Spectrum of the Hilbert *L*-matrix

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F. Š.: Asymptotic spectral properties of the Hilbert *L*-matrix, `arXiv:2202.04116`

Contents

1 *L*-matrices and *L*-operators

2 The Hilbert *L*-operator

3 Finite Hilbert *L*-matrix

L-matrix vs. Hankel matrix

- L-matrix $\mathcal{L}$:

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- Goal 1: Definition of an *L*-operator on $\ell^2(\mathbb{N}_0)$.

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- Goal 1: Definition of an *L*-operator on $\ell^2(\mathbb{N}_0)$.
- Goal 2: Spectral analysis.

The standard construction

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- Operators $A_{\max}$ and $A_{\min}$:

$$A_{\max}x := \mathcal{A} \cdot x \quad \text{and} \quad A_{\min}x := \mathcal{A} \cdot x,$$

where x is from the respective domain:

$$\operatorname{Dom} A_{\max} := \{x \in \ell^2 \mid \mathcal{A} \cdot x \in \ell^2\}$$

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If $\mathcal{A}$ is an L-matrix, then the standard construction is applicable iff $a \in \ell^2(\mathbb{N}_0)$.

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$$b_n := \frac{1}{a_n - a_{n+1}}, \qquad \mathcal{J} := \begin{pmatrix} b_0 & -b_0 & & & & \\ -b_0 & b_0 + b_1 & -b_1 & & & \\ & -b_1 & b_1 + b_2 & -b_2 & & \\ & & -b_2 & b_2 + b_3 & -b_3 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

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Definition (*L*-operator)

To the *L*-matrix $\mathcal{L}$, s.t. $a_n \neq a_{n+1}$, $\forall n \in \mathbb{N}_0$, we associate the *L-operator* $L := J^{-1}$.

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Example:

- Consider the *L*-matrix

$$\mathcal{L} = \begin{pmatrix} H_1 & H_2 & H_3 & \dots \\ H_2 & H_2 & H_3 & \dots \\ H_3 & H_3 & H_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}.$$

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- Then $b_n = -(n+1)$ and the Jacobi operator J corresponds to the Laguerre polynomials (up to a sign).

Definition of the L-operator (cont.)

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- Consider the L-matrix

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- Then $b_n = -(n+1)$ and the Jacobi operator J corresponds to the Laguerre polynomials (up to a sign).
- One can use well-known properties of the Laguerre polynomials to show that the spectrum of the L-operator $L = J^{-1}$ is simple and $\sigma(L) = \sigma_{ac}(L) = (-\infty, 0]$.

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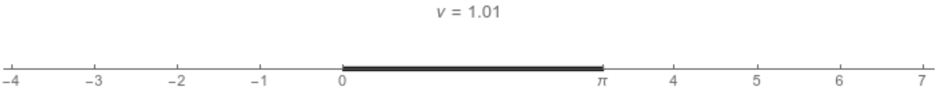
The generalized Hilbert matrix

The Hankel matrix $H(\nu)$ with

$$a_n = a_n(\nu) = \frac{1}{n+\nu}, \qquad \nu \in \mathbb{R} \setminus (-\mathbb{N}_0),$$

is called the *generalized Hilbert matrix*.



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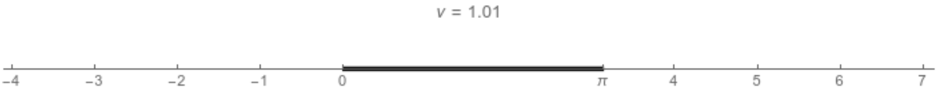
Theorem [M. Rosenblum, 1958]

For $\nu \in \mathbb{R} \setminus (-\mathbb{N}_0)$, one has $\sigma(H(\nu)) = \sigma_{ac}(H(\nu)) \cup \sigma_d(H(\nu))$, where

$$\sigma_{ac}(H(\nu)) = [0, \pi] \quad \text{and} \quad \sigma_d(H(\nu)) = \begin{cases} \emptyset, & \frac{1}{2} \leq \nu, \\ \left\{ \frac{\pi}{\sin \pi\nu} \right\}, & -\frac{1}{2} \leq \nu < \frac{1}{2}, \\ \left\{ \pm \frac{\pi}{\sin \pi\nu} \right\}, & \nu < -\frac{1}{2}. \end{cases}$$



