A functional model and tridiagonalisation for symmetric anti-linear operators

Frantisek Štampach
(Czech Technical University in Prague)

joint with Alexander Pushnitski
(King's College London)

Workshop on Operator Theory, Complex Analysis, and Applications
Universidade da Madeira, Funchal
July 22-26, 2024

Based on: A. Pushnitski, F. Štampach: arXiv:2402.01237 (2024).

Contents

Diagonalisation of self-adjoint operators with simple spectrum

- Let $\mathcal{H}$ be a complex separable Hilbert space.

Diagonalisation of self-adjoint operators with simple spectrum

- Let $\mathcal{H}$ be a complex separable Hilbert space.
- Let A be bounded self-adjoint linear operator on $\mathcal{H}$ with a cyclic vector $\delta \in \mathcal{H}$, i.e.

$$\overline{\operatorname{span}\{A^n\delta \mid n \geq 0\}} = \mathcal{H}.$$

(We take δ normalized, i.e. $\|\delta\| = 1$.)

Diagonalisation of self-adjoint operators with simple spectrum

- Let $\mathcal{H}$ be a complex separable Hilbert space.
- Let A be bounded self-adjoint linear operator on $\mathcal{H}$ with a cyclic vector $\delta \in \mathcal{H}$, i.e.

$$\overline{\operatorname{span}\{A^n\delta \mid n \geq 0\}} = \mathcal{H}.$$

(We take δ normalized, i.e. $\|\delta\| = 1$.)

Theorem [Spectral theorem, self-adjoint case, simple spectrum]

There is a unique probability measure μ on $\mathbb{R}$ and a unitary operator $U : \mathcal{H} \to L^2(\mu)$ such that

$$U\delta = 1 \quad \text{and} \quad UAU^{-1} = M_\lambda.$$

Here $M_\lambda f(\lambda) = \lambda f(\lambda)$.

Diagonalisation of self-adjoint operators with simple spectrum

- Let $\mathcal{H}$ be a complex separable Hilbert space.
- Let A be bounded self-adjoint linear operator on $\mathcal{H}$ with a cyclic vector $\delta \in \mathcal{H}$, i.e.

$$\overline{\operatorname{span}\{A^n\delta \mid n \geq 0\}} = \mathcal{H}.$$

(We take δ normalized, i.e. $\|\delta\| = 1$.)

Theorem [Spectral theorem, self-adjoint case, simple spectrum]

There is a unique probability measure μ on $\mathbb{R}$ and a unitary operator $U : \mathcal{H} \to L^2(\mu)$ such that

$$U\delta = 1 \quad \text{and} \quad UAU^{-1} = M_\lambda.$$

Here $M_\lambda f(\lambda) = \lambda f(\lambda)$.

- "Proof": $\mu := \langle \delta, E_A \delta \rangle$, where E_A is the projection-valued spectral measure of A.

Diagonalisation of self-adjoint operators with simple spectrum

- Let $\mathcal{H}$ be a complex separable Hilbert space.
- Let A be bounded self-adjoint linear operator on $\mathcal{H}$ with a cyclic vector $\delta \in \mathcal{H}$, i.e.

$$\overline{\operatorname{span}\{A^n\delta \mid n \geq 0\}} = \mathcal{H}.$$

(We take δ normalized, i.e. $\|\delta\| = 1$.)

Theorem [Spectral theorem, self-adjoint case, simple spectrum]

There is a unique probability measure μ on $\mathbb{R}$ and a unitary operator $U : \mathcal{H} \to L^2(\mu)$ such that

$$U\delta = 1 \quad \text{and} \quad UAU^{-1} = M_\lambda.$$

Here $M_\lambda f(\lambda) = \lambda f(\lambda)$.

- "Proof": $\mu := \langle \delta, E_A\delta \rangle$, where E_A is the projection-valued spectral measure of A.
- M_λ is a "functional model" of A acting on the "model space" $L^2(\mu)$.

