Spectral bounds for 1D discrete Schrödinger operators with complex potentials

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Joint with: O. O. Ibrogimov, D. Krejčiřík, A. Laptev

Workshop on Operator Theory, Complex Analysis, and Applications

June 22, 2021

Contents

Motivation: the (continuous) Schrödinger operator on the line

Theorem (Abramov, Aslanyan, Davies [JPA, 2001])

For a complex valued $V \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, one has

$$\sigma_p\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V\right) \setminus [0,\infty) \subset \left\{\lambda \in \mathbb{C} \;\middle|\; 4|\lambda| \leq \|V\|^2_{L^1(\mathbb{R})}\right\}.$$

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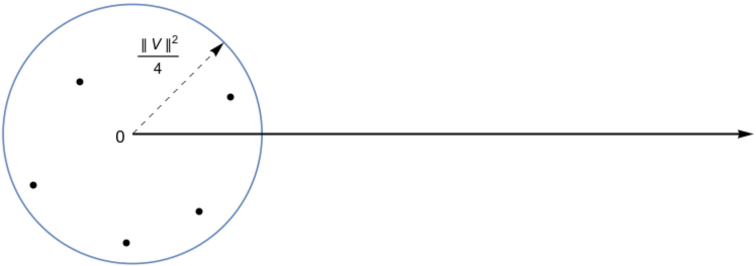

The discrete Schrödinger operator on $\mathbb{Z}$

- Difference operators on $\ell^2(\mathbb{Z})$:

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$$(D^*D\psi)_n = -\psi_{n-1} + 2\psi_n - \psi_{n+1}, \qquad n \in \mathbb{Z}.$$

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where $\lambda = k^{-1} + k$.
- The Joukowski map:
$$\lambda(k) = k^{-1} + k$$
is 1–1 mapping of the punctured unit disk $0 < |k| < 1$ onto $\mathbb{C} \setminus [-2, 2]$.

The spectral enclosure for ℓ^1-potentials

Theorem (Ibrogimov, F. Š. [IEOT, 2019])

Let $v \in \ell^1(\mathbb{Z})$. Then

$$\sigma_{\mathrm{p}}(H_0 + V) \subset \left\{ \lambda \in \mathbb{C} \setminus (-2, 2) \;\middle|\; |\lambda^2 - 4| \leq \|v\|^2_{\ell^1(\mathbb{Z})} \right\}.$$

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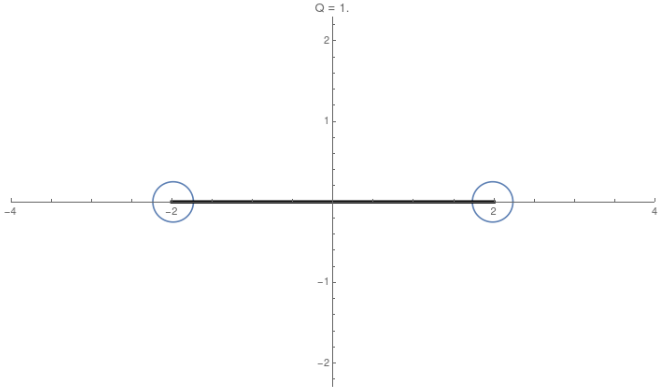
- Proof based on the Birmann–Schwinger principle.

Geometry of the spectral enclosure

The boundary curve for $Q := \|v\|_{\ell^1(\mathbb{Z})}$: $$|\lambda^2 - 4| = Q^2.$$

...it is the Cassini oval with two foci at ± 2.



The optimality

- Delta potential:
$$v_n := \omega\delta_{n,0}, \quad \forall n \in \mathbb{Z},$$
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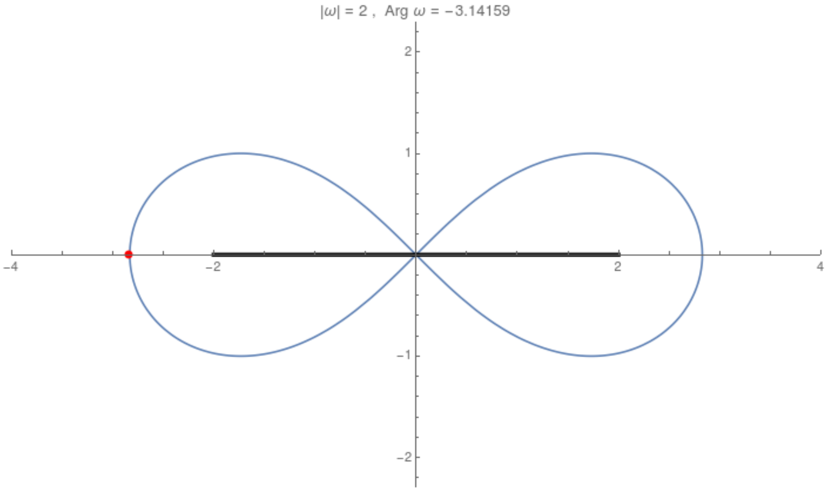
- Moreover, for any $Q > 0$, one has

