On the localization of spectra of complex sampling Jacobi matrices and open problems

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Quantum circle

Contents

Sampling matrices

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Problem:

Localization of $\operatorname{spec}(J_{a,b}(\Delta_n))$ in terms of a, b.

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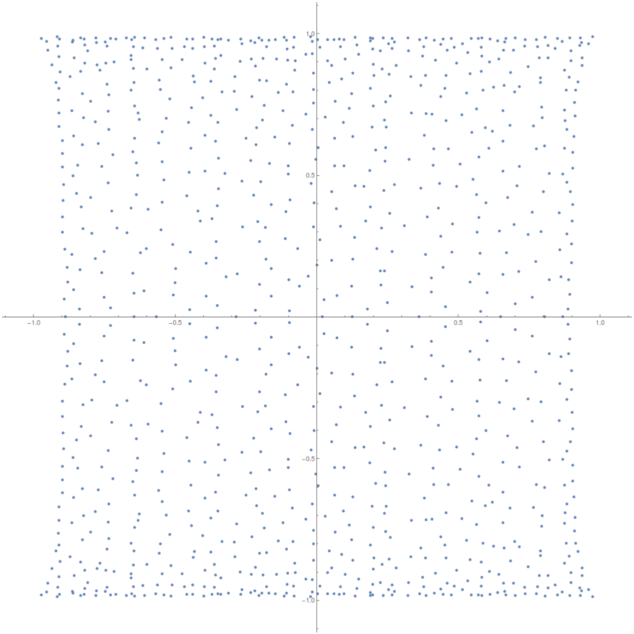

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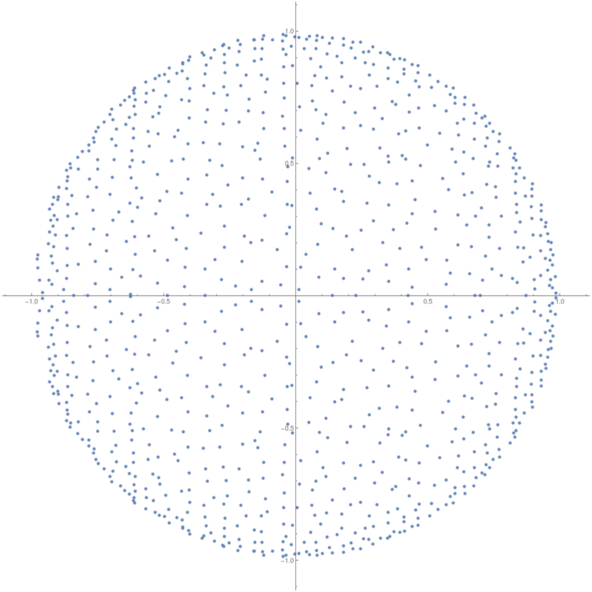

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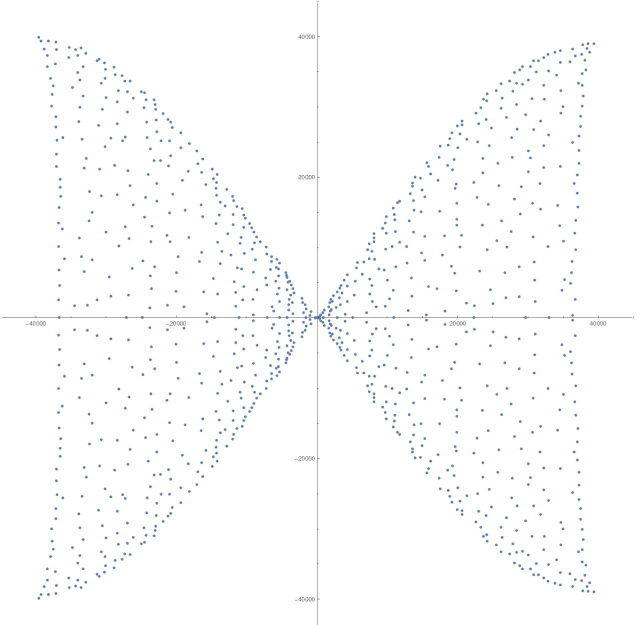

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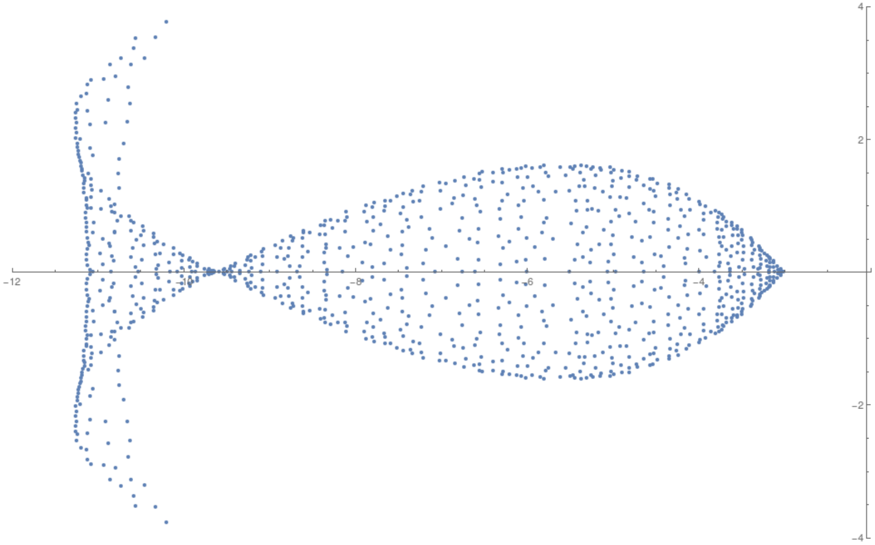

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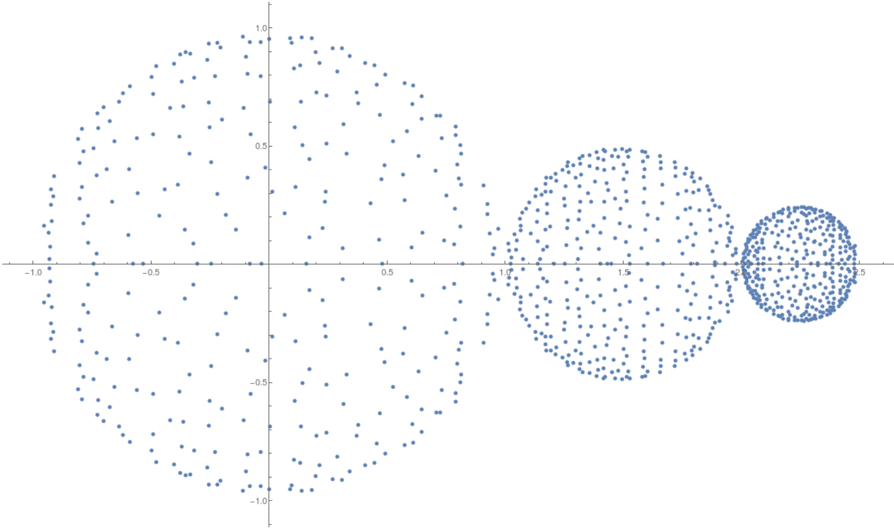

