Introduction to the Theory of Orthogonal Polynomials

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Notes of 3 lectures given at CoW&MP, Kruh u Jilemnice, Czech Republic

May 18-24, 2014

Contents

1. Basics from the theory of measure and integral, definition of orthogonal polynomials, examples, tree-term recurrence, Favard's theorem (regular lecture).
2. Christoffel-Darboux kernel and formula, zeros of orthogonal polynomials, properties of the very classical orthogonal polynomials (regular lecture).
3. Orthogonal polynomials and spectral theory of Jacobi operators, interesting comments, criteria on uniqueness of the measure of orthogonality, the case of non-uniqueness of measure of orthogonality, Markov's theorem, Navanlinna parametrization (informative chitchat lecture).

Sources used: Akhiezer's, Chihara's and Ismail's monograph, Koornwinder's lecture notes, papers cited later.

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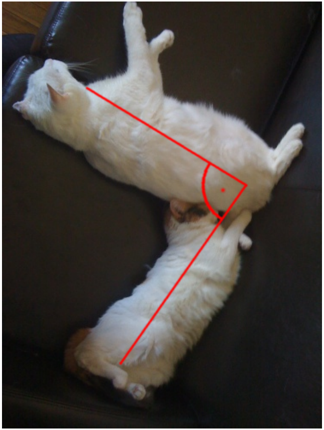

By the way orthogonal polynomials brings together many mathematical and physical branches:

- **Complex analysis** (Bieberbach conjecture, moment problem, Padé approximation)
- **Functional analysis** (Fourier-Plancherel transform, spectral analysis of Jacobi operators)
- **Numerical mathematics** (approximation theory, quadrature, differential equations)
- **Number theory** (continued fractions, proofs of irrationality of numbers)
- **Quantum mechanics** (harmonic oscilator and its deformations, Schrödinger operator with spherically symmetric potential, coherent states)
- **Integrable systems** (solitons, Toda equation)
- Random matrix theory, Riemann-Hilbert problem, Radon transform, Zonal spherical harmonics, group representation theory, coding theory, electrostatic problems, ...

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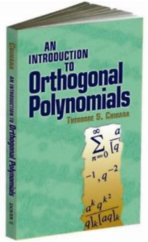

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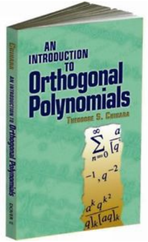
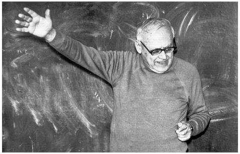


- T. S. Chihara: *An Introduction to Orthogonal Polynomials*, Gordon and Breach, 1978, reprinted Dover, 2011.

- N. I. Akhiezer: *The Classical Moment Problem and Some Related Questions in Analysis*, Oliver & Boyd, 1965.



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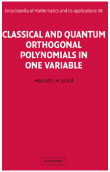

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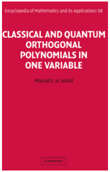

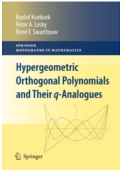

- R. Koekoek, P. A. Lesky R. F. Swarttouw: *Hypergeometric orthogonal polynomials and their q-analogues*, Springer-Verlag, 2010.
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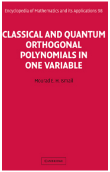

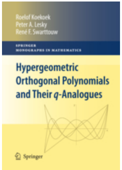

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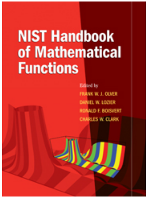
- NIST Digital Library of Mathematical Functions, in particular Chp. 18 on Orthogonal polynomials
- `http://dlmf.nist.gov`



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$$p_0(x) = 1, \quad p_n(x) = x^n - \sum_{k=0}^{n-1} \frac{\langle p_k, x^n \rangle}{\langle p_k, p_k \rangle} p_k(x).$$

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- Orthonormal polynomials ($h_n = 1$) vs. Monic orthogonal polynomials ($k_n = 1$).

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- These examples are special cases of the inner product of the form

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)g(x)\mathrm{d}\mu(x)$$

where μ is a (positive) measure.

