New explicitly diagonalizable Hankel matrices

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joint with P. Šťovíček

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Contents

The Hankel matrix

- The semi-infinite matrix H with entries

$$H_{m,n} = h_{m+n},$$

i.e.,

$$H = \begin{pmatrix} h_0 & h_1 & h_2 & h_3 & \dots \\ h_1 & h_2 & h_3 & h_4 & \dots \\ h_2 & h_3 & h_4 & h_5 & \dots \\ h_3 & h_4 & h_5 & h_6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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- If $h \in \ell^2(\mathbb{N}_0)$ and real, then H determines a densely defined symmetric operator on $\ell^2(\mathbb{N}_0)$.
- Although the general spectral theory of Hankel operators if deeply developed, only very few concrete interesting (=not of finite rank) Hankel matrices with “explicitly” solvable spectral problem.

The Hilbert matrix

- The Hilbert matrix:

$$(H_0)_{m,n} = \frac{1}{m+n+1},$$

i.e.,

$$H_0 = \begin{pmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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- The (generalized) Hilbert matrix:

$$(H_\lambda)_{m,n} = \frac{1}{m+n+1+\lambda},$$

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History of the Hilbert matrix

- Hilbert's inequality (1908): There is $M > 0$ such that

$$0 \leq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m a_n}{m+n+1} \leq M \sum_{n=0}^{\infty} a_n^2,$$

for all real $a \in \ell^2(\mathbb{N}_0)$.

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$$\|H_\lambda\| = \pi, \ \ \lambda \geq -1/2, \qquad \text{and} \qquad \|H_\lambda\| = \frac{\pi}{|\sin(\lambda\pi)|}, \ \ -1 < \lambda < -1/2.$$

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- Rosenblum 1958: A complete explicit spectral representation of H_λ for $\lambda > -1$.
  (Rosenblum applied ideas of the commutator method.)

An alternative proof to Rosenblum's approach

- The Hilbert matrix H_0 commutes with the Jacobi matrix

$$J = \begin{pmatrix} \beta_0 & \alpha_0 & & & \\ \alpha_0 & \beta_1 & \alpha_1 & & \\ & \alpha_1 & \beta_2 & \alpha_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where

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- The associated sequence of ON polynomials $P = \{P_n\}_{n\in\mathbb{N}_0}$, is unambiguously defined as the formal eigenvector of J:

$$JP(x) = xP(x)$$

  normalized such that $P_0(x) = 1$.

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- In this case, P_n is a particular case of the Continuous dual Hahn polynomials - a three-parameter family of hypergeometric OG polynomials listed in the Askey scheme.

An alternative proof to Rosenblum's approach

- As a result, we know that $\{P_n\}_{n\in\mathbb{N}_0}$ is an ONB of $L^2((0,\infty),\rho(x)\mathrm{d}x)$, where

$$\rho(x) = \frac{\pi\sinh(\pi\sqrt{x})}{\cosh^2(\pi\sqrt{x})}.$$

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- Moreover, the unitary mapping

$$U:\ell^2(\mathbb{N}_0)\to L^2((0,\infty),\rho(x)\mathrm{d}x):e_n\mapsto P_n$$

diagonalizes the Jacobi operator J, i.e, $UJU^{-1}=T_x$.

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$$UH_0U^{-1}=T_f$$

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- Determination of f using a generating function formula for P_n:

$$f(x)=f(x)P_0(x)=T_fUe_0=UH_0e_0=\sum_{n=0}^{\infty}\frac{P_n(x)}{n+1}=$$

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$$f(x) = f(x)P_0(x) = T_f U e_0 = UH_0 e_0 = \sum_{n=0}^{\infty}\frac{P_n(x)}{n+1} = \frac{\pi}{\cosh(\pi x)}.$$

A summary of the commutator method

- In total, this approach shows that H_0 is unitarily equivalent to the multiplication operator by function

$$f(x) = \frac{\pi}{\cosh(\pi x)}$$

acting on $L^2((0,\infty),\rho(x)\mathrm{d}x)$. This yields the spectral representation of H_0. Particularly,

$$\mathrm{spec}(H_0) = \mathrm{spec}_{\mathrm{ac}}(H_0) = [0,\pi].$$

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2. Finding the spectral mapping f.