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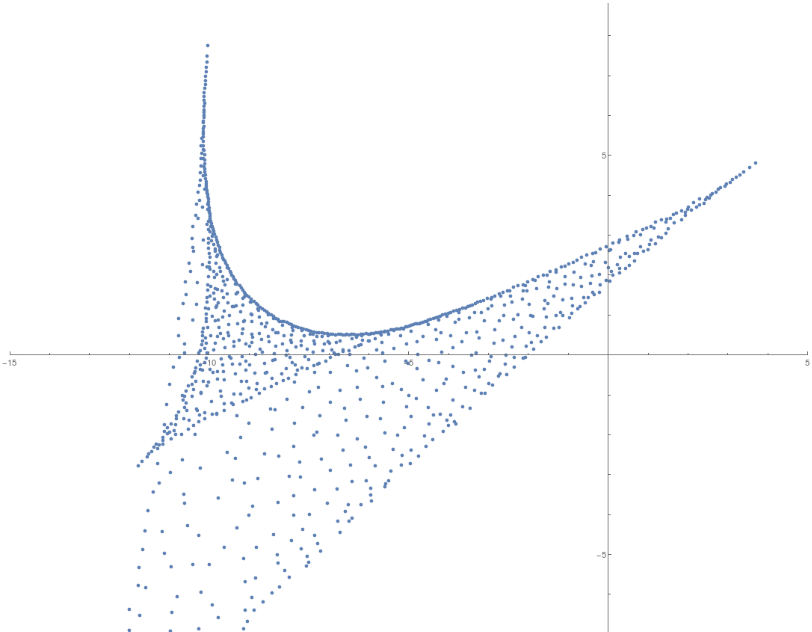

It seems the eigenvalues are somewhat localized ...

Estimations for the localization domain

- One has

$$\|J(\Delta_n)\| \leq \|b\|_\infty + 2\|a\|_\infty, \qquad \forall n,\ \forall \Delta_n,\ \forall a, b \in C([0,1]).$$

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Gerschrogin circle theorem:

Let $A = (a_{i,j}) \in \mathbb{C}^{n,n}$ and

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Applying Gerschrogin's theorem we obtain much better localization:

$$\boxed{\operatorname{spec}\left(J_{a,b}(\Delta_n)\right) \subset \bigcup_{0 \leq t \leq 1} D(b(t), 2a(t)) \qquad \forall n,\ \forall \Delta_n}$$

Weaker formulation and the optimal localization

- Let $\Delta = \{\Delta_n\}_{n=1}^{\infty}$ be a sequence of partitions of $[0,1]$. Put

$$\boxed{\Lambda_{a,b}(\Delta) := \{z \in \mathbb{C} \mid \liminf_{n\to\infty} \operatorname{dist}(z, \operatorname{spec}(J_{a,b}(\Delta_n))) = 0\}.}$$

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- So, $\lambda \in \Lambda_{a,b}(\Delta)$ iff

$$\exists \{n_k\} \subset \mathbb{N} \quad \exists \lambda_k \in \operatorname{spec}(J_{a,b}(\Delta_{n_k}))) \text{ such that } \lim_{k\to\infty} \lambda_{n_k} = \lambda.$$

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Conjecture:

For all $a, b \in C([0, 1])$ and Δ a sequence of partitions of $[0, 1]$, it holds

$$\Lambda_{a,b}(\Delta) \subset \mathcal{S}_{a,b} := \bigcup_{0\leq t\leq 1} [b(t) - 2a(t), b(t) + 2a(t)]$$

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Equivalently the statement says: $\forall\epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \ge n_0,$ one has

$$\mathrm{spec}(J_{a,b}(\Delta_n)) \subset \mathcal{U}_\epsilon(\mathcal{S}_{a,b}).$$

Let’s take a look on pictures...

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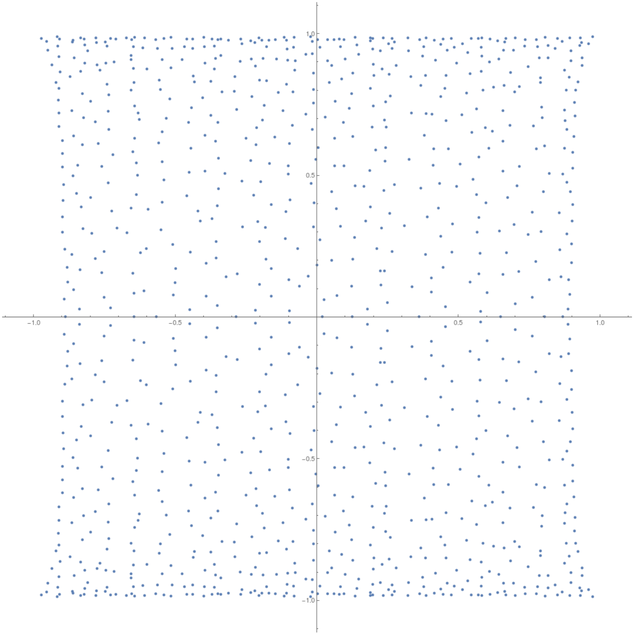

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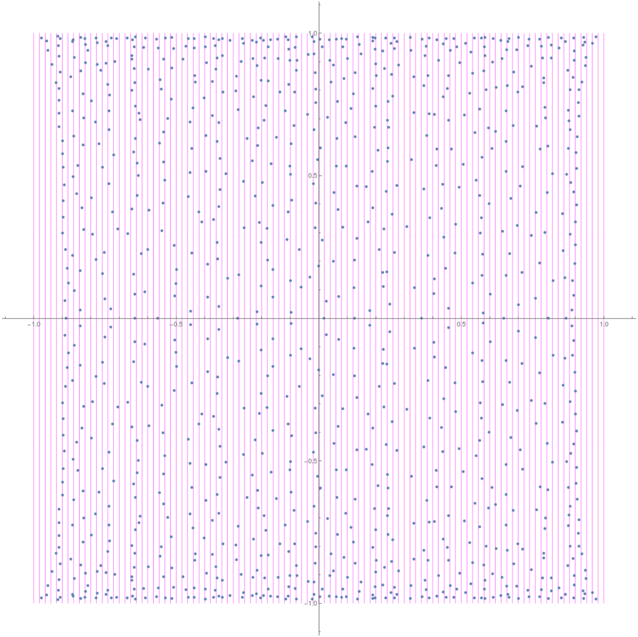