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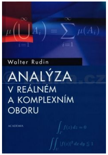
- Very nice introduction on the general theory of measures and integral calculus is given in

W. Rudin: *Real and complex analysis*, in czech, Academia, 2003.



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- **Example:** For $x_k = k$, $w_k = 1/k!$, $k \in \mathbb{Z}_+$ we have

$$\int_{\mathbb{R}} 1\mathrm{d}\mu(x) = \sum_{k=0}^{\infty} \frac{1}{k!} = \mathrm{e}.$$

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(The number of mass points is at most countable.)
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- So if $A \subset \mathbb{R} \setminus \operatorname{supp} \mu$ then $\mu(A) = 0$ and one does not need to "integrate outside the support",
$$\int_{\mathbb{R}} f(x) \mathrm{d}\mu(x) = \int_{\operatorname{supp} \mu} f(x) \mathrm{d}\mu(x),$$
for any measurable function f. (Examples!)

- Let μ be positive Borel measure on $\mathbb{R}$ of infinite support such that

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- The orthogonality relation then reads

$$\int_{\mathbb{R}} p_m(x)p_n(x)\mathrm{d}\mu(x) = h_n \delta_{m,n}, \quad m, n \in \mathbb{Z}_+.$$

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- In principle, $H_n(x)$ can be computed for any $n \in \mathbb{Z}_+$ by using the Gram-Schmidt procedure. Several first are

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x,$$

$$\dots$$

$$H_{10}(x) = 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240$$

However, it is a difficult task to derive the explicit formula for H_n from the very definition. The formula reads

$$H_n(x) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{k!(n-2k)!} (2x)^{n-2k}.$$

- Laguerre polynomials L_n : orthogonal on $(0, \infty)$ w. r. t. the weight function e^{-x}, thus $w(x) = e^{-x}\chi_{(0,\infty)}(x)$ (normalized by $k_n = (-1)^n/n!$ or $h_n = 1$).

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More examples

- Laguerre polynomials L_n : orthogonal on $(0,\infty)$ w. r. t. the weight function e^{-x}, thus $w(x) = e^{-x}\chi_{(0,\infty)}(x)$ (normalized by $k_n = (-1)^n/n!$ or $h_n = 1$).
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- Charlier polynomials $C_n^{(a)}$: orthogonal on $\mathbb{Z}_+$ w. r. t. the weights $w_k = a^k/k!$ where $a > 0$ (normalized by $C_n^{(a)}(0) = (-a)^n$).

Theorem

Monic orthogonal polynomials p_n satisfy

$$\begin{aligned} p_{n+1}(x) &= (x - a_n)p_n(x) - b_{n-1}p_{n-1}(x), \quad \text{for } n \geq 1, \\ p_1(x) &= (x - a_0)p_0(x) \end{aligned}$$

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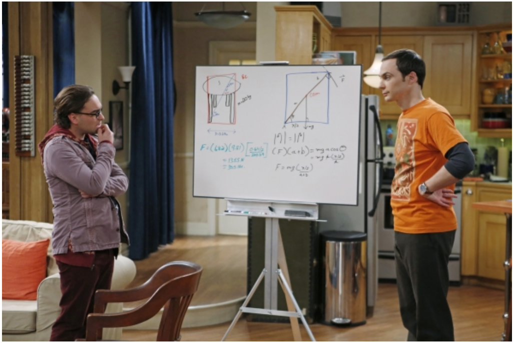

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If polynomials p_n of degree n satisfy

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- If sequences $\{a_n\}$ and $\{b_n\}$ are bounded then the measure μ is unique.
- The measure μ is unique if

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{b_n}} = \infty \qquad \text{(Carleman's condition)}$$

Christoffel-Darboux kernel

Definition

Let p_n be OPs w. r. t. measure μ. The Christoffel-Darboux kernel kernel is the function

$$K_n(x,y) := \sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{h_k}$$

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Linear map $\Pi_n : \mathcal{P} \to \mathcal{P}_n$ defined by formula

$$(\Pi_n f) := \int_{\mathbb{R}} K_n(x,y)f(y)\mathrm{d}\mu(y)$$

is an orthogonal projection onto $\mathcal{P}_n$ (*Proof:* whiteboard).

