On the Hilbert L-matrix

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Contents

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3. Open problems: Determine the numbers

$$\nu_0 := \inf\{\nu > 0 \mid \|L_\nu\| = 4\} \qquad \text{and} \qquad \|L_\nu\|, \ \text{ for } \nu < 1/2.$$

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If $\mathcal{A}$ is an L-matrix, then the standard construction is applicable iff its parameter seq. $a \in \ell^2(\mathbb{N}_0)$.

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$$b_n := \frac{1}{a_n - a_{n+1}}, \qquad \mathcal{J} := \begin{pmatrix} b_0 & -b_0 & & & & \\ -b_0 & b_0 + b_1 & -b_1 & & & \\ & -b_1 & b_1 + b_2 & -b_2 & & \\ & & -b_2 & b_2 + b_3 & -b_3 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

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Definition (L-operator)

To the L-matrix $\mathcal{L}$, s.t. $a_n \neq a_{n+1}$, $\forall n \in \mathbb{N}_0$, we associate the *$L$-operator* $L := J^{-1}$.

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Example:

- Consider the *L*-matrix

$$\mathcal{L} = \begin{pmatrix} H_1 & H_2 & H_3 & \dots \\ H_2 & H_2 & H_3 & \dots \\ H_3 & H_3 & H_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}.$$

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- Then $b_n = -(n+1)$ and the Jacobi operator J corresponds to the Laguerre polynomials (up to the sign).
- One can use well-known properties of the Laguerre polynomials to show that the spectrum of the L-operator $L = J^{-1}$ is simple and $\sigma(L) = \sigma_{ac}(L) = (-\infty, 0]$.

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- For general ν, OGPs unknown but a closely related study has been done in [Ismail, Letessier, Valent, *SIAM J. Math. Anal.* (1989)]

Spectral analysis of J_ν

- Spectral analysis of J_ν is possible in terms of the regularized hypergeometric functions with unit argument:

$$ {}_3\tilde{F}_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix} \,\middle|\, 1 \right) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n}{n!\, \Gamma(b_1+n)\Gamma(b_2+n)}. $$

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The function is analytic in $\Re(b_1 + b_2 - a_1 - a_2 - a_3) > 0$.

- One has

$$ \boxed{J_\nu \phi(z;\nu) = \left(\frac{1}{4} - z^2\right)\phi(z;\nu) + \chi(z;\nu) e_0,} $$

where

$$ \phi_n(z;\nu) := \frac{\Gamma(n+\nu)\Gamma(n+\nu+1)}{\Gamma(z+3/2)}\, {}_3\tilde{F}_2\left(\begin{array}{c} z-1/2, n+\nu, n+\nu \\ n+\nu+z+1/2, n+\nu+z+1/2 \end{array}\middle|\, 1\right) $$

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Spectrum of J_ν for general ν

- Asymptotic analysis of the involved functions, etc. (many details omitted), yields the spectrum of J_ν for general ν.

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- Since $L_\nu = J_\nu^{-1}$ the result readily translates to L_ν...

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A closer look at $\sigma_p(J_\nu)$ for $\nu > 0$

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$$\nu \mapsto {}_3F_2\left(\begin{matrix} -1/2, 1/2, 3/2 \\ 1, \nu + 1/2 \end{matrix} \middle| 1\right) \qquad (= c \cdot \chi(0;\nu))$$

has a unique positive zero ν_0 which is located in $(0, 1/2)$; numerically $\nu_0 \approx 0.3491$.

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2. We have
$$\sigma_p(J_\nu) = \begin{cases} \emptyset, & \text{if } \nu \geq \nu_0, \\ 1/4 - x_0^2(\nu), & \text{if } 0 < \nu < \nu_0, \end{cases}$$

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$$\sigma_p(J_\nu) = \begin{cases} \emptyset, & \text{if } \nu \geq \nu_0, \\ 1/4 - x_0^2(\nu), & \text{if } 0 < \nu < \nu_0, \end{cases}$$
where $x_0(\nu)$ is the unique zero of the function
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A closer look at $\sigma_p(J_\nu)$ for $\nu > 0$

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The point spectrum of L_ν for $\nu > 0$

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$$\|L_\nu\| = \frac{4}{1 - 4x_0^2(\nu)}.$$

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...to show an animation.

The point spectrum of L_ν for $\nu < 0$

Conjecture

Suppose $\nu < 0$ and $-\nu \notin \mathbb{N}$. Then $\sigma(L_\nu)$ consists of exactly one negative eigenvalue and none or exactly one eigenvalue of L_ν greater than 4.

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More precisely, there are numbers $-2 < \nu_3 < \nu_2 < -1 < \nu_1 < 0$ such that

$$\sigma_p(L_\nu) = \begin{cases} \{\lambda_-(\nu)\}, & \text{for } \nu \in (\nu_3, \nu_2) \cup (\nu_1, 0), \\ \{\lambda_-(\nu), \lambda_+(\nu)\}, & \text{otherwise,} \end{cases}$$

where $\lambda_-(\nu) < 0$ and $\lambda_+(\nu) > 4$.

...to show an animation.

Thank you!