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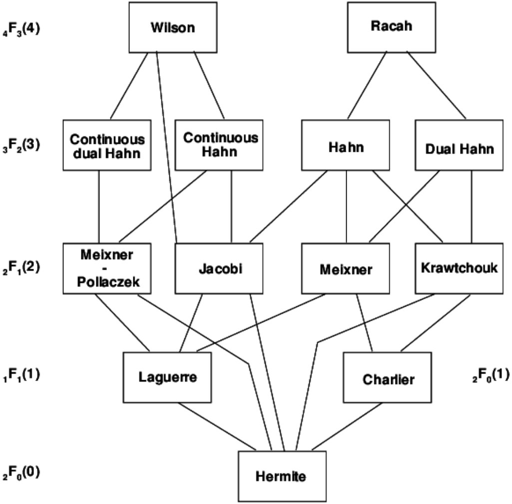
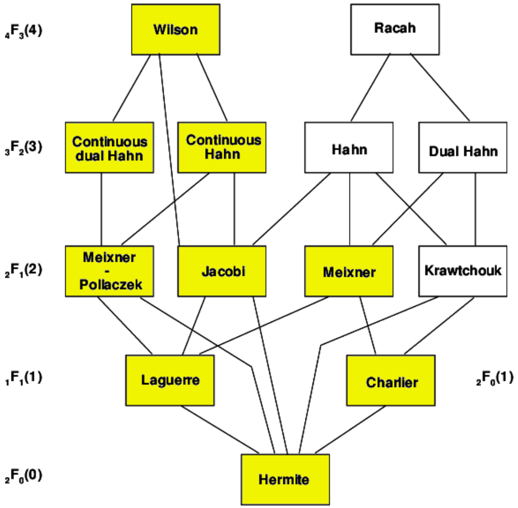
The Askey scheme



The Askey scheme - semi-infinite Jacobi matrices



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Let $H = (h_{m+n})$ be a Hankel matrix with rank $H > 1$ and $h \in \ell^2(\mathbb{N}_0)$.

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Contents

A class of Jacobi matrices

We consider the class of Jacobi matrices $J = J(a, b, c; \sigma, k)$ with:

$$\begin{aligned} \alpha_n &= -\sqrt{(n+1)(n+a+1)(n+b+1)(n+c+1)}, \\ \beta_n &= (k^{-1}+k)n(n+\sigma), \end{aligned}$$

for $a, b, c > -1$, $\sigma \in \mathbb{R}$, and $k \in (0, 1)$.

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Theorem

The Jacobi matrix J commutes with a non-trivial Hankel matrix if and only if α_n is a polynomial function of n.

Characterization of commuting Hankel matrices

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Let J is the Jacobi matrix with

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Then, up to a constant multiplier, the Hankel matrix $H_{m,n} = h_{m+n}$ with entries

$$h_n = \frac{k^n \Gamma(n+a+1)}{\Gamma\big(n+a+\omega(a,\sigma)+1\big)}\; {}_2F_1\big(n+a+1, \omega(a,\sigma)-1; n+a+\omega(a,\sigma)+1; k^2\big),$$

where

$$\omega(a,\sigma) = \frac{-2k^2+(1+k^2)(\sigma-a)}{1-k^2},$$

is the only Hankel matrix with $h \in \ell^2(\mathbb{N}_0)$ commuting with J.