Theorem

Assume p_n are monic OPs ($k_n = 1$) then it holds

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Corollary

$$\sum_{k=0}^{n}\frac{p_k^2(x)}{h_k} = \frac{1}{h_n}\left(p_{n+1}'(x)p_n(x) - p_n'(x)p_{n+1}(x)\right).$$

Proofs: whiteboard

Zeros of OPs

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Let p_n be OPs w. r. t. μ (of degree n). Then p_n has n distinct zeros in $\operatorname{supp}\mu$.

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Exercise for students #2: Prove the last Theorem. Hint: WLOG take $k_n = 1$ and use that for all $x \in \mathbb{R}$ it holds

$$p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x) = h_n \sum_{k=0}^{n} \frac{p_k^2(x)}{h_k} > 0,$$

as it follows from the Christoffel-Darboux formula.

Graphs of Chebyshev OPs of the second kind

- Chebyshev polynomials of the second kind U_n are orthogonal on $[-1, 1]$ w.r.t the weight function $\sqrt{1-x^2}$.

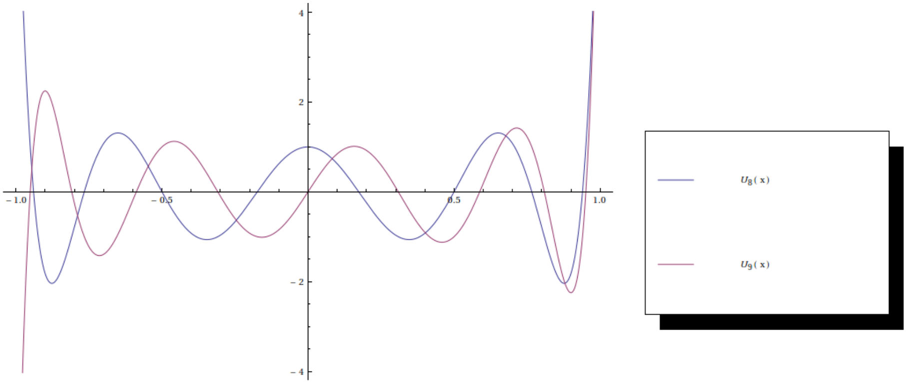

Figure : Alternating zeros of Chebyshev polynomials U_8 and U_9.

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Recurrence relation:

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- From which it further follows

$$n\int_a^b q_{n-1}(x)^2 w_1(x)\mathrm{d}x = \xi_n \int_a^b p_n(x)^2 w(x)\mathrm{d}x.$$

(Details on whiteboard.)

Derivation of relations for Laguerre polynomials

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hint 2: Ask me for the advice!

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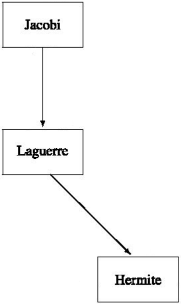
3. The polynomials are orthogonal w. r. t. $0 < w \in C^\infty$ on an open interval and there exists a polynomial Y such that the Rodriguez formula
$$p_n(x) = \text{const.}w(x)^{-1}\frac{d^n}{dx^n}\left[Y(x)^n w(x)\right].$$