Tridiagonalisation of self-adjoint operators with simple spectrum

- Jacobi matrix:

$$J=\begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ 0 & 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad a_n>0,\ b_n\in\mathbb{R}.$$

Tridiagonalisation of self-adjoint operators with simple spectrum

- Jacobi matrix:

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ 0 & 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad a_n > 0,\ b_n \in \mathbb{R}.$$

Theorem [Tridiagonalization, self-adjoint case, simple spectrum]

There exist a unique Jacobi matrix J and a unitary operator $V : \mathcal{H} \to \ell^2(\mathbb{N}_0)$ such that

$$V\delta = \delta_0 \quad \text{and} \quad VAV^{-1} = J,$$

where $\delta_0 = (1, 0, 0, \dots)^T$.

Tridiagonalisation of self-adjoint operators with simple spectrum

- Jacobi matrix:

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ 0 & 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad a_n > 0, \ b_n \in \mathbb{R}.$$

Theorem [Tridiagonalization, self-adjoint case, simple spectrum]

There exist a unique Jacobi matrix J and a unitary operator $\boxed{V : \mathcal{H} \to \ell^2(\mathbb{N}_0)}$ such that

$$\boxed{V\delta = \delta_0 \quad \text{and} \quad VAV^{-1} = J,}$$

where $\delta_0 = (1, 0, 0, \dots)^T$. The sequences $a_n > 0$ and $b_n \in \mathbb{R}$ are uniquely defined by these conditions.

Tridiagonalisation of self-adjoint operators with simple spectrum

- Jacobi matrix:

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ 0 & 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad a_n > 0,\ b_n \in \mathbb{R}.$$

Theorem [Tridiagonalization, self-adjoint case, simple spectrum]

There exist a unique Jacobi matrix J and a unitary operator $V : \mathcal{H} \to \ell^2(\mathbb{N}_0)$ such that

$$V\delta = \delta_0 \quad \text{and} \quad VAV^{-1} = J,$$

where $\delta_0 = (1, 0, 0, \dots)^T$. The sequences $a_n > 0$ and $b_n \in \mathbb{R}$ are uniquely defined by these conditions.

- J is a "functional model" of A acting on the "model space" $\ell^2(\mathbb{N}_0)$.

Sketch of the proof

Sketch of the proof

- Apply Gram–Schmidt process to monomials $1, \lambda, \lambda^2, \dots$ w.r.t. the inner product

$$[p, q] := \langle p(A)\delta, q(A)\delta \rangle = \int_{\mathbb{R}} p(\lambda)\overline{q(\lambda)}\mathrm{d}\mu(\lambda) \quad \text{(i.e. in } L^2(\mu)),$$

where μ be the spectral measure of A.

Sketch of the proof

- Apply Gram–Schmidt process to monomials $1, \lambda, \lambda^2, \dots$ w.r.t. the inner product
$$[p, q] := \langle p(A)\delta, q(A)\delta \rangle = \int_{\mathbb{R}} p(\lambda)\overline{q(\lambda)} \mathrm{d}\mu(\lambda) \quad \text{(i.e. in } L^2(\mu)),$$
where μ be the spectral measure of A.
- The resulting sequence $\{p_n\}_{n=0}^{\infty}$ of orthonormal polynomials satisfies the *3-term recurrence*:
$$\boxed{a_{n-1}p_{n-1}(\lambda) + b_n p_n(\lambda) + a_n p_{n+1}(\lambda) = \lambda p_n(\lambda), \quad n \geq 0,}$$
with a convention $a_{-1} = p_{-1} = 0$ and normalised by conditions $a_n > 0$ and $p_0 = 1$.

Sketch of the proof

- Apply Gram–Schmidt process to monomials $1, \lambda, \lambda^2, \dots$ w.r.t. the inner product

$$[p,q] := \langle p(A)\delta, q(A)\delta \rangle = \int_{\mathbb{R}} p(\lambda)\overline{q(\lambda)}\mathrm{d}\mu(\lambda) \quad \text{(i.e. in } L^2(\mu)),$$

  where μ be the spectral measure of A.
- The resulting sequence $\{p_n\}_{n=0}^{\infty}$ of orthonormal polynomials satisfies the *3-term recurrence*:

$$\boxed{a_{n-1}p_{n-1}(\lambda) + b_n p_n(\lambda) + a_n p_{n+1}(\lambda) = \lambda p_n(\lambda), \quad n \geq 0,}$$

  with a convention $a_{-1} = p_{-1} = 0$ and normalised by conditions $a_n > 0$ and $p_0 = 1$.
- From the 3-term recurrence, we read off a_n and b_n, and define J.

Sketch of the proof

- Apply Gram–Schmidt process to monomials $1, \lambda, \lambda^2, \ldots$ w.r.t. the inner product

$$[p, q] := \langle p(A)\delta, q(A)\delta \rangle = \int_{\mathbb{R}} p(\lambda)\overline{q(\lambda)}\mathrm{d}\mu(\lambda) \quad \text{(i.e. in } L^2(\mu)),$$

  where μ be the spectral measure of A.
- The resulting sequence $\{p_n\}_{n=0}^{\infty}$ of orthonormal polynomials satisfies the *3-term recurrence*:

$$\boxed{a_{n-1}p_{n-1}(\lambda) + b_n p_n(\lambda) + a_n p_{n+1}(\lambda) = \lambda p_n(\lambda), \quad n \geq 0,}$$

  with a convention $a_{-1} = p_{-1} = 0$ and normalised by conditions $a_n > 0$ and $p_0 = 1$.
- From the 3-term recurrence, we read off a_n and b_n, and define J.
- $\{p_n(A)\delta\}_{n=0}^{\infty}$ is ONB of $\mathcal{H}$, and the unitary map $V : \mathcal{H} \to \ell^2(\mathbb{N}_0)$ is defined by the correspondence

$$V : p_n(A)\delta \mapsto \delta_n,$$

  where $\{\delta_n\}_{n=0}^{\infty}$ is the standard ONB of $\ell^2(\mathbb{N}_0)$.

Sketch of the proof

- Apply Gram–Schmidt process to monomials $1, \lambda, \lambda^2, \dots$ w.r.t. the inner product

$$[p, q] := \langle p(A)\delta, q(A)\delta \rangle = \int_{\mathbb{R}} p(\lambda)\overline{q(\lambda)}\mathrm{d}\mu(\lambda) \quad \text{(i.e. in } L^2(\mu)\text{)},$$

  where μ be the spectral measure of A.
- The resulting sequence $\{p_n\}_{n=0}^{\infty}$ of orthonormal polynomials satisfies the *3-term recurrence*:

$$\boxed{a_{n-1}p_{n-1}(\lambda) + b_n p_n(\lambda) + a_n p_{n+1}(\lambda) = \lambda p_n(\lambda), \quad n \geq 0,}$$

  with a convention $a_{-1} = p_{-1} = 0$ and normalised by conditions $a_n > 0$ and $p_0 = 1$.
- From the 3-term recurrence, we read off a_n and b_n, and define J.
- $\{p_n(A)\delta\}_{n=0}^{\infty}$ is ONB of $\mathcal{H}$, and the unitary map $V : \mathcal{H} \to \ell^2(\mathbb{N}_0)$ is defined by the correspondence

$$V : p_n(A)\delta \mapsto \delta_n,$$

  where $\{\delta_n\}_{n=0}^{\infty}$ is the standard ONB of $\ell^2(\mathbb{N}_0)$.
- Then

$$V\delta = \delta_0 \quad \text{and} \quad VAV^{-1} = J.$$

Contents

Anti-linear operators

- Let B be bounded anti-linear operator on $\mathcal{H}$:

$$B(x+y) = Bx + By \quad \text{and} \quad B(\alpha x) = \overline{\alpha} Bx \qquad (\forall x, y \in \mathcal{H}; \forall \alpha \in \mathbb{C})$$

Anti-linear operators

- Let B be bounded anti-linear operator on $\mathcal{H}$:

$$B(x+y) = Bx + By \quad \text{and} \quad B(\alpha x) = \overline{\alpha} Bx \qquad (\forall x, y \in \mathcal{H}; \forall \alpha \in \mathbb{C})$$

- We assume B has a (normalized) cyclic vector δ:

$$\overline{\operatorname{span}\{B^n \delta \mid n \geq 0\}} = \mathcal{H}.$$

Anti-linear operators

- Let B be bounded anti-linear operator on $\mathcal{H}$:

$$B(x+y) = Bx + By \quad \text{and} \quad B(\alpha x) = \overline{\alpha} Bx \qquad (\forall x, y \in \mathcal{H}; \forall \alpha \in \mathbb{C})$$

- We assume B has a (normalized) cyclic vector δ:

$$\overline{\operatorname{span}\{B^n \delta \mid n \geq 0\}} = \mathcal{H}.$$

- We assume B is symmetric:

$$\langle Bx, y \rangle = \langle By, x \rangle \qquad (\forall x, y \in \mathcal{H})$$

Anti-linear operators

- Let B be bounded anti-linear operator on $\mathcal{H}$:

$$B(x+y) = Bx + By \quad \text{and} \quad B(\alpha x) = \overline{\alpha} Bx \qquad (\forall x, y \in \mathcal{H}; \forall \alpha \in \mathbb{C})$$

- We assume B has a (normalized) cyclic vector δ:

$$\overline{\text{span}\{B^n\delta \mid n \geq 0\}} = \mathcal{H}.$$

- We assume B is symmetric:

$$\langle Bx, y\rangle = \langle By, x\rangle \qquad (\forall x, y \in \mathcal{H})$$

Examples:

Anti-linear operators

- Let B be bounded anti-linear operator on $\mathcal{H}$:

$$B(x+y) = Bx + By \quad \text{and} \quad B(\alpha x) = \overline{\alpha} Bx \qquad (\forall x, y \in \mathcal{H}; \forall \alpha \in \mathbb{C})$$

- We assume B has a (normalized) cyclic vector δ:

$$\overline{\operatorname{span}\{B^n \delta \mid n \geq 0\}} = \mathcal{H}.$$

- We assume B is symmetric:

$$\langle Bx, y \rangle = \langle By, x \rangle \qquad (\forall x, y \in \mathcal{H})$$

Examples:

- Anti-linear Jacobi operator: $B = J\mathcal{C}$, where J is the Jacobi matrix with complex entries and $\mathcal{C}$ the complex conjugation on $\ell^2(\mathbb{N}_0)$.

Anti-linear operators

- Let B be bounded anti-linear operator on $\mathcal{H}$:
$$B(x+y) = Bx + By \quad \text{and} \quad B(\alpha x) = \overline{\alpha} Bx \qquad (\forall x, y \in \mathcal{H}; \forall \alpha \in \mathbb{C})$$
- We assume B has a (normalized) cyclic vector δ:
$$\overline{\operatorname{span}\{B^n \delta \mid n \geq 0\}} = \mathcal{H}.$$
- We assume B is symmetric:
$$\langle Bx, y \rangle = \langle By, x \rangle \qquad (\forall x, y \in \mathcal{H})$$

Examples:

- Anti-linear Jacobi operator: $B = J\mathcal{C}$, where J is the Jacobi matrix with complex entries and $\mathcal{C}$ the complex conjugation on $\ell^2(\mathbb{N}_0)$.
- Complex symmetric operators: Let $\mathcal{C}$ be a conjugation on $\mathcal{H}$, i.e. $\mathcal{C}$ anti-unitary and $\mathcal{C}^2 = I$, and A a linear bounded $\mathcal{C}$-symmetric operator on $\mathcal{H}$; i.e. $A^* = \mathcal{C}A\mathcal{C}$. Then $B := A\mathcal{C}$ is anti-linear symmetric operator.

Anti-linear literature

- Complex symmetric operators ($\mathcal{C}/\mathcal{T}/\mathcal{J}$-symmetric, self-transpose):
  Bender, Câmara, Dereziński, Garcia, Gazeau, Georgescu, Krejčiřík, Ptak, Putinar, Shapiro, Siegl, Znojil,... `(very incomplete list)`
- Functional properties of anti-linear operators:
  Gérard, Herbut, Kaplansky, Müller, Pushnitski, Treil, Uhlmann, Vujičić,...
- Complex Jacobi matrix and the moment problem:
  Huhtanen, Perämäki, Ruotsalainen, Zagorodnyuk,...

The spectral data of B

- Recall B is bounded anti-linear symmetric operator with normalized cyclic vector δ.

The spectral data of *B*

- Recall B is bounded anti-linear symmetric operator with normalized cyclic vector δ.
- B^2 is a linear non-negative operator and $|B| := \sqrt{B^2}$.

