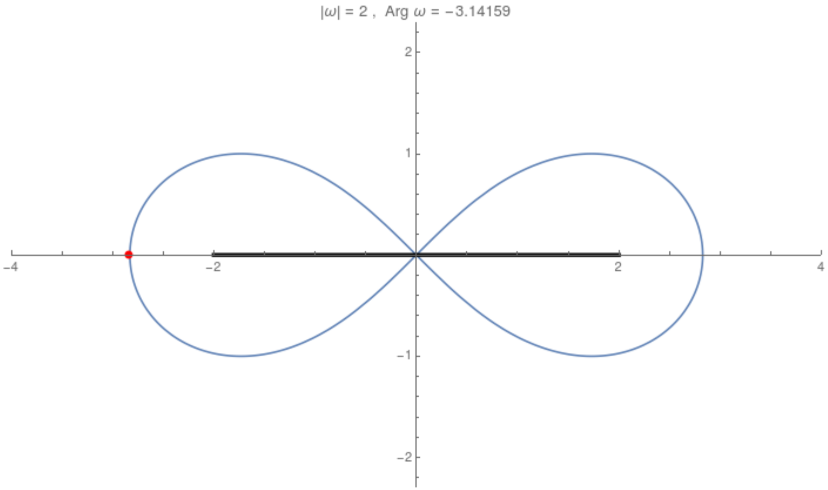
- Moreover, for any $Q > 0$, one has

$$\{\lambda_\omega \mid \omega = Qe^{\mathrm{i}\theta}, -\pi < \theta \leq \pi\} = \{\lambda \in \mathbb{C} \mid |\lambda^2 - 4| = Q^2\}.$$

Numerical illustration: the delta potential demonstrates optimality



ℓ^p-potentials, $p > 1$

Theorem (ℓ^p-potential)

Let $1 < p \leq \infty$ and $v \in \ell^p(\mathbb{Z})$. Denote by $q \in [1, \infty)$ the corresponding Hölder exponent, i.e.,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then

$$\sigma(H_V) \subset \left\{ k + \frac{1}{k} \;\middle|\; k \in \mathbb{C} \setminus \{0\},\ |k| \leq 1 \text{ and } \left| k - \frac{1}{k} \right| \left(\frac{1 - |k|^q}{1 + |k|^q} \right)^{1/q} \leq \|v\|_{\ell^p(\mathbb{Z})} \right\}.$$

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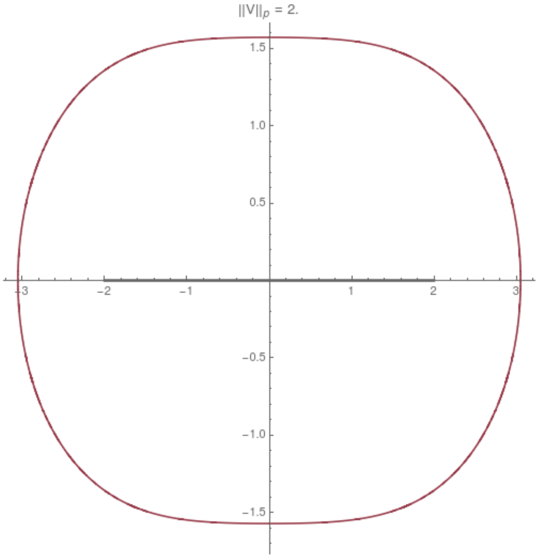
Remarks:

- The proof is based again on Birman–Schwinger principle and uses either the Schur test or discrete Young's inequality.
- No optimality result.
- The interval $[-2, 2]$ always involved in the spectral enclosure $\Rightarrow$ no consequences for embedded eigenvalues.

ℓ^p-potentials: plots of the spectral enclosure for $p = 2$



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- Free discrete Dirac operator D_0:

$$D_0 := \begin{pmatrix} m & d \\ d^* & -m \end{pmatrix}$$

acting on $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$, where $m \geq 0$ and d^* is the adjoint operator to d.

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  acting on $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$, where $m \geq 0$ and d^* is the adjoint operator to d.
- Considered potentials:

$$V = \begin{pmatrix} V^{1,1} & V^{1,2} \\ V^{2,1} & V^{2,2} \end{pmatrix},$$

  where $V^{i,j}$ act on $\ell^2(\mathbb{Z})$ as diagonal operators determined by doubly infinite complex sequences.