Towards the Askey scheme - the very classical polynomials



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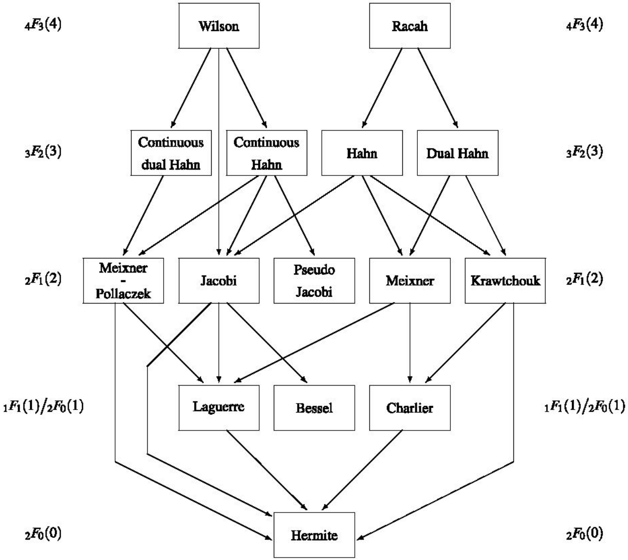
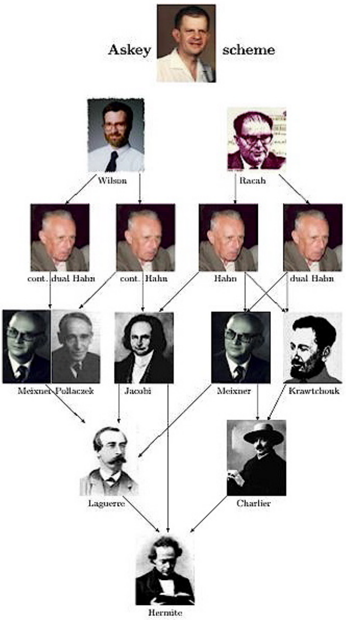
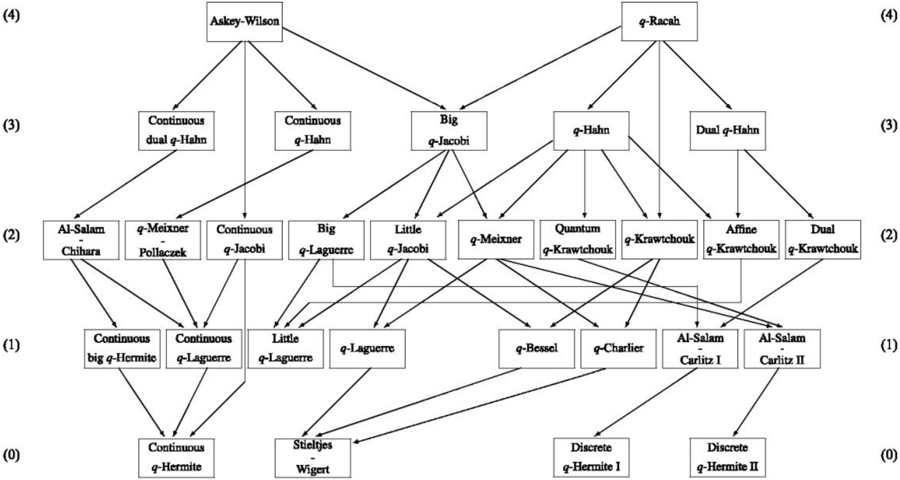


Askey scheme with heads



q-Askey scheme



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- Matrix J determines (not uniquely in general) a densely defined linear operator on $\ell^2(\mathbb{Z}_+)$.

- We saw any sequence of monic OPs $\{p_n\}$ is a solution of the recurrence

$$p_{n+1}(x) = (x - a_n)p_n(x) - b_{n-1}p_{n-1}(x), \quad (p_{-1}(x) := 0),$$

and vice versa.

- Consequently, by setting $\sqrt{b_0 \dots b_{n-1}}P_n(x) := p_n(x)$ one arrives at the equation

$$\sqrt{b_{n-1}}P_{n-1}(x) + a_nP_n(x) + \sqrt{b_n}P_{n+1}(x) = xP_n(x), \quad (P_{-1}(x) = 0).$$

- Hence sequence $P(x) := \{P_n(x)\}$ is a formal solution of the eigenvalue equation

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where J is semi-infinite symmetric Jacobi matrix with diagonal sequence $\{a_n\}$ and off-diagonal sequence $\{\sqrt{b_n}\}$.

- Matrix J determines (not uniquely in general) a densely defined linear operator on $\ell^2(\mathbb{Z}_+)$.
- So one can guess there is close connection between the spectral analysis of tridiagonal linear operators and corresponding OPs. Indeed, there is relation between spectral measure of J (under some assumptions) and the measure of orthogonality μ for OPs.

Problem n.1: The measure μ is given (e.g. by its density $w(x)$) and the goal is to recover sequences $\{a_n\}$ and $\{b_n\}$ (or at least some of their properties as asymptotics, periodicity, etc.) from the three-term recurrence relation of corresponding OPs - a.k.a. inverse spectral problem.