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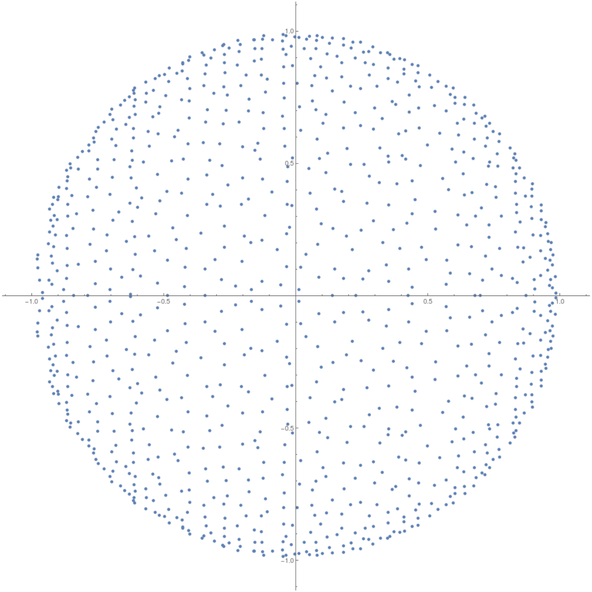

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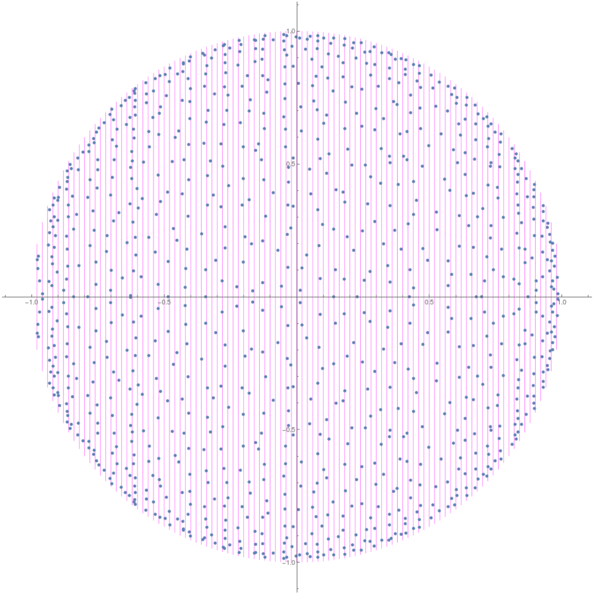

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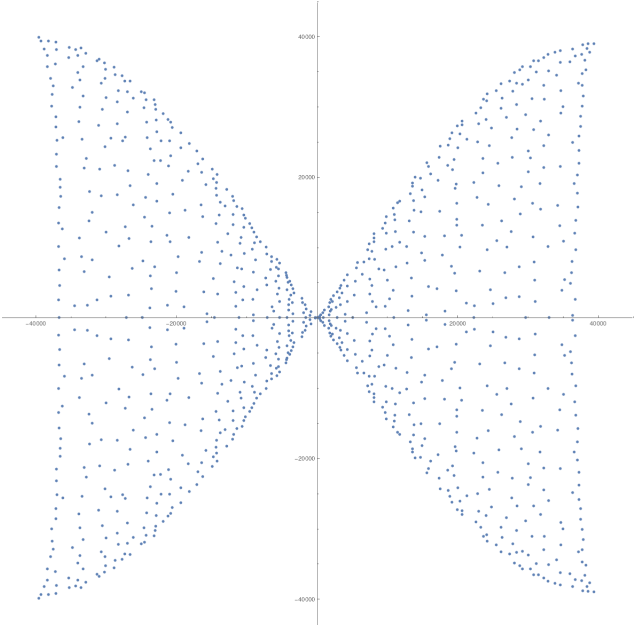

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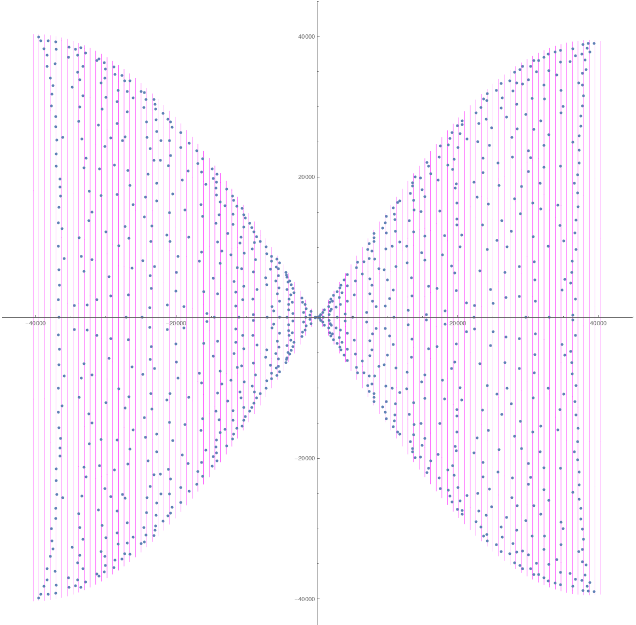

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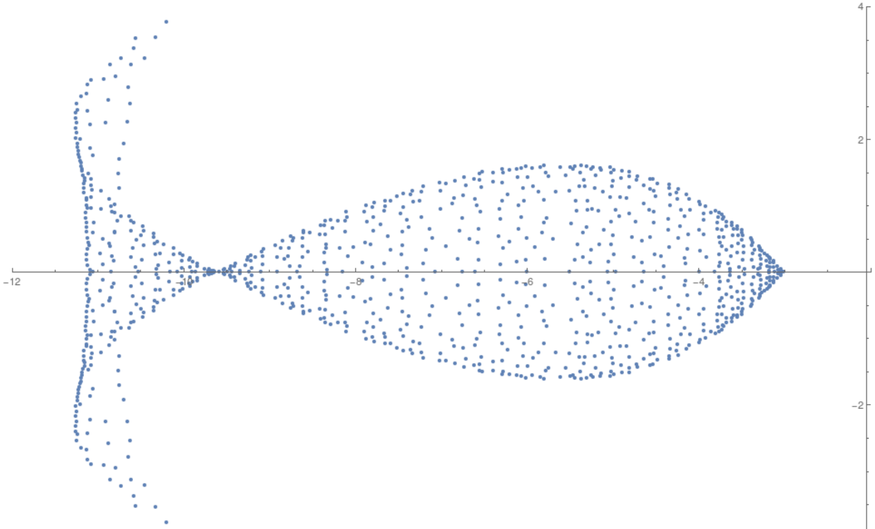