$$\{\lambda_\omega \mid \omega = Qe^{\mathrm{i}\theta}, -\pi < \theta \leq \pi\} = \{\lambda \in \mathbb{C} \mid |\lambda^2 - 4| = Q^2\}.$$

Numerical illustration: the delta potential demonstrates optimality



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Theorem (Frank, Laptev, Seiringer [OTAA, 2011])

Let $V \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ be complex valued and $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ be the Schrödinger operator acting on $L^2(\mathbb{R}_+)$ with *Dirichlet* boundary condition at 0. Then

$$\sigma_d\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V\right) \subset \left\{\lambda \in \mathbb{C} \,\middle|\, 4|\lambda| \leq h\left(\cot\frac{\arg\lambda}{2}\right)\|V\|^2_{L^1(\mathbb{R}_+)}\right\},$$

where

$$h(a) := \sup_{y\geq 0}\left|e^{\mathrm{i}a\,y} - e^{-y}\right|^2$$

Moreover, the bound is sharp. (The cases of Neumann and Robin b.c. also therein.)

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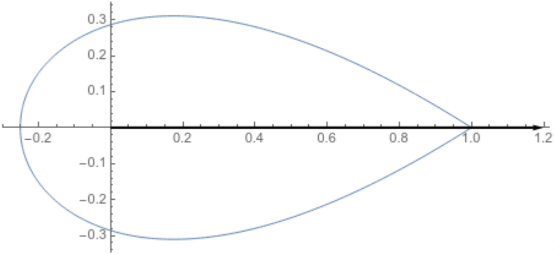

Discrete Schrödinger operators on ℕ

- Difference operators on $\ell^2(\mathbb{N})$:

$$(D\psi)_n := \begin{cases} \psi_{n-1} - \psi_n, & n > 1, \\ -\psi_1, & n = 1, \end{cases} \qquad (D^*\psi)_n = \psi_{n+1} - \psi_n, \qquad n \in \mathbb{N}.$$

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- The discrete *Dirichlet* and *Neumann* Laplacians on $\mathbb{N}$:

$$D^*D = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad DD^* = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

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- We denote $J_0 := 2 - D^*D$ and $J_1 := 2 - DD^*$. Then

$$J_0 = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad J_1 := \begin{pmatrix} 1 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

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- For $a \in \mathbb{C}$, we put
$$J_a := \begin{pmatrix} a & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$
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$$\boxed{J_a + V} = \begin{pmatrix} a + v_1 & 1 & & & \\ 1 & v_2 & 1 & & \\ & 1 & v_3 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Spectral properties of J_a

- Spectrum of J_a and its parts:

$$\sigma_{\mathrm{c}}(J_a) = [-2, 2], \quad \sigma_{\mathrm{r}}(J_a) = \emptyset, \quad \text{and} \quad \sigma_{\mathrm{p}}(J_a) = \begin{cases} \emptyset, & \text{if } |a| \leq 1, \\ \{a + a^{-1}\}, & \text{if } |a| > 1. \end{cases}$$

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$$\boxed{(J_a - \lambda)^{-1}_{m,n} = \frac{(k-a)k^{m+n-1} - (k^{-1}-a)k^{|n-m|+1}}{(1-ak)(k^{-1}-k)}} \qquad m,n \in \mathbb{N},$$

where $\lambda = k^{-1} + k \notin \sigma(J_a)$ for $0 < |k| < 1$.