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Examples:

- Quantum mechanics: discrete Schrödinger operators on $\ell^2(\mathbb{Z}_+)$;

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- Statistical physics: equation of motion for magnetization in kinetic Ising chain;

$$\frac{dm_n}{dt} = -m_n + \frac{\sqrt{\gamma_{n-1}\gamma_n}}{2} m_{n-1} + \frac{\sqrt{\gamma_n\gamma_{n+1}}}{2} m_{n+1}.$$

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Problem 4. A moment problem

Mourad Ismail

Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

Borzov et al. [1] defined polynomials $\{\widetilde{H}_n(x|q)\}$ recursively by

$$\widetilde{H}_0(x|q) = 1, \quad \widetilde{H}_1(x|q) = 2x,$$

$$\widetilde{H}_{n+1}(x|q) = 2x\widetilde{H}_n(x|q) - (q^{-n} - q^n)\widetilde{H}_{n-1}(x|q), \quad 0 < q < 1. \tag{1}$$

These polynomials generalize Hermite polynomials since

Figure : Open problem: M. Ismal, JCAM, 2005

Intermezzo - Is it still actual? (cntd.)

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Finding a measure of orthogonality

P.D. Siafarikas

Department of Mathematics, University of Patras, 26500 Patras, Greece

Find the measure of orthogonality of the polynomials:

$$P_{n+1}(x) + P_{n-1}(x) + \frac{2b}{n+1}P_n(x) = xP_n(x),$$

$$P_{-1}(x) = 0, \quad P_0(x) = 1, \quad b \neq 0.$$

Figure : Open problem: P. D. Siafarikas, JCAM, 2001

Criteria for boundedness of the measure of orthogonality μ

Let p_n be the sequence of monic OPs generated by the three-term recurrence

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Proof: only indicated on whiteboard (in terms of operators)

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Let $\lim_{n\to\infty} a_n = a \in \mathbb{R}$ and $\lim_{n\to\infty} b_n = b \in \mathbb{R}$ then $\operatorname{supp}\mu$ is a bounded set which is composed of interval $[a - 2\sqrt{b}, a + 2\sqrt{b}]$ and possibly at most countably many point being outside $[a - 2\sqrt{b}, a + 2\sqrt{b}]$ with the only possible limit points $a \pm 2\sqrt{b}$.

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Remark: Blumenthal (1898) proved a part of the above theorem, but he asserted there can be at most finitely many point of $\operatorname{supp}\mu$ in the complement of $[a - 2\sqrt{b}, a + 2\sqrt{b}]$. Chihara (1968) proved the assertion is false (chain sequences approach and Szögo's theorem).

Nowadays one can find numerous other proofs in the literature. However, the concrete example illustrating the invalidity of Blumenthal's assertion was missing until 2000, ...

Criteria for uniqueness of the measure of orthogonality μ

Theorem

The measure of orthogonality of the monic OPs $\{p_n\}$ is unique iff there exists at least one $z \in \mathbb{C} \setminus \mathbb{R}$ such that

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Remarks:

- If the measure of orthogonality μ is unique then

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  if x is a mass point of the measure μ.
- In the case of non-uniqueness the sum above is the value of the largest possible jumps of a measure μ at x and there always exists a measure realizing this jump.

Recall the *n*th moment of Borel measure μ on $\mathbb{R}$ is defined as

$$m_n := \int_{\mathbb{R}} x^n d\mu(x), \qquad \text{(provided the integral exists).}$$

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Theorem (Carleman, 1926)

The measure of orthogonality μ of monic OPs p_n is unique if one of the following condition holds:

1. $$\sum_{n=0}^{\infty} \frac{1}{\sqrt[2n]{m_{2n}}} = \infty,$$

2. $$\sum_{n=0}^{\infty} \frac{1}{\sqrt{b_n}} = \infty.$$

- Hermite:

$$m_{2n} = \int_{\mathbb{R}} x^{2n} e^{-x^2} \mathrm{d}x = \Gamma\left(n + \frac{1}{2}\right)$$

Since $\Gamma(n + 1/2) \leq n!$ we have

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt[2n]{m_{2n}}} \geq \sum_{n=0}^{\infty} \frac{1}{\sqrt[n]{n!}} = \infty \quad \text{(Stirling's formula)}$$

At the same time $b_n = (n + 1)/2$ hence also $\sum_n b_n^{-1/2} = \infty$. Consequently, the measure of orthogonality of Hermite OPs is **unique**. (In other words: “Gaussian normal distribution is uniquely determined by its moments.”)