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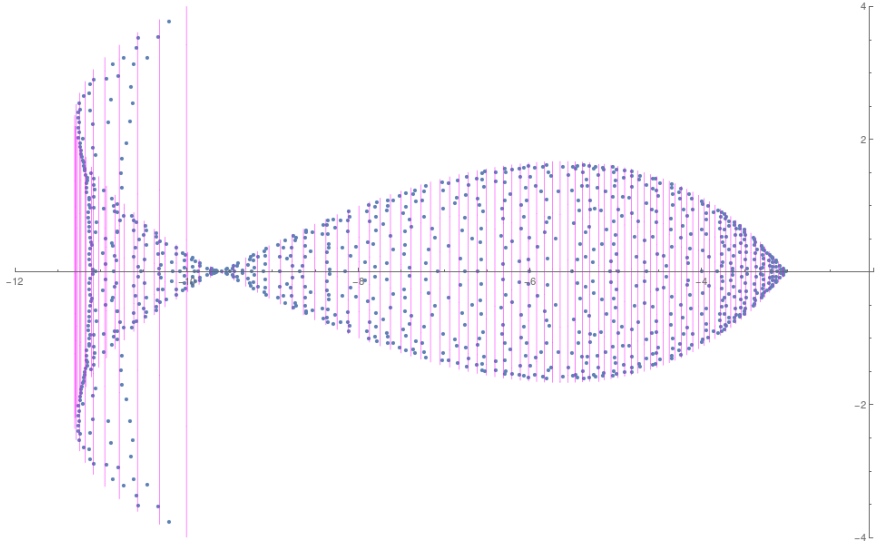

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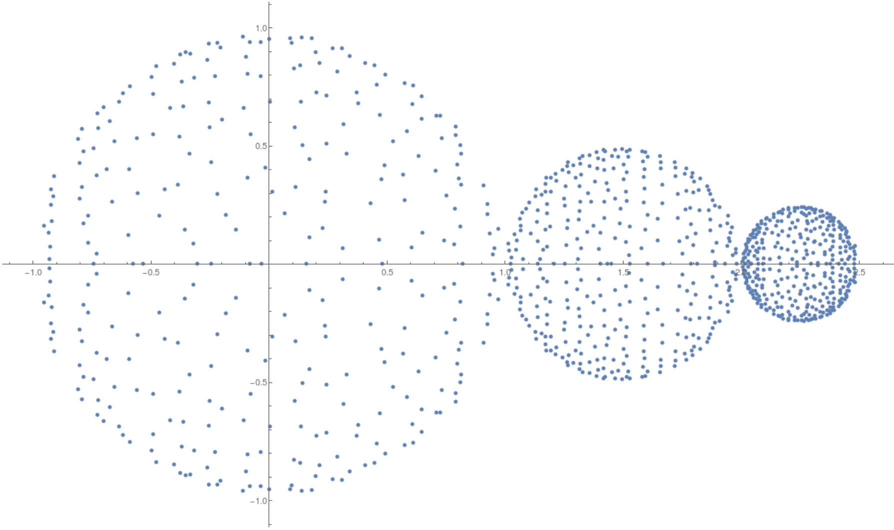

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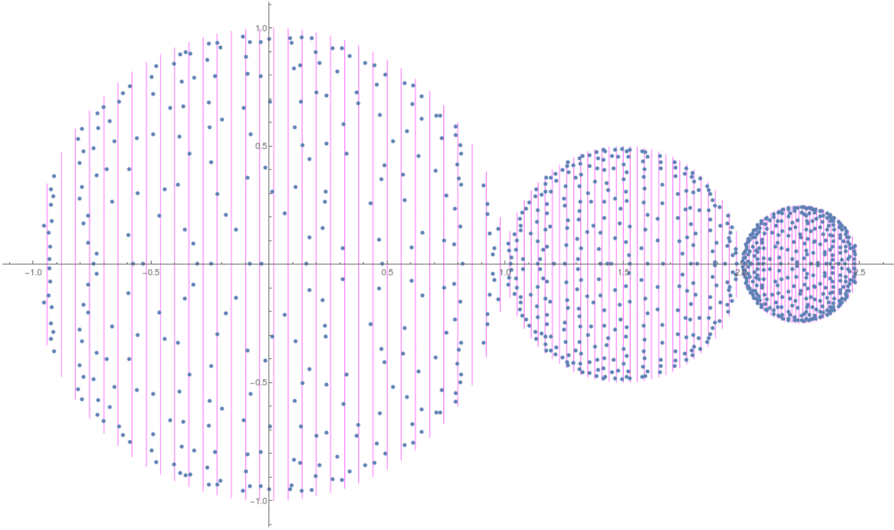

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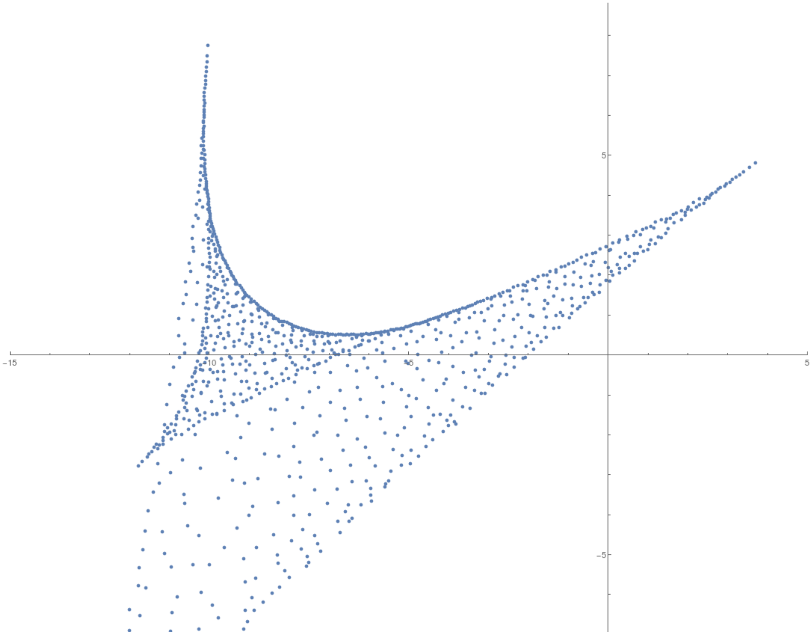

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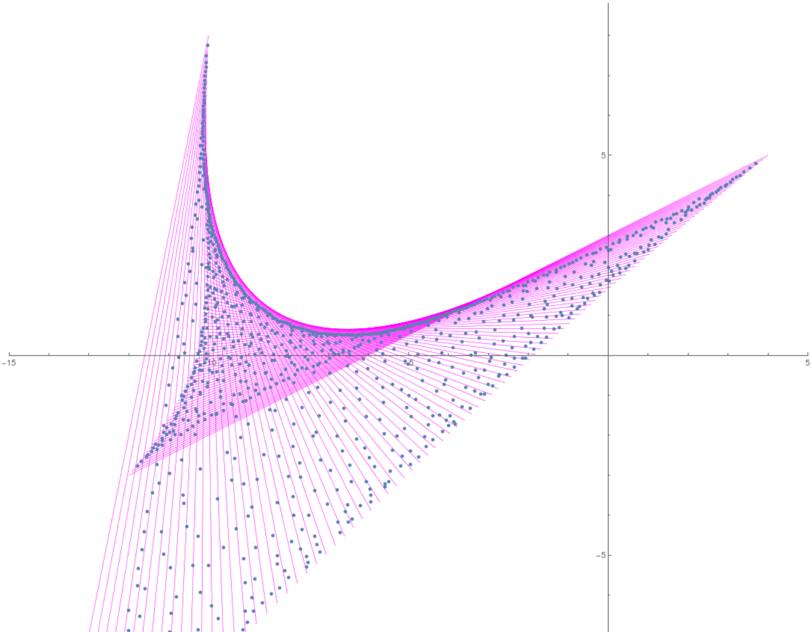

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$$\boxed{A_n^{(m)} = \bigoplus_{i=1}^{m} J_{n_i}(a_i, b_i) + \sum_{i=1}^{m-1} x_i \left(e_{N_i} e_{N_i+1}^T + e_{N_i+1} e_{N_i}^T \right)}$$

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4. However, it is to say that picture is very incomplete now and several pieces are missing!

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Asymptotic eigenvalue distribution of $A_n^{(m)}$

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$$\boxed{\lim_{n\to\infty} \frac{\det\left(A_{N(n)}^{(m)} - z\right)}{\prod_{j=1}^m a_j^{n_j(n)} U_{n_j(n)}\left(\frac{b_j - z}{2a_j}\right)} = \prod_{j=1}^{m-1}\left[1 - f\left(\frac{b_j - z}{2a_j}\right) f\left(\frac{b_{j+1} - z}{2a_{j+1}}\right)\right]}$$

where $f(z) = z - \sqrt{z-1}\sqrt{z+1}$ and $U_n(\cdot)$ stands for the Chebyshev polynomials of the 2nd kind, and the convergence is local uniform in $z \in \mathbb{C} \setminus \cup_{j=1}^m [b_j - 2a_j, b_j + 2a_j]$.

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Corollary:

$$\text{“The set of limit points of } \operatorname{spec}\left(A_{N(n)}^{(m)}\right), \text{ as } n \to \infty\text{”} = \bigcup_{j=1}^{m}[b_j - 2a_j, b_j + 2a_j]$$

The limit of eigenvalue-counting measures of $A_n^{(m)}$

- In case of matrices $A_{N(n)}^{(m)}$, we can prove much more.

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$$\boxed{w-\lim_{n\to\infty} \mu_n^{(m)} = \sum_{j=1}^{m} \ell_j \omega_{a_j, b_j}}$$

where $\omega_{a,b}$ is the absolutely continuous measure supported on $[b-2a, b+2a]$ with density

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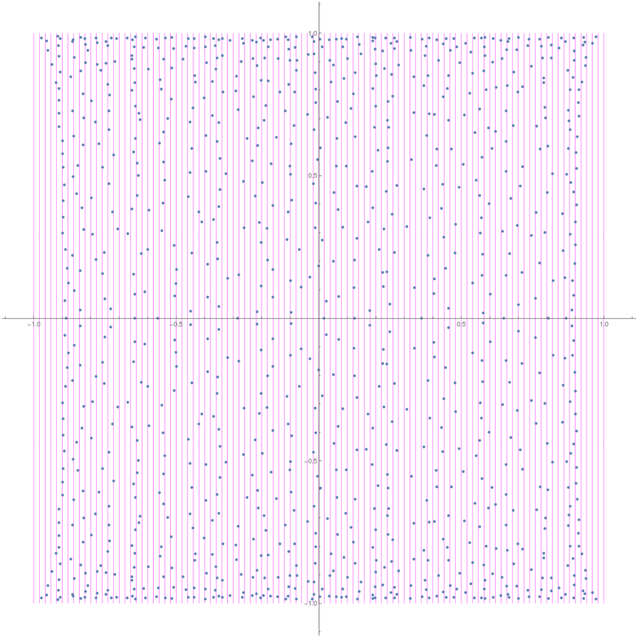
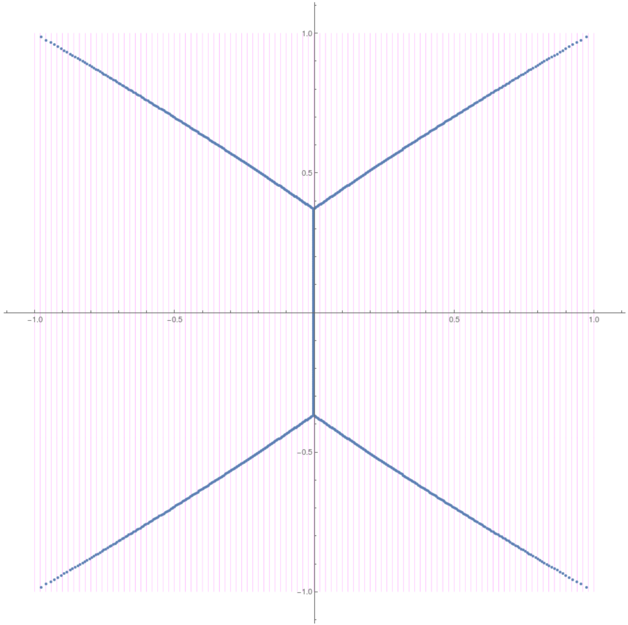
See the pictures ...

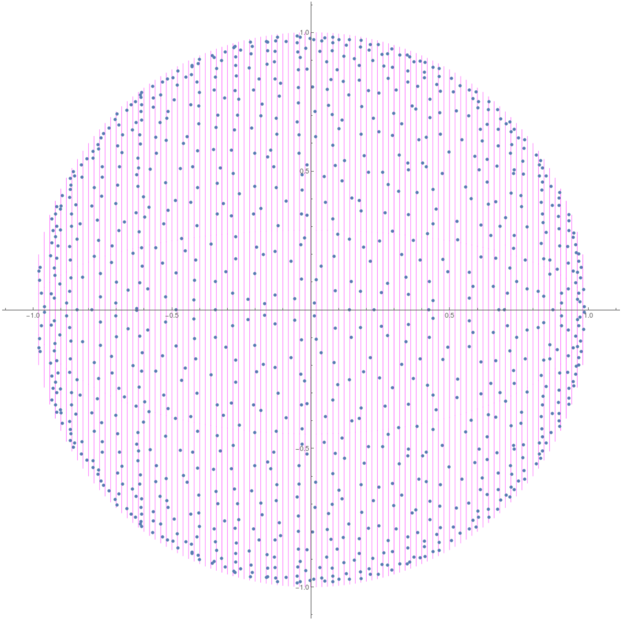
The square



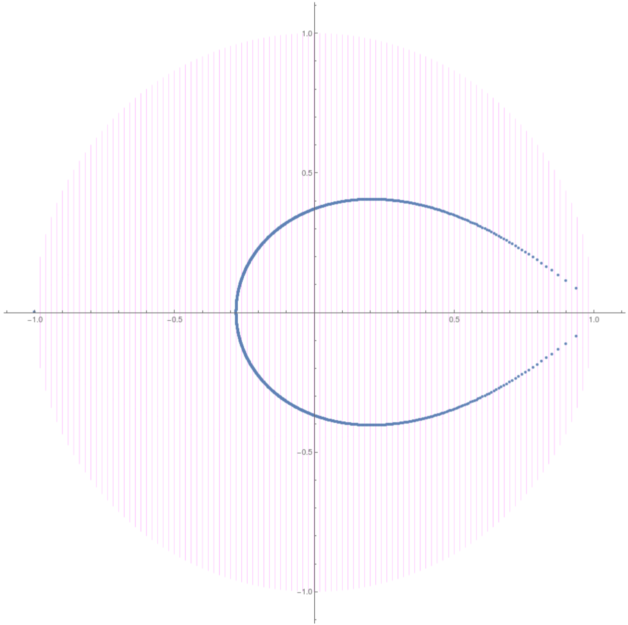
The square - uniform grid



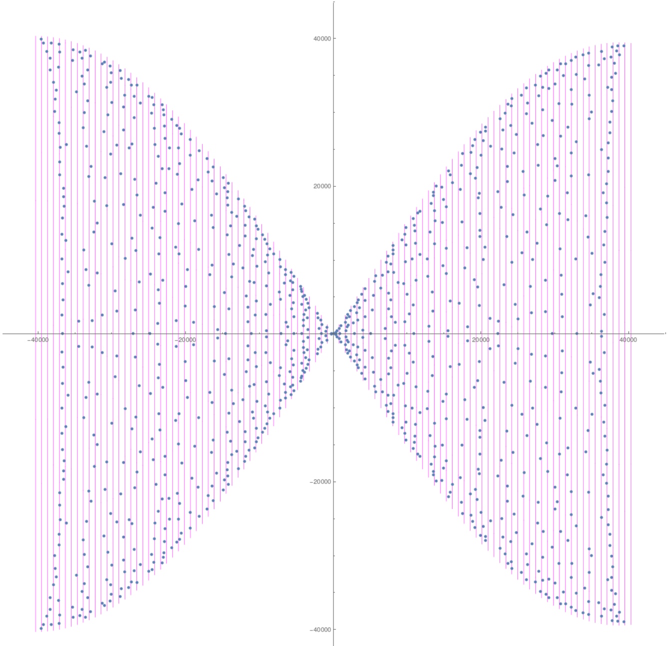
The circle



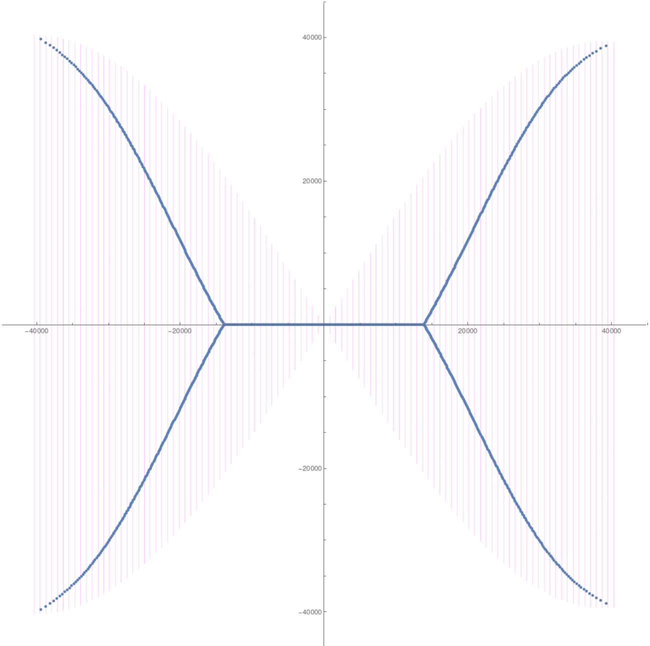
The circle - uniform grid



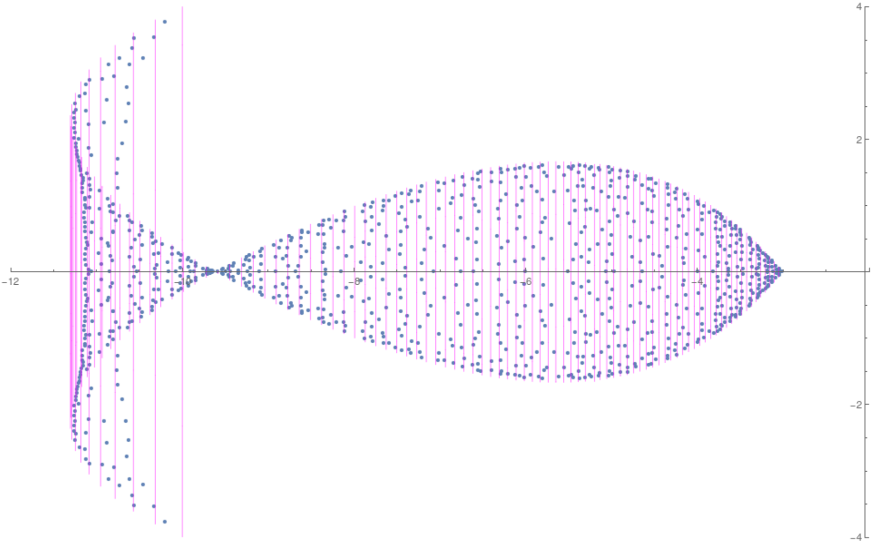
The butterfly



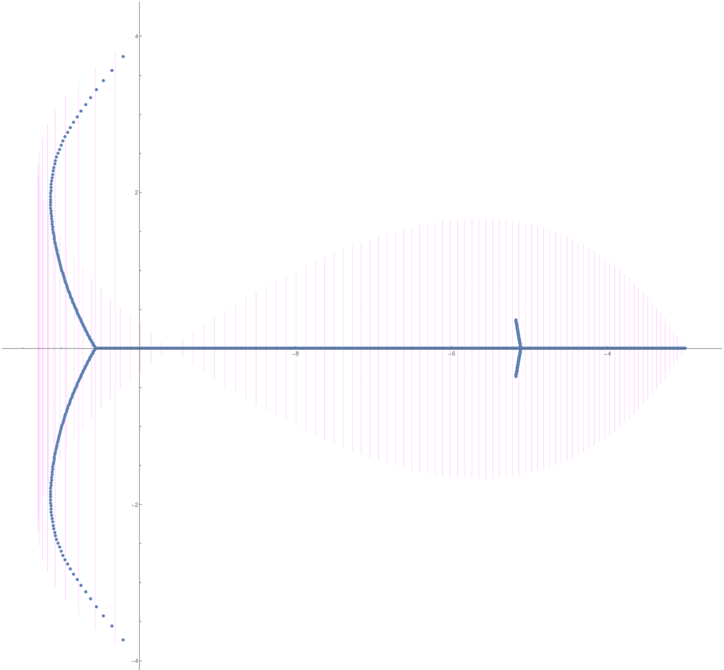
The butterfly- uniform grid



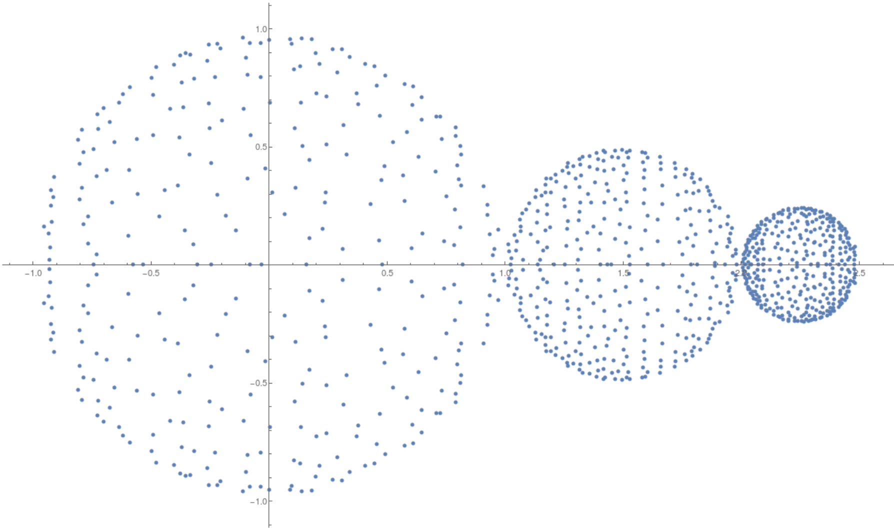
The fish



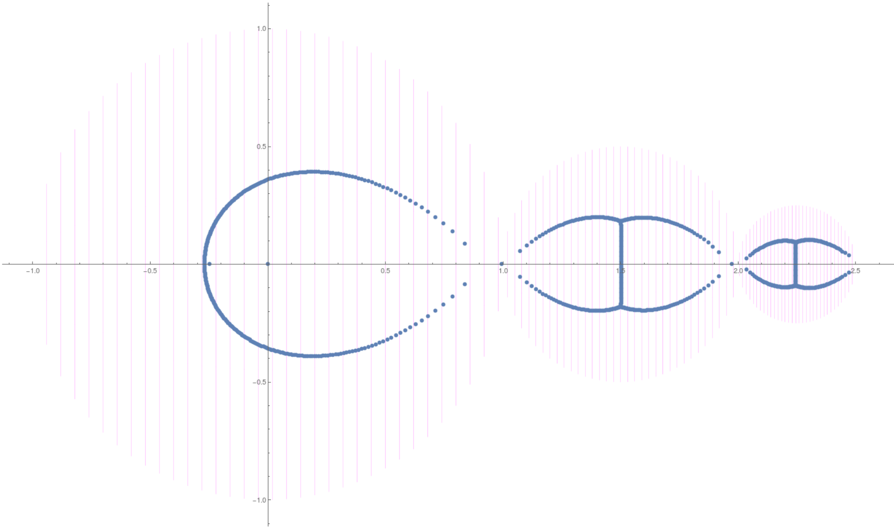
The fish - uniform grid



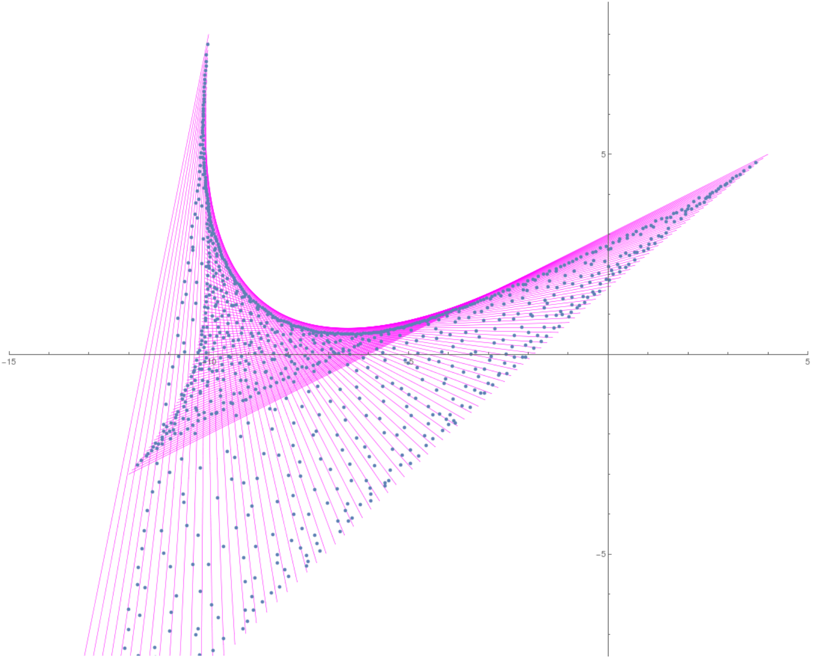
Fallen snowman



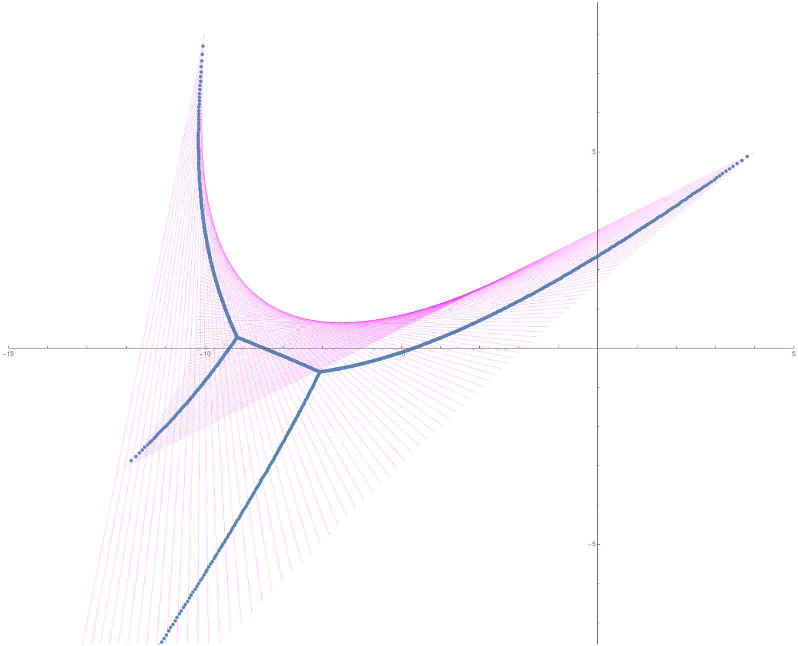
Fallen snowman - uniform grid



The random object



The random object - uniform grid



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Except few very special examples, all these questions remain open . . .

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- The limiting set of spectra $\mathrm{spec}(T_n(b))$:

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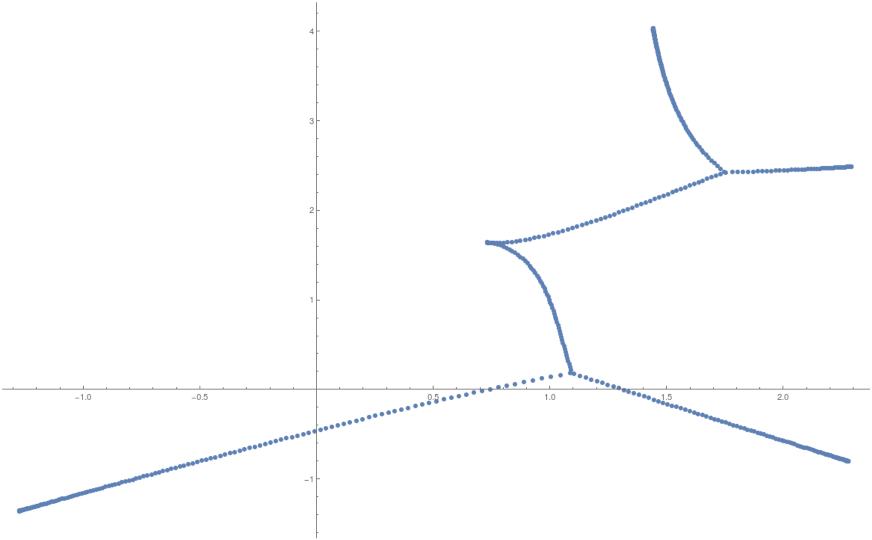
Theorem (Schmidt, Spitzer, Ullman):

$\Lambda(b)$ is a connected set that equals the union of a finite number of pairwise disjoint open analytic arcs and a finite number of the so called exceptional points (roughly speaking: branching points and endpoints).

An example (7-diagonal Toeplitz)



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Theorem (Hirschman):

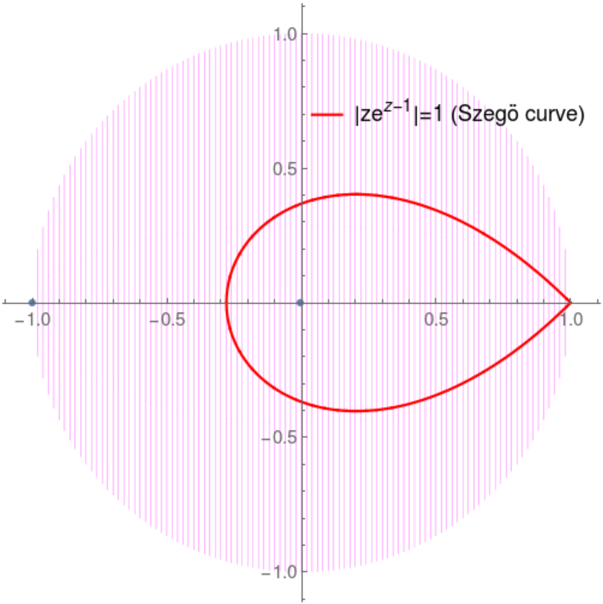
The sequence of eigenvalue-counting measures of $T_n(b)$ converges weakly to a measure μ supported on $\Lambda(b)$. In addition,

$$\boxed{\mathrm{d}\mu(\lambda) = \frac{1}{2\pi}\frac{1}{g(\lambda)}\left|\frac{\partial g(\lambda)}{\partial n_1} + \frac{\partial g(\lambda)}{\partial n_2}\right| \mathrm{d}s(\lambda),}$$

for $\lambda \in \Lambda(b)$ a nonexceptional point (for such points, the outer normal vector derivatives $\partial g/\partial n_1$ and $\partial g/\partial n_2$ with respect to the two components separated by the respective arc of $\Lambda(b)$ exist) Here, $\mathrm{d}s$ stands for the arc length measure.

Contents

Unit disk and the Szegö curve



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- Two measures μ and ν are called equipotential iff

$$U^{\mu}(z) = U^{\nu}(z), \quad \forall z \in \mathbb{C} \setminus (\operatorname{supp} \mu \cup \operatorname{supp} \nu).$$

Theorem

Let μ_n be the eigenvalue-counting measures of $J_{a,b}(\Delta_n)$ with uniform partitions Δ_n. Then there is a neighborhood U of ∞ such that

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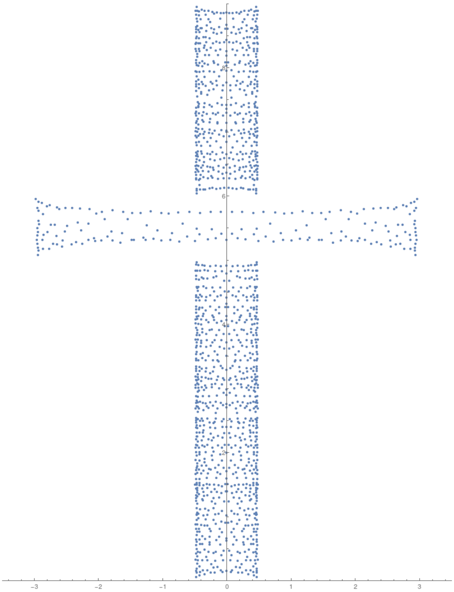
Corollary

If the Conjecture stating $\Lambda_{a,b}(\Delta) \subset \mathcal{S}_{a,b}$ holds true and the weak* limit μ of measures μ_n exists. Then the measures μ and σ are equipotential.



Veselé Velikonoce