The spectral data of B

- Recall B is bounded anti-linear symmetric operator with normalized cyclic vector δ.
- B^2 is a linear non-negative operator and $|B| := \sqrt{B^2}$.
- Spectral measure ν:

$$\langle f(|B|)\delta, \delta\rangle = \int_0^\infty f(s)\mathrm{d}\nu(s), \quad \forall f \in C(\mathbb{R}).$$

The spectral data of B

- Recall B is bounded anti-linear symmetric operator with normalized cyclic vector δ.
- B^2 is a linear non-negative operator and $|B| := \sqrt{B^2}$.
- Spectral measure ν:

$$\langle f(|B|)\delta, \delta\rangle = \int_0^\infty f(s)\mathrm{d}\nu(s), \quad \forall f \in C(\mathbb{R}).$$

Proposition [Phase function ψ]

There exists a unique complex-valued phase function $\psi \in L^\infty(\nu)$ such that

$$\langle f(|B|)B\delta, \delta\rangle = \int_0^\infty sf(s)\psi(s)\mathrm{d}\nu(s), \quad \forall f \in C(\mathbb{R}).$$

and $\psi(0) = 1$ if $\nu(\{0\}) > 0$. The function ψ satisfies $|\psi(s)| \leq 1$, ν-a.e. $s > 0$.

The spectral data of B

- Recall B is bounded anti-linear symmetric operator with normalized cyclic vector δ.
- B^2 is a linear non-negative operator and $|B| := \sqrt{B^2}$.
- Spectral measure ν:

$$\langle f(|B|)\delta, \delta\rangle = \int_0^\infty f(s)\mathrm{d}\nu(s), \quad \forall f \in C(\mathbb{R}).$$

Proposition [Phase function ψ]

There exists a unique complex-valued phase function $\psi \in L^\infty(\nu)$ such that

$$\langle f(|B|)B\delta, \delta\rangle = \int_0^\infty sf(s)\psi(s)\mathrm{d}\nu(s), \quad \forall f \in C(\mathbb{R}).$$

and $\psi(0) = 1$ if $\nu(\{0\}) > 0$. The function ψ satisfies $|\psi(s)| \leq 1$, ν-a.e. $s > 0$.

- (ν, ψ) is called spectral data of B.

The functional model $\mathcal{B}$

Proposition

The vector δ is of maximal type for $|B|$ and the spectrum of $|B|$ has

$$\begin{cases} \text{multiplicity 1 on } S_1 := \{s \geq 0 : |\psi(s)| = 1\}, \\ \text{multiplicity 2 on } S_2 := \{s \geq 0 : |\psi(s)| < 1\}. \end{cases}$$

The functional model $\mathcal{B}$

Proposition

The vector δ is of maximal type for $|B|$ and the spectrum of $|B|$ has

$$\begin{cases} \text{multiplicity 1 on } S_1 := \{s \geq 0 : |\psi(s)| = 1\}, \\ \text{multiplicity 2 on } S_2 := \{s \geq 0 : |\psi(s)| < 1\}. \end{cases}$$

Define

$$L^2(\nu;\mathbb{C}^2) := \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : \int_0^\infty \left(|f_1(s)|^2 + |f_2(s)|^2 \right) \mathrm{d}\nu(s) < \infty \right\}.$$

The functional model $\mathcal{B}$

Proposition

The vector δ is of maximal type for $|B|$ and the spectrum of $|B|$ has

$$\begin{cases} \text{multiplicity 1 on } S_1 := \{s \geq 0 : |\psi(s)| = 1\}, \\ \text{multiplicity 2 on } S_2 := \{s \geq 0 : |\psi(s)| < 1\}. \end{cases}$$

Define

$$L^2(\nu;\mathbb{C}^2) := \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : \int_0^\infty \left(|f_1(s)|^2 + |f_2(s)|^2 \right) \mathrm{d}\nu(s) < \infty \right\}.$$

Definition [model space, model operator]

We define the model operator $\mathcal{B}$ on the subspace

$$\boxed{\mathcal{M}(\nu) := \{ f \in L^2(\nu;\mathbb{C}^2) : f_2 \equiv 0 \text{ on } S_1 \}}$$