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- All OPs from the Askey scheme: **unique** measure of orthogonality.

Example of non-unique orthogonality measure

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$$\int_0^\infty x^k x^{-\ln x}\sin(2\pi\ln x)\mathrm{d}x = 0, \quad \forall k \in \mathbb{Z}_+,$$
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- Moreover, note the function
$$f_\theta(x) = \frac{\sin(2\pi \ln x)}{1 + \theta \sin(2\pi \ln x)}$$
is in $L^2(\mathbb{R}_+, \mathrm{d}\mu_\theta)$ and is orthogonal to all polynomials.

The previous example is due to Thomas Joannes Stieltjes:

- Dutch mathematician, born in 1856 in Zwolle, died in 1894 in Toulouse at the age of 38!
- 1877 - Assistant at Leiden observatory
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- His work is also seen as important as a first step towards the theory of Hilbert spaces.

Stieltjes transform of a measure and associated OPs

- Let μ be finite Borel measure on $\mathbb{R}$. The Stieltjes (Chauchy) transform is given by the formula
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- Assume μ is a probability measure ($m_0 = 1$). It can be shown polynomials p_n and $p_{n-1}^{(1)}$ are related by formula
$$p_{n-1}^{(1)}(x) = \int_{\mathbb{R}} \frac{p_n(x) - p_n(y)}{x-y} \mathrm{d}\mu(y).$$

Theorem (essentially due to Markov)

Suppose the measure of orthogonality μ ($m_0 = 1$) of monic OPs p_n is unique. Then it holds

$$\lim_{n\to\infty} \frac{p_{n-1}^{(1)}(x)}{p_n(x)} = \int_{\mathbb{R}} \frac{d\mu(x)}{x - z}$$

and the convergence is local uniform on $\mathbb{C} \setminus \mathbb{R}$.

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- The theorem as stated has been proved then by Hamburger in 1920. In the respective paper he treated the complete convergence of continued fractions.

Measures in case of non-uniqueness - Nevanlinna functions

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converge locally uniformly on $\mathbb{C}$.

- Consequently four Nevanlinna functions

$$A(z) := z \sum_{n=0}^{\infty} \frac{p_n^{(1)}(0) p_n^{(1)}(z)}{b_1 b_2 \dots b_n}, \quad B(z) := -1 + z \sum_{n=1}^{\infty} \frac{p_{n-1}^{(1)}(0) p_n(z)}{\sqrt{b_0} b_1 \dots b_{n-1}},$$

$$C(z) := 1 + z \sum_{n=1}^{\infty} \frac{p_n(0) p_{n-1}^{(1)}(z)}{\sqrt{b_0} b_1 \dots b_{n-1}}, \quad D(z) := z \sum_{n=0}^{\infty} \frac{p_n(0) p_n(z)}{b_0 b_1 \dots b_{n-1}}$$

are well defined entire functions.

Theorem (Nevanlinna, 1922)

All the measures of orthogonality for OPs in the case of non-uniqueness are parametrized via homeomorphism $\varphi \mapsto \mu_\varphi$ of $\mathcal{P} \cup \{\infty\}$ onto the set of all measures of orthogonality given by

$$\int_{\mathbb{R}} \frac{d\mu_\varphi(x)}{x - z} = -\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}$$

where $\mathcal{P}$ is the set of holomorphic functions in the upper half-plane $\{z \in \mathbb{C} \mid \Im z > 0\}$ with nonnegative imaginary part (Pick functions).

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- μ_t are also very closely related with spectral measures of all self-adjoint extensions of the corresponding Jacobi operator.

End of story - Starring: