A Borg-Marchenko type theorem for Schrödinger operators with complex potentials

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Mathematical aspects of the physics with non-self-adjoint operators
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Contents

An abstract framework

- Given: $A = A^*$ on a Hilbert space $\mathcal{H}$,
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  for special classes of operators A.
- Direct problem: Given A (and δ), examine μ.
- Inverse problem: a) Uniqueness: Sometimes μ determines A.
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Example 1 - Jacobi matrices

- Set $\mathcal{H} := \ell^2(\mathbb{N}_0)$,

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Theorem

a) Uniqueness: J is uniquely determined by μ.
b) Surjectivity: Given a probability measure μ with compact and infinite support in $\mathbb{R}$, there exists J such that μ is the spectral measure of J.

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- Set $\mathcal{H} := L^2(\mathbb{R}_+)$ and $A := H$ the Schrödinger operator in $L^2(\mathbb{R}_+)$ given by

$$\boxed{Hf(x) = -f''(x) + q(x)f(x)}$$

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- Put formally $\delta := \delta_0$ the Dirac delta function at 0. Then

$$\langle (H-\lambda)^{-1}\delta_0, \delta_0\rangle = \mathcal{G}(0,0;\lambda) = \int_{\mathbb{R}} \frac{\mathrm{d}\sigma(t)}{t-\lambda},$$

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(Surjectivity of the spectral mapping $H \mapsto \sigma$ open.)

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where $b_n \in \mathbb{C}$, $a_n > 0$, a_n, b_n bounded.

- Then $\Lambda : J \mapsto (\nu, \psi)$ is a bijection, i.e.

Theorem (Pushnitski–Š., 2024)

a) Uniqueness: J is uniquely determined by (ν, ψ).
b) Surjectivity: Let ν is a probability meas. on $[0, \infty)$ with compact and infinite support.
Let $\psi \in L^\infty(\nu)$ be such that $\psi(0) = 1$ and $|\psi(s)| \leq 1$ for ν-a.e. $s \geq 0$.
Then (ν, ψ) is a spectral data for some J.

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Definition of the spectral data of Schrödinger operators 1/2

- "Hermitization" of H:

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- There is a unique 2×2 matrix-valued spectral measure Σ on $\mathbb{R}$ such that

$$M(\lambda) = \int_{\mathbb{R}} \frac{\mathrm{d}\Sigma(t)}{t - \lambda}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

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Proposition

There exists a unique *even scalar measure* ν on $\mathbb{R}$ and a unique *odd function* $\psi \in L^{\infty}(\nu)$ satisfying $|\psi(t)| \leq 1$ for ν-a.e. $t \in \mathbb{R}$ such that the spectral measure Σ of $\mathbf{H}$ has the structure

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- (ν, ψ) is the spectral data of H.
- If $q = 0$, then

$$\nu(s) = \frac{\mathrm{d}s}{2\pi\sqrt{s}}, \quad \psi(s) = 1, \quad s > 0.$$

Properties of spectral data (ν, ψ) of H

Proposition

The spectrum of $|H|$ has

$$\text{multiplicity one on } S_1 := \{s > 0 : |\psi(s)| = 1\},$$
$$\text{multiplicity two on } S_2 := \{s > 0 : |\psi(s)| < 1\}.$$

(The Borel representative of $\psi \in L^\infty(\nu)$ is fixed such that $|\psi(s)| \leq 1$ for all $s \in \mathbb{R}$.)

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2. $H \geq 0$ if and only if $\psi(s) = 1$ ν-a.e. $s > 0$.
3. If $H = H^*$, then for all $t \in \mathbb{R}$, we have

$$\boxed{\mathrm{d}\sigma(t) = (1 + \psi(t))\,\mathrm{d}\nu(t).}$$

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Thank you!