The functional model $\mathcal{B}$

Proposition

The vector δ is of maximal type for $|B|$ and the spectrum of $|B|$ has

$$\begin{cases} \text{multiplicity 1 on } S_1 := \{s \geq 0 : |\psi(s)| = 1\}, \\ \text{multiplicity 2 on } S_2 := \{s \geq 0 : |\psi(s)| < 1\}. \end{cases}$$

Define

$$L^2(\nu;\mathbb{C}^2) := \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : \int_0^\infty \left(|f_1(s)|^2 + |f_2(s)|^2 \right) \mathrm{d}\nu(s) < \infty \right\}.$$

Definition [model space, model operator]

We define the model operator $\mathcal{B}$ on the subspace

$$\mathcal{M}(\nu) := \{ f \in L^2(\nu;\mathbb{C}^2) : f_2 \equiv 0 \text{ on } S_1 \}$$

by

$$(\mathcal{B}f)(s) := \begin{pmatrix} \psi(s) & \sqrt{1-|\psi(s)|^2} \\ \sqrt{1-|\psi(s)|^2} & -\overline{\psi(s)} \end{pmatrix} \begin{pmatrix} s\overline{f_1(s)} \\ s\overline{f_2(s)} \end{pmatrix}.$$

The functional model $\mathcal{B}$

Definition [model space, model operator]

We define the model operator $\mathcal{B}$ on the subspace

$$\mathcal{M}(\nu) := \{f \in L^2(\nu;\mathbb{C}^2) : f_2 \equiv 0 \text{ on } S_1\}$$

by

$$(\mathcal{B}f)(s) := \begin{pmatrix} \psi(s) & \sqrt{1-|\psi(s)|^2} \\ \sqrt{1-|\psi(s)|^2} & -\overline{\psi(s)} \end{pmatrix} \begin{pmatrix} s\overline{f_1(s)} \\ s\overline{f_2(s)} \end{pmatrix}.$$

Example:

The functional model $\mathcal{B}$

Definition [model space, model operator]

We define the model operator $\mathcal{B}$ on the subspace

$$\mathcal{M}(\nu) := \{f \in L^2(\nu;\mathbb{C}^2) : f_2 \equiv 0 \text{ on } S_1\}$$

by

$$(\mathcal{B}f)(s) := \begin{pmatrix} \psi(s) & \sqrt{1-|\psi(s)|^2} \\ \sqrt{1-|\psi(s)|^2} & -\overline{\psi(s)} \end{pmatrix} \begin{pmatrix} \overline{sf_1(s)} \\ \overline{sf_2(s)} \end{pmatrix}.$$

Example: Suppose the spectrum of $|B|$ is simple, i.e. $|\psi(s)| = 1$ for ν-a.e. $s \geq 0$.

The functional model $\mathcal{B}$

Definition [model space, model operator]

We define the model operator $\mathcal{B}$ on the subspace

$$\mathcal{M}(\nu) := \{f \in L^2(\nu;\mathbb{C}^2) : f_2 \equiv 0 \text{ on } S_1\}$$

by

$$(\mathcal{B}f)(s) := \begin{pmatrix} \psi(s) & \sqrt{1-|\psi(s)|^2} \\ \overline{\sqrt{1-|\psi(s)|^2}} & -\overline{\psi(s)} \end{pmatrix} \begin{pmatrix} s\overline{f_1(s)} \\ s\overline{f_2(s)} \end{pmatrix}.$$

Example: Suppose the spectrum of $|B|$ is simple, i.e. $|\psi(s)| = 1$ for ν-a.e. $s \geq 0$. Then $\mathcal{M}(\nu) \simeq L^2(\nu)$ and $\mathcal{B}$ acts (after a trivial identification) as

$$(\mathcal{B}f)(s) = s\psi(s)\overline{f(s)}.$$

Notice the analogy between $\mathcal{B}$ and M_s.

A spectral theorem for symmetric anti-linear operators

- Assumption: Let B be a bounded symmetric anti-linear operator on $\mathcal{H}$ with a cyclic vector δ and (ν, ψ) the spectral data of B.

A spectral theorem for symmetric anti-linear operators

- Assumption: Let B be a bounded symmetric anti-linear operator on $\mathcal{H}$ with a cyclic vector δ and (ν, ψ) the spectral data of B.
- Model: Let $\mathcal{B}$ be the model operator on $\mathcal{M}(\nu)$ constructed from (ν, ψ).