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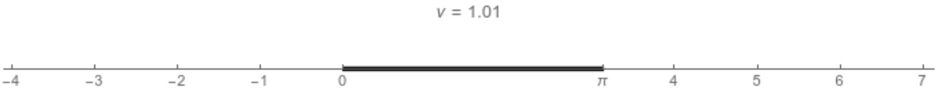

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Open problems: Determine the numbers

$$\nu_0 := \inf\{\nu > 0 \mid \|L(\nu)\| = 4\} \qquad \text{and} \qquad \|L(\nu)\|, \text{ for } \nu < 1/2.$$

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- For $\nu = 1$, J_1 corresponds to a subfamily of the Continuous dual Hahn OGPs.
- For general ν, OGPs unknown but a closely related study has been done by [Ismail, Letessier, Valent, 1989]

Spectral analysis of J_ν

- Spectral analysis of J_ν is possible in terms of the regularized hypergeometric functions with unit argument:

$$ {}_3\tilde{F}_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix} \,\middle|\, 1\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n}{n!\,\Gamma(b_1+n)\Gamma(b_2+n)}, $$

where $(a)_n = a(a+1)\ldots(a+n-1)$. The function is analytic in

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- One has

$$ \boxed{J_\nu \phi(z;\nu) = \left(\frac{1}{4} - z^2\right)\phi(z;\nu) + \chi(z;\nu)e_0,} $$

where

$$ \phi_n(z;\nu) := \frac{\Gamma(n+\nu)\Gamma(n+\nu+1)}{\Gamma(z+3/2)}\, {}_3\tilde{F}_2\left(\begin{matrix} z-1/2, n+\nu, n+\nu \\ n+\nu+z+1/2, n+\nu+z+1/2 \end{matrix} \middle| 1\right) $$

and

$$ \chi(z;\nu) := \frac{(z+1/2)\Gamma(\nu)\Gamma(\nu+1)}{\Gamma(z+1/2)}\, {}_3\tilde{F}_2\left(\begin{matrix} \nu-1, \nu+1, z+1/2 \\ z+\nu+1/2, z+\nu+1/2 \end{matrix} \middle| 1\right). $$

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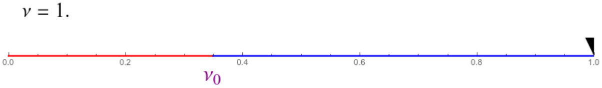
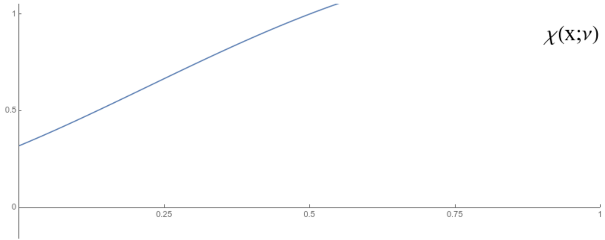
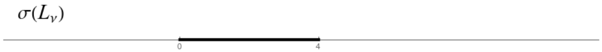

ν = 1.
0.0
0.2
0.4
0.6
0.8
1.0
ν0


1
0.5
0
0.25
0.5
0.75
1
χ(x;ν)


σ(Lν)
0
4

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Conjecture

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Suppose $\nu < 0$ and $-\nu \notin \mathbb{N}$. Then $\sigma(L(\nu))$ consists of exactly one negative eigenvalue and none or exactly one eigenvalue of $L(\nu)$ greater than 4.

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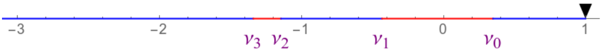
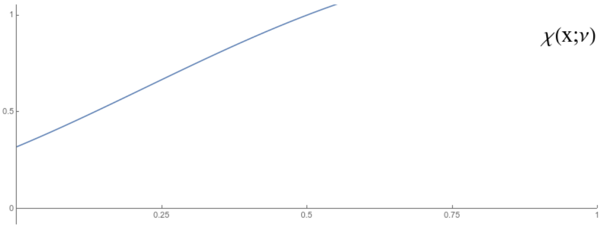
Suppose $\nu < 0$ and $-\nu \notin \mathbb{N}$. Then $\sigma(L(\nu))$ consists of exactly one negative eigenvalue and none or exactly one eigenvalue of $L(\nu)$ greater than 4.
More precisely, there are numbers $-2 < \nu_3 < \nu_2 < -1 < \nu_1 < 0$ such that

$$\sigma_p(L(\nu)) = \begin{cases} \{\lambda_-(\nu)\}, & \text{for } \nu \in (\nu_3, \nu_2) \cup (\nu_1, 0), \\ \{\lambda_-(\nu), \lambda_+(\nu)\}, & \text{otherwise,} \end{cases}$$

where $\lambda_-(\nu) < 0$ and $\lambda_+(\nu) > 4$.

$\nu = 1.$





$\sigma(L_\nu)$

Contents

1 L-matrices and L-operators

2 The Hilbert L-operator

3 Finite Hilbert L-matrix

Truncations of the Hilbert matrix and the L-matrix

- We denote by $L_n(\nu)$ and $H_n(\nu)$ the $n \times n$ sections of $L(\nu)$ and $H(\nu)$, i.e.,

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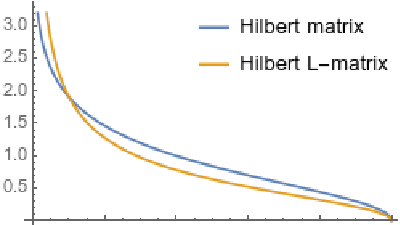

A sketch of the proof

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- By the Argument Principle, we have

$$\#\{\mu\in\sigma(L_n(\nu))\mid x<\mu<4\}=\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_x}\frac{\partial_\xi \det(1-\xi L_n(\nu))}{\det(1-\xi L_n(\nu))}\,\mathrm{d}\xi,$$

where γ_x is a simple closed counter-clockwise oriented curve crossing the real line at the points $1/4$ and $1/x$.

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$$\lim_{n\to\infty}\frac{1}{\log n}\frac{\partial_\xi \det(1-\xi L_n(\nu))}{\det(1-\xi L_n(\nu))}=\frac{\operatorname{sign}(\Im\xi)}{2\mathrm{i}\sqrt{\xi-1/4}}.$$
- By the Argument Principle, we have
$$\#\{\mu\in\sigma(L_n(\nu))\mid x<\mu<4\}=\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_x}\frac{\partial_\xi \det(1-\xi L_n(\nu))}{\det(1-\xi L_n(\nu))}\,\mathrm{d}\xi,$$
where γ_x is a simple closed counter-clockwise oriented curve crossing the real line at the points $1/4$ and $1/x$.
- Squeezing the integration curve towards the real line and computing the resulting integral yields the result.

Small eigenvalues

Theorem [H. Widom, H. S. Wilf, 1966, for $\nu = 1$; Kalyabin, 2001, for $\nu > 0$]

For $\nu > 0$, we have

$$\lambda_{1,n}(\nu) = \frac{2^{15/4}\pi^{3/2}}{(1+\sqrt{2})^{2\nu-2}} \frac{\sqrt{n}}{(1+\sqrt{2})^{4n}} \left(1 + o(1)\right), \quad n \to \infty.$$

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Theorem [F. Š., 2022]

For $\nu > 0$ and $j \in \mathbb{N}$ fixed, we have

$$\mu_{j,n}(\nu) = \frac{1}{4n^2}\left[1 + \frac{i_j}{\sqrt[3]{3}} n^{-2/3} + o\left(n^{-2/3}\right)\right], \quad n \to \infty,$$

where $i_1 < i_2 < \dots$ are positive zeros of the Airy function.

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Theorem [A. Maté, P. Nevai, V. Totik, 1986]

Let $\{\gamma_n\}_{n=1}^{\infty}$ be a positive sequence such that

$$\gamma_n = c^2 n^{2\delta}\left(1 + o\left(n^{-2/3}\right)\right), \quad n \to \infty,$$

where $c, \delta > 0$. Then zeros $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ of polynomial Q_n defined by

$$Q_0(x) = 0, \quad Q_1(x) = x, \quad \text{and} \quad Q_{n+1}(x) = xQ_n(x) - \gamma_n Q_{n-1}(x), \quad n \in \mathbb{N},$$

fulfill

$$x_{n-j+1,n} = 2cn^{\delta}\left[1 - 6^{-1/3}\delta^{2/3} i_j\, n^{-2/3} + o\left(n^{-2/3}\right)\right], \quad n \to \infty,$$

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- Perturbation arguments $\Rightarrow$ the first two terms in the asymptotic expansions of large eigenvalues of $L_n^{-1}(\nu)$ determined by those of $B_n(\nu)$ and $B_{n-1}(\nu)$.
- Polynomials $Q_{2n}(x) := \det(x^2 - B_n(\nu))$ satisfy the M. N. T. recursion with
$$\gamma_n = (\lfloor (n-1)/2 \rfloor + \nu)(\lfloor (n+1)/2 \rfloor + \nu).$$

Large eigenvalues

Theorem [N. G. de Bruijn, H. S. Wilf, 1962]

One has

$$\lambda_{n,n}(2) \equiv \|H_n(2)\| = \pi - \frac{\pi^5}{2\log^2 n} + O\left(\frac{\log\log n}{\log^3 n}\right), \quad n \to \infty.$$

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- The method of their proof admits a generalization...

Theorem [H. S. Wilf, 1970]

Let $\mathcal{K} : (0,\infty)\times(0,\infty)\to\mathbb{R}$ be symmetric, homogeneous of degree -1, and decreasing kernel such that

$$\mathcal{K}(x,1) = O\left(x^{-1/2-\delta}\right), \quad x\to\infty,$$

for some $\delta > 0$. Then the norm of matrix $K_n := (\mathcal{K}(i,j))_{i,j=1}^{n}$ satisfies

$$\|K_n\| = A - \frac{B\pi^2}{\log^2 n} + O\left(\frac{\log\log n}{\log^3 n}\right), \quad n\to\infty,$$

where

$$A = \int_0^\infty \frac{\mathcal{K}(x,1)}{\sqrt{x}}\,\mathrm{d}x \quad \text{and} \quad B = \int_1^\infty \frac{\log^2 x}{\sqrt{x}}\,\mathcal{K}(x,1)\,\mathrm{d}x.$$

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- Wilf's theorem is applicable to $L_n\equiv L_n(1)$:

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$$\sum_{k=1}^{n}\left(\frac{x_1 + \cdots + x_k}{k}\right)^2 \leq \|L_n\| \sum_{k=1}^{n} x_k^2.$$

Large eigenvalues cont.

Theorem [F. Š., 2022]

For $j \in \mathbb{N}$ fixed, we have

$$\mu_{n-j+1,n} = 4 - \frac{16\pi^2 j^2}{\log^2 n} + \frac{32\pi^2 j^2 (\gamma + 6\log 2)}{\log^3 n} + O\left(\frac{1}{\log^4 n}\right), \quad n \to \infty,$$

where γ is the Euler–Mascheroni constant.

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- The method gives expansions to arbitrary order: $\|L_n\| - 4 =$

$$= -\frac{16\pi^2}{\log^2 n} + \frac{32\pi^2\kappa}{\log^3 n} - \frac{16\pi^2(3\kappa^2 - 4\pi^2)}{\log^4 n} + \frac{32\pi^2\left[6\kappa(\kappa^2 - 4\pi^2) - 13\pi^2\zeta(3)\right]}{3\log^5 n} + \dots,$$

where $\kappa := \gamma + 6\log 2$.

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$$\sum_{n=0}^{\infty} q_n(\xi)t^n = (1-t)^{-1/2+\mathrm{i}\xi}\ {}_2F_1\left(\left.\begin{matrix} 1/2+\mathrm{i}\xi,\ 1/2+\mathrm{i}\xi \\ 1 \end{matrix}\ \right|\ t\right).$$

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- Following steps of the Darboux method with some special care of the remainder term yields: $(\forall K > 0)(\exists C_K > 0)(\forall \xi \in (0, K])(\forall n \in \mathbb{N})$

$$q_n(\xi) = \frac{1}{\xi\sqrt{n}}\left[\Im\left(\frac{\Gamma(1+2\mathrm{i}\xi)}{\Gamma^3(1/2+\mathrm{i}\xi)}\right)\cos(\xi\log n) + \Re\left(\frac{\Gamma(1+2\mathrm{i}\xi)}{\Gamma^3(1/2+\mathrm{i}\xi)}\right)\sin(\xi\log n)\right] + R_n(\xi),$$

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- The desired asymptotic expansion of the large eigenvalues of L_n can be computed from the expression in [...].

Thank you!