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- For $\sigma = a+1$ and $k \to 1$, we arrive at the generalized Hankel matrix

$$h_n = \frac{1}{n+a+1}.$$

Stieltjes–Carlitz polynomials

- Following the lines of the commutator method we seek for Jacobi matrices with

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- These corresponds to the particular values of the parameters

	p	q	r	s
a	$-1/2$	$1/2$	$-1/2$	$1/2$
σ	$1/(k^2+1)$	$(1+2k^2)/(k^2+1)$	$k^2/(k^2+1)$	$(2+k^2)/(k^2+1)$

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- Basic elements of the theory of elliptic functions:

$$K = K(k) := \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}, \qquad K' = K'(k) := K\big(\sqrt{1-k^2}\big),$$

and

$$q = q(k) := \exp\big(-\pi K'(k)/K(k)\big).$$

Four new diagonalizable Hankel matrices

We introduce four Hankel matrices $H^{(p)}$, $H^{(q)}$, $H^{(r)}$, $H^{(s)}$, depending on a parameter $k \in (0,1)$,

$$H^{(j)}_{m,n} = h^{(j)}_{m+n}, \qquad j = p, q, r, s,$$

for $m, n \in \mathbb{N}_0$, where

$$h_n^{(p)} := \frac{k^n \Gamma(n+1/2)}{(n+1)!}\,{}_2F_1\left(\left.\begin{array}{c} n+1/2, 1/2 \\ n+2 \end{array}\right| k^2\right) = \frac{4k^n}{\sqrt{\pi}} \int_0^1 t^{2n} \sqrt{\frac{1-t^2}{1-k^2t^2}}\,\mathrm{d}t,$$

$$h_n^{(q)} := \frac{k^n \Gamma(n+3/2)}{(n+1)!}\,{}_2F_1\left(\left.\begin{array}{c} n+3/2, -1/2 \\ n+2 \end{array}\right| k^2\right) = \frac{2k^n}{\sqrt{\pi}} \int_0^1 t^{2n+2} \sqrt{\frac{1-k^2t^2}{1-t^2}}\,\mathrm{d}t,$$

$$h_n^{(r)} := \frac{k^n \Gamma(n+1/2)}{n!}\,{}_2F_1\left(\left.\begin{array}{c} n+1/2, -1/2 \\ n+1 \end{array}\right| k^2\right) = \frac{2k^n}{\sqrt{\pi}} \int_0^1 t^{2n} \sqrt{\frac{1-k^2t^2}{1-t^2}}\,\mathrm{d}t,$$

$$h_n^{(s)} := \frac{k^n \Gamma(n+3/2)}{(n+2)!}\,{}_2F_1\left(\left.\begin{array}{c} n+3/2, 1/2 \\ n+3 \end{array}\right| k^2\right) = \frac{4k^n}{\sqrt{\pi}} \int_0^1 t^{2n+2} \sqrt{\frac{1-t^2}{1-k^2t^2}}\,\mathrm{d}t.$$

Diagonalization of $H^{(p)}$, $H^{(q)}$, $H^{(r)}$, $H^{(s)}$.

Theorem

Each of the Hankel matrices $H^{(j)}$, $j = p, q, r, s$, represents a positive trace class operator on $\ell^2(\mathbb{N}_0)$ with simple eigenvalues which are as follows:

$$\nu_m^{(p)} = \frac{4\sqrt{\pi}}{k}\,\frac{q^{m+1/2}}{1+q^{2m+1}}, \quad m \geq 0,$$

$$\nu_m^{(q)} = \frac{2\sqrt{\pi}}{k}\,\frac{q^{m+1/2}}{1+q^{2m+1}}, \quad m \geq 0,$$

$$\nu_m^{(r)} = 2\sqrt{\pi}\,\frac{q^{m}}{1+q^{2m}}, \quad m \geq 0,$$

$$\nu_m^{(s)} = \frac{4\sqrt{\pi}}{k^2}\,\frac{q^{m}}{1+q^{2m}}, \quad m \geq 1.$$

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Moreover, the corresponding eigenvectors and their ℓ^2-norms are expressible in terms of the Stieltjes–Carlitz polynomials and elliptic integrals (not displayed).

Thank you!