The spectral enclosure for ℓ^1-potentials

Theorem

Let $a \in \mathbb{C}$ and $v \in \ell^1(\mathbb{N})$. Then

$$\sigma_{\mathrm{p}}(J_a + V) \subset \left\{ \lambda \in \mathbb{C} \setminus (-2,2) \;\middle|\; |\lambda^2 - 4| \leq g_a(\lambda) \, \|v\|^2_{\ell^1(\mathbb{N})} \right\},$$

where

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Remark:

Dirichlet: $g_0(\lambda) = \sup_{n \in \mathbb{N}} \left| 1 - k^{2n} \right|$ Neumann: $g_1(\lambda) = \sup_{n \in \mathbb{N}} \left| 1 + k^{2n-1} \right|$

Geometry of the optimal spectral enclosures

The boundary curve for $Q := \|v\|_{\ell^1(\mathbb{N}_0)}$:

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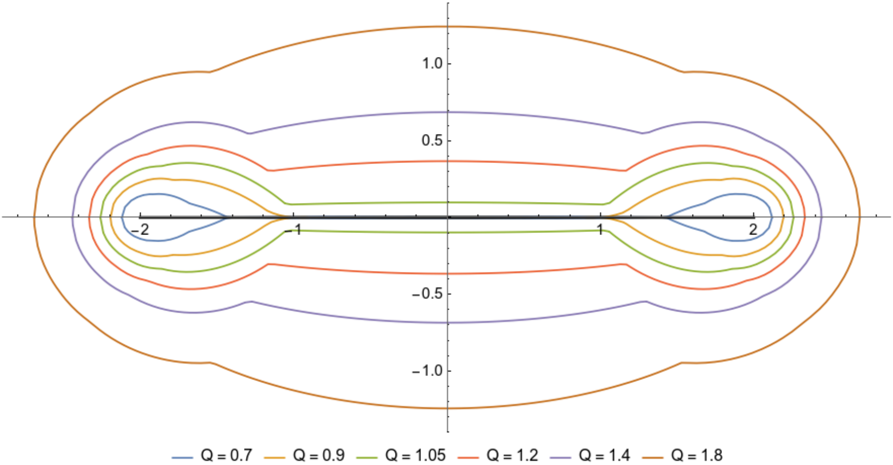
The boundary curve for $Q := \|v\|_{\ell^1(\mathbb{N})}$: $|\lambda^2 - 4| = g_0(\lambda)\, Q^2.$ Dirichlet



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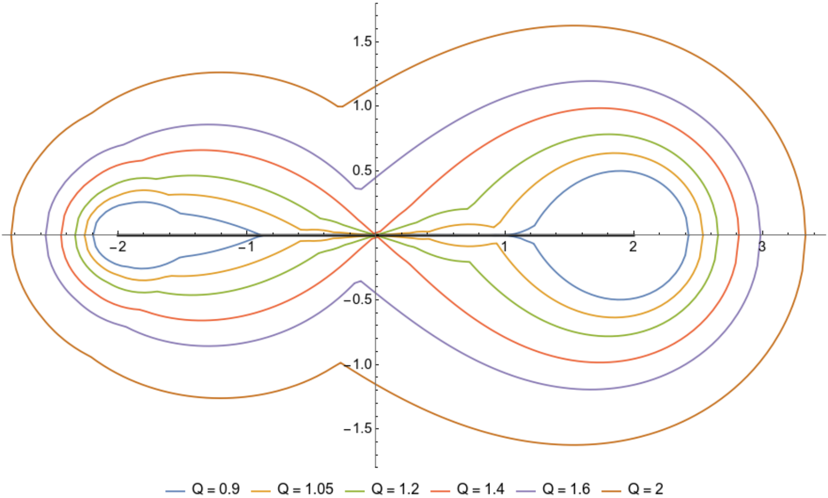

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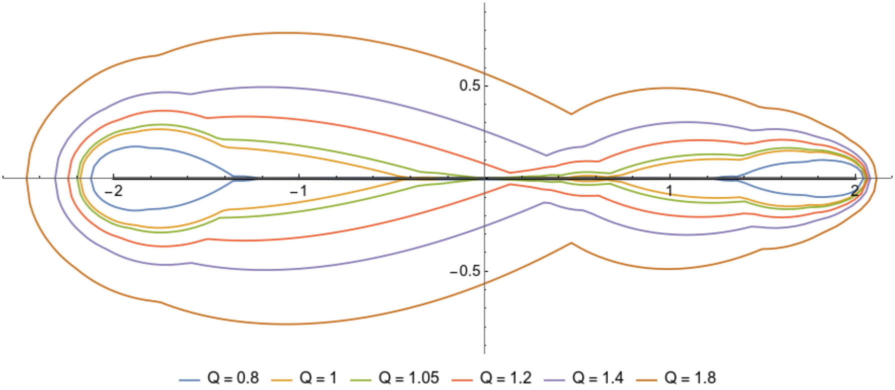

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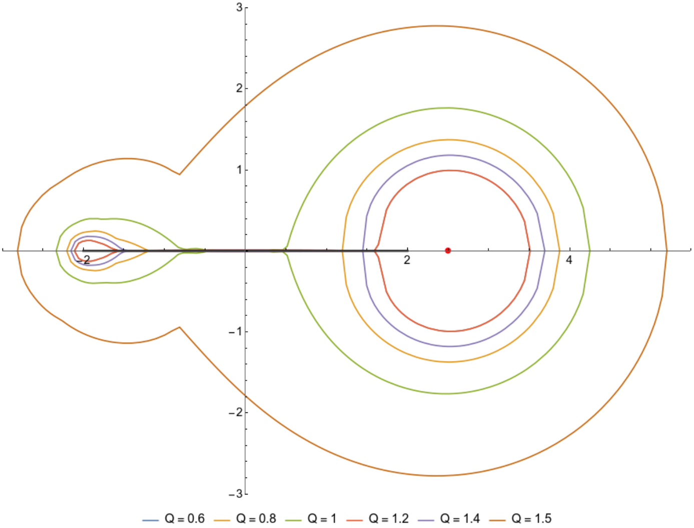

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Theorem

If the matrix K with elements

$$K_{m,n} = \sqrt{|v_n|}\,\min(m,n)\sqrt{|v_m|}, \quad m,n \in \mathbb{N},$$

satisfies $\|K\| < 1$, then $\sigma(J_0 + V) = \sigma_c(J_0 + V) = \sigma(J_0) = [-2, 2].$

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satisfies $\|K\| < 1$, then $\boxed{\sigma(J_0 + V) = \sigma_c(J_0 + V) = \sigma(J_0) = [-2,2].}$

Equivalently, if there exists $c < 1$ such that for all $\psi \in \ell^2(\mathbb{N})$ it holds

$$\sum_{n=1}^{\infty} |v_n||\psi_n|^2 \leq c \sum_{n=1}^{\infty} |\psi_n - \psi_{n-1}|^2 \qquad (\textit{Hardy})$$

where $\psi_0 := 0$, then $\boxed{\sigma(J_0 + V) = \sigma_c(J_0 + V) = \sigma(J_0) = [-2,2].}$

Spectral stability: perturbations not producing discrete eigenvalues

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Proofs is based on the B.–S. principle and ideas from [Hansmann, Krejčiřík, JAM2021].

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Remark: No similar stability for H_0 or J_1 (H_0, J_1 are critical).

Discrete Hardy inequalities

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The classical discrete Hardy inequality [Hardy, Landau, 1921]

For all $\psi \in \ell^2(\mathbb{N})$, one has

$$\sum_{n=1}^{\infty} |\psi_n - \psi_{n-1}|^2 \geq \sum_{n=1}^{\infty} \frac{1}{4n^2} |\psi_n|^2.$$

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For all $\psi \in \ell^2(\mathbb{N})$, one has

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where

$$w_n = 2 - \sqrt{1 - \frac{1}{n}} - \sqrt{1 + \frac{1}{n}} = \frac{1}{4n^2} + \frac{5}{64n^4} + \frac{21}{512n^6} + \dots > \frac{1}{4n^2}.$$

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Moreover, the improved weight is optimal ... [next slide].

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Even more explicit Hardy weights:

For all $\psi \in \ell^2(\mathbb{N})$ and $q \in (0, 1/2]$, one has

$$\sum_{n=1}^{\infty} |\psi_n - \psi_{n-1}|^2 \geq \sum_{n=1}^{\infty} \left[2 - \left(1 - \frac{1}{n}\right)^q - \left(1 + \frac{1}{n}\right)^q \right] |\psi_n|^2.$$

For $q \in (0, 1/2)$, (Opt1) holds.

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for all $n \in \mathbb{N}$ and some $c < 1$ and $q \in (0, 1/2]$, then

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$$\boxed{\sigma_d(J_0 + V) = \emptyset.}$$

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Thank you!