2×2-block Laurent matrix representation of D_0

- By using a suitable orthonormal basis of $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$, D_0 can be represented by the 2×2-block tridiagonal Laurent matrix:

$$D_0 = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & a^T & b & a & & & \\ & & a^T & b & a & & \\ & & & a^T & b & a & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$b := \begin{pmatrix} -m & 1 \\ 1 & m \end{pmatrix} \quad \text{and} \quad a := \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

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- The considered perturbation of D_0:

$$V = \bigoplus_{n \in \mathbb{Z}} v_n, \quad \text{where} \quad v_n := \begin{pmatrix} v_n^{11} & v_n^{12} \\ v_n^{21} & v_n^{22} \end{pmatrix}.$$

Facts about D_0

- The spectrum:

$$\sigma(D_0) = \sigma_{\mathrm{ess}}(D_0) = \left[-\sqrt{m^2+4}, -m\right] \cup \left[m, \sqrt{m^2+4}\right].$$

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$$\lambda^2 = m^2 + 2 - k - k^{-1}$$

determines a one-to-two mapping $\lambda = \lambda(k)$ from $0 < |k| < 1$ onto $\rho(D_0)$.

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- The 2×2-block Laurent matrix representation of the resolvent:

$$(D_0 - \lambda)^{-1}_{m,n} = T_{n-m}(k),$$

  where

$$T_0(k) = \frac{1}{k^{-1} - k} \begin{pmatrix} \lambda - m & 1 - k \\ 1 - k & \lambda + m \end{pmatrix},$$

$$T_j(k) = T_{-j}^T(k) = \frac{k^j}{k^{-1} - k} \begin{pmatrix} \lambda - m & 1 - k \\ 1 - k^{-1} & \lambda + m \end{pmatrix}, \quad j \geq 1.$$

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Let $V \in \ell^1(\mathbb{Z}, \mathbb{C}^{2\times 2})$. Then

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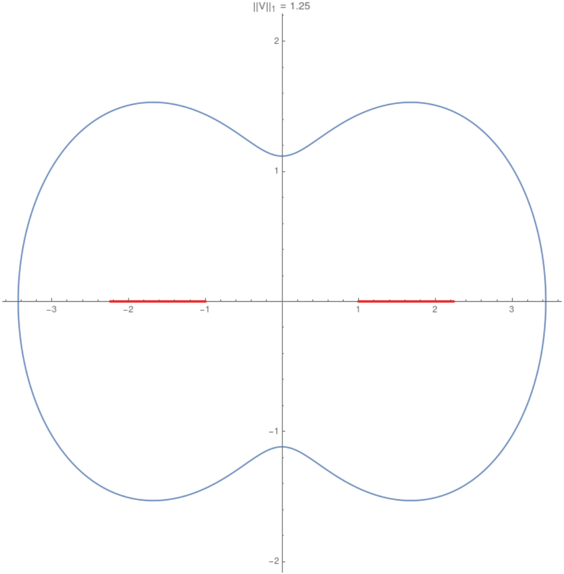
Remark: The Banach space $\ell^p(\mathbb{Z}, \mathbb{C}^{2\times 2})$ is equipped with the norm

$$\|V\|_p = \left(\sum_{n\in\mathbb{Z}} |v_n|^p \right)^{1/p}, \quad \text{if } 1 \leq p < \infty, \qquad \|V\|_\infty = \sup_{n\in\mathbb{Z}} |v_n|,$$

where $|v_n|$ denotes the operator norm of the matrix $v_n \in \mathbb{C}^{2\times 2}$.

Geometry of the boundary curve for $m = 1$



Embedded eigenvalues

Corollary:

Let $V \in \ell^1(\mathbb{Z}, \mathbb{C}^{2\times 2})$. If $2\|V\|_1^2 < (m^2 + 2 - m\sqrt{m^2+4})$ then the union of two intervals

$$(-\tau_+, -\tau_-) \cup (\tau_-, \tau_+),$$

where

$$\tau_\pm = \sqrt{2 + m^2 - 2\|V\|_1^2 \pm 2\sqrt{1 - (m^2+2)\|V\|_1^2 + \|V\|_1^4}},$$

is free of embedded eigenvalues of H_V.

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- The 2×2 complex matrices $T_0(k)$ and $T_1(k)$ appear in the formula for the resolvent $(D_0 - \lambda)^{-1}$.
- Their spectral norms can be expressed explicitly but lead to complicated expressions:

$$|T_1(k)|^2 = |k|^2 \frac{|\lambda + m|^2 + |\lambda - m|^2 + (|k| + |k|^{-1})|\lambda^2 - m^2|}{|\lambda^2 - m^2||\lambda^2 - m^2 - 4|},$$

$$|T_0(k)|^2 = \text{“...even more complicated :(.”}$$

Optimality of the improved spectral enclosure for ℓ^1-potential

- We were able to prove only a "partial" optimality, i.e., only a part of the boundary of the spectral enclosure ca be approached by an eigenvalue of D_V for some $V \in \ell^1(\mathbb{Z}, \mathbb{C}^{2\times 2})$.

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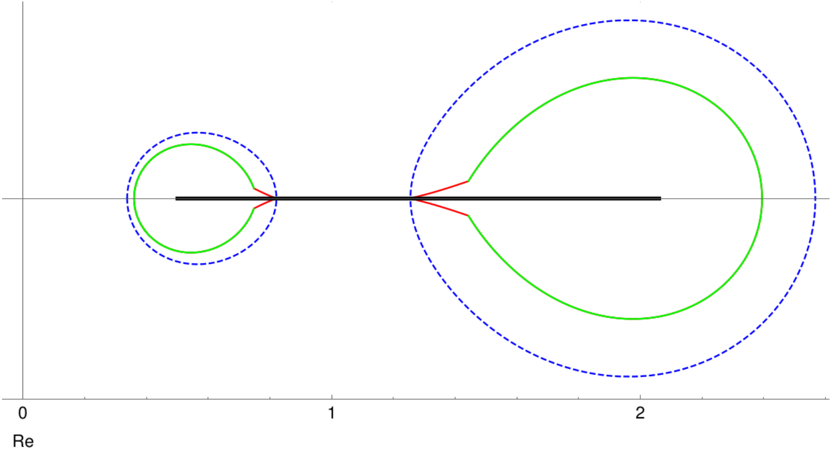

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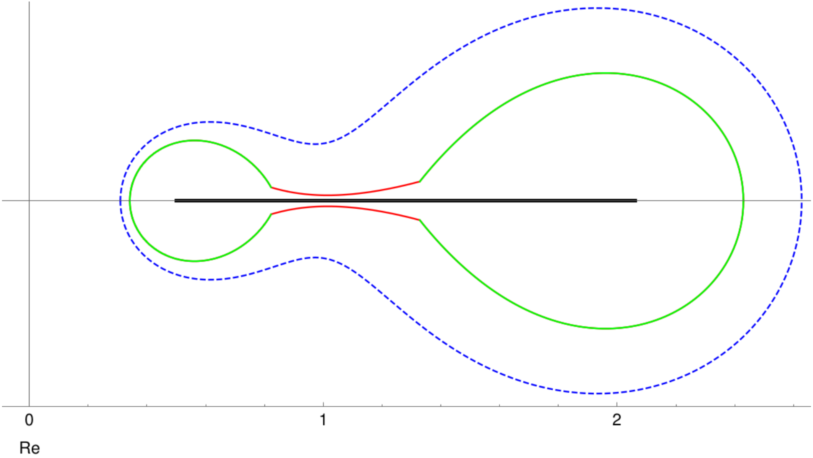

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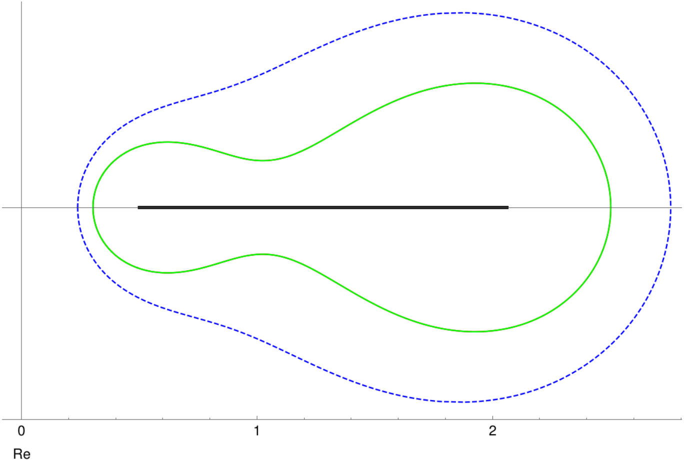

Spectral enclosures for ℓ^p–potentials, $p > 1$

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Let $1 < p \leq \infty$, q the Hölder dual index to p, and assume $V \in \ell^p(\mathbb{Z}, \mathbb{C}^{2\times 2})$. Then $\sigma(D_V) \setminus \sigma(D_0)$ is a subset of:

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Thank you!