An Inverse spectral problem for complex Jacobi matrices

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Jacobi matrix: $J = \begin{bmatrix} b_0 & a_0 & & \\ a_0 & b_1 & a_1 & \\ & a_1 & b_2 & a_2 \\ & & \ddots & \ddots & \ddots \end{bmatrix}$ on $\ell^2(\mathbb{N}_0)$

$\{\delta_n\}_{n=0}^{\infty}$ stand. basis of $\ell^2(\mathbb{N}_0)$

0. SA case – motivation: Assump.: a) J hermitian: a_n, b_n real,

b) J bounded: a_n, b_n bounded,

c) J non-deg.: $a_n \neq 0$

$\oplus$ normalization: $a_n > 0$,

Thm. ("Direct SP"): J sat. (a)–(c) $\Rightarrow$ δ_0 is a cyclic vector of J.

Consequently, J is unitarily equiv. to ~~unique~~ mult. oper. by x on $L^2(\mu)$, where $\mu(\Delta) := \langle \chi_\Delta(J)\delta_0, \delta_0 \rangle$.

Remark: In OPRL lang.: "Favard's theorem".

Spectral mapping: $\underset{\substack{\uparrow \\ \text{(a)–(c)}}}{J} \longmapsto \underset{\substack{\uparrow \\ \text{prob. meas. on } \mathbb{R},\ \operatorname{supp}\mu \text{ compact \& infinite}}}{\mu}$

Thm. ("Inverse SP"):

1) Injectivity: The spec. meas. μ of J determines J sat. (a)–(c) uniquely.

2) Surjectivity: Given a prob. meas. μ on $\mathbb{R}$ with $\operatorname{supp}\mu$ comp. & infinite. Then $\exists\, J$ sat. (a)–(c) s.t. μ is the spectral measure of J.

In total, $J \xrightarrow{1-1} \mu$.

1. NSA case – spectral data : Assump.:

a) $a_n, b_n \in \mathbb{C}$

b) a_n, b_n bounded

c) $a_n > 0$

($J \sim J' \Leftrightarrow |a_n| = |a_n'|$ & pick a repre.)

Measure ν: $|J| := \sqrt{J^* J}$

Def. $\nu(\Delta) := \langle \chi_\Delta(|J|)\delta_0, \delta_0 \rangle$, $\Delta \in \mathcal{B}_{\mathbb{R}}$

Thm. ("Direct SP"): The spec. of $|J|$ has multiplicity ≤ 2 and δ_0 is a vector of maximal type of $|J|$: $\forall \Delta \in \mathcal{B}_{\mathbb{R}}$: $\chi_\Delta(|J|)\delta_0 = 0 \Rightarrow \chi_\Delta(|J|) = 0$.

Phase func. ψ:

Thm: J sat. (a)-(c). $\Rightarrow \exists_1 \psi \in L^\infty(\nu)$, $|\psi(s)| = 1$ ν-a.e. $s > 0$, $\psi(0) = 1$

s.t. $\forall f \in C(\mathbb{R})$: $\langle J f(|J|)\delta_0, \delta_0 \rangle = \int_0^\infty s f(s) \psi(s) d\nu(s)$.

Rem.: Let $|J|$ has simple & discr. spec., denote $\{s_k\}_{k=0}^\infty$ s.v. of J.

Then $\exists$ normalized vector $x^{(k)}$ given uniquely up to a sign s.t.

$J x^{(k)} = s_k \overline{x^{(k)}}$.

We have $\nu = \sum_{k=0}^\infty |x_0^{(k)}|^2 \delta_{s_k}$, $\psi(s_k) = \frac{x_0^{(k)}}{\overline{x_0^{(k)}}}$.

Spectral mapping: $J \mapsto (\nu, \psi) =: \Lambda(J)$.

2. Inverse SP:

Thm. ("Inverse SP"):

1) Injectivity: J sat. (a)-(c) is determined uniquely by $\Lambda(J) \equiv (\nu, \psi)$.

2) Surjectivity: Given a prob. meas. ν on $[0, \infty)$ with $\operatorname{supp}\nu$ compact & infinite, given ψ Borel meas., $|\psi(s)| = 1$ ν-a.e. $s > 0$, $\psi(0) = 1$. Then $\exists J$ sat. (a)-(c) s.t. $\Lambda(J) = (\nu, \psi)$.

In total, $\Lambda : J \xleftrightarrow{1-1} (\nu, \varphi)$.

"Proof":

0) $\begin{pmatrix} 0 & J \\ J^* & 0 \end{pmatrix}$ on $\ell^2(\mathbb{N}_0) \oplus \ell^2(\mathbb{N}_0) \xleftrightarrow[\text{unit.}]{1-1} \mathbb{J} := \begin{bmatrix} B_0 & A_0 & & \\ A_0^* & B_1 & A_1 & \\ & A_1^* & B_2 & A_2 \\ & & \ddots & \ddots & \ddots \end{bmatrix}$ on $\ell^2(\mathbb{N}_0; \mathbb{C}^2)$, $\quad B_n = \begin{pmatrix} 0 & b_n \\ \overline{b_n} & 0 \end{pmatrix}$, $\quad A_n = \begin{pmatrix} 0 & a_n \\ \overline{a_n} & 0 \end{pmatrix}$

$\mathbb{J} = \mathbb{J}^* \rightsquigarrow$ spec. meas. $\mathbb{M}$ (2×2-matrix valued)

$$d\mathbb{M}(s) = \frac{1}{2} \begin{cases} \begin{pmatrix} 1 & \varphi(s) \\ \overline{\varphi}(s) & 1 \end{pmatrix} d\nu(s), & s \geq 0 \\ \begin{pmatrix} 1 & -\varphi(-s) \\ -\overline{\varphi}(-s) & 1 \end{pmatrix} d\tilde{\nu}(s), & s < 0 \end{cases}$$

here $\tilde{\nu}(+\Delta) := \nu(-\Delta)$

1) Injectivity:

$J \longrightarrow \Lambda(J) \equiv (\nu, \varphi) \longrightarrow \mathbb{M} \xrightarrow{\text{theory of MOPRL}} [\mathbb{J}] \longrightarrow \mathbb{J} \in [\mathbb{J}]$ the only with $B_n = \begin{pmatrix} 0 & b_n \\ \overline{b_n} & 0 \end{pmatrix}$, $A_n = \begin{pmatrix} 0 & a_n \\ a_n & 0 \end{pmatrix}$, a_n, b_n sat. (a)-(c).

$\longrightarrow J$

2) Surjectivity:

$(\nu, \varphi) \longrightarrow \underset{\text{non-deg.}}{\mathbb{M}} \longrightarrow [\mathbb{J}] \longrightarrow \exists\, \mathbb{J}' \in [\mathbb{J}]$ with A_n, B_n off-diag.

$\longrightarrow (a_n, b_n) \longrightarrow J$.

3. Special classes: J sat. (a)–(c).

Thm.: 1) $|J|$ has simple spectrum $\Leftrightarrow$ $|\varphi(s)|=1$ ν-a.e. $s>0$,

2) $J=J^*$ $\Leftrightarrow$ φ real. ν-a.e.,

3) $b_n=0$ $\Leftrightarrow$ $\varphi(s)=0$ ν-a.e. $s>0$,

4) $JJ^*=J^*J$ and $|J|$ has simple spec. $\Rightarrow$ $J=|J|\varphi(|J|)$.

Thm.: $J=J^*$: $d\mu(s)=\frac{1}{2}(1+\varphi(s))\,d\nu(s)$, $s\geq 0$,

$d\tilde{\mu}(s)=\frac{1}{2}(1-\varphi(s))\,d\nu(s)$, $s>0$.

4. "Anti-symmetric" OGPs: J sat. (a)–(c).

Def. $\{q_n\}_{n=0}^{\infty}$, $q_0(s)=1$, $a_{n-1}q_{n-1}(s)+b_nq_n(s)+a_nq_{n+1}(s)=s\,\overline{q_n(s)}$, $n\geq 0$ $(q_{-1}:=0)$

$$Jq(s)=s\,\overline{q(s)}.$$

Thm.:

$$\frac{1}{2}\int\left\langle\begin{pmatrix}1+\operatorname{Re}\varphi(s) & -i\operatorname{Im}\varphi(s)\\ i\operatorname{Im}\varphi(s) & 1-\operatorname{Re}\varphi(s)\end{pmatrix}\begin{pmatrix}q_m(s)\\ q_m(-s)\end{pmatrix},\begin{pmatrix}q_n(s)\\ q_n(-s)\end{pmatrix}\right\rangle_{\mathbb{C}^2}d\nu(s)=\delta_{m,n}$$

Rem.: $J=J^*$ $\Rightarrow$ rel. μ & (ν,φ) $\longrightarrow$ simplifies to the standard form:

$$\int_{\mathbb{R}}p_n(x)p_m(x)\,d\mu(x)=\delta_{m,n}$$

Example: $a_n=1$, $b_n=\omega\in\mathbb{C}$ $\Rightarrow$ $\nu_\omega=\ldots$, $\varphi_\omega=\ldots$

More examples? MOPRL with $B_n=\begin{pmatrix}0 & b_n\\ \overline{b_n} & 0\end{pmatrix}$, $A_n=\begin{pmatrix}0 & a_n\\ \overline{a_n} & 0\end{pmatrix}$ with known $\mathbb{M}$?

5. Related & on-going works:

- Hankel matrices – [P. Gerard, A. Pushnitski]
- Block Jacobi matrices with complex entries
- Sturm-Liouville (Schrödinger) operators

works in progress...