A spectral theorem for symmetric anti-linear operators

- Assumption: Let B be a bounded symmetric anti-linear operator on $\mathcal{H}$ with a cyclic vector δ and (ν, ψ) the spectral data of B.
- Model: Let $\mathcal{B}$ be the model operator on $\mathcal{M}(\nu)$ constructed from (ν, ψ).

Theorem [Pushnitski–Š.]

There exists a unitary map $U : \mathcal{H} \to \mathcal{M}(\nu)$ such that

$$\boxed{U\delta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad UBU^{-1} = \mathcal{B}.}$$

Contents

Tridiagonalisation of symmetric anti-linear operators

- Anti-linear Jacobi operator: $J_{\mathcal{C}} := J\mathcal{C}$, where

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ 0 & 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad a_n > 0, \; b_n \in \mathbb{C}.$$

Tridiagonalisation of symmetric anti-linear operators

- Anti-linear Jacobi operator: $\boxed{J_{\mathcal{C}} := J\mathcal{C}}$, where

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ 0 & 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad a_n > 0, \; b_n \in \mathbb{C}.$$

- Assumption: Let B be a bounded symmetric anti-linear operator on $\mathcal{H}$ with a cyclic vector δ.

Tridiagonalisation of symmetric anti-linear operators

- Anti-linear Jacobi operator: $J_{\mathcal{C}} := J\mathcal{C}$, where

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ 0 & 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad a_n > 0, \; b_n \in \mathbb{C}.$$

- Assumption: Let B be a bounded symmetric anti-linear operator on $\mathcal{H}$ with a cyclic vector δ.

Theorem [Pushnitski–Š.]

There exist bounded sequences $a_n > 0$ and $b_n \in \mathbb{C}$ and a unitary map $V : \mathcal{H} \to \ell^2(\mathbb{N}_0)$ such that

$$V\delta = \delta_0 \quad \text{and} \quad VBV^{-1} = J_{\mathcal{C}}.$$

Tridiagonalisation of symmetric anti-linear operators

- Anti-linear Jacobi operator: $\boxed{J_{\mathcal{C}} := J\mathcal{C}}$, where

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ 0 & 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad a_n > 0, \; b_n \in \mathbb{C}.$$

- Assumption: Let B be a bounded symmetric anti-linear operator on $\mathcal{H}$ with a cyclic vector δ.

Theorem [Pushnitski–Š.]

There exist bounded sequences $a_n > 0$ and $b_n \in \mathbb{C}$ and a unitary map $V : \mathcal{H} \to \ell^2(\mathbb{N}_0)$ such that

$$\boxed{V\delta = \delta_0 \quad \text{and} \quad VBV^{-1} = J_{\mathcal{C}}.}$$

The sequences $a_n > 0$ and $b_n \in \mathbb{C}$ are uniquely defined by these conditions.

Sketch of the proof

Sketch of the proof

- Apply Gram–Schmidt process to monomials $1, s, s^2, \dots$ w.r.t. the inner product
$$[p,q] := \langle p(B)\delta, q(B)\delta\rangle = \int_0^\infty \left\langle \begin{pmatrix} 1 & \psi(s) \\ \frac{1}{\psi(s)} & 1 \end{pmatrix} \begin{pmatrix} p^e(s) \\ p^o(s) \end{pmatrix}, \begin{pmatrix} q^e(s) \\ q^o(s) \end{pmatrix} \right\rangle \mathrm{d}\nu(s),$$
where
$$p^e(s) := \frac{p(s)+p(-s)}{2}, \qquad p^o(s) := \frac{p(s)-p(-s)}{2}$$
and (ν, ψ) are spectral data of B.

Sketch of the proof

- Apply Gram–Schmidt process to monomials $1, s, s^2, \ldots$ w.r.t. the inner product
$$[p, q] := \langle p(B)\delta, q(B)\delta \rangle = \int_0^\infty \left\langle \begin{pmatrix} \frac{1}{\psi(s)} & \psi(s) \\ & 1 \end{pmatrix} \begin{pmatrix} p^e(s) \\ p^o(s) \end{pmatrix}, \begin{pmatrix} q^e(s) \\ q^o(s) \end{pmatrix} \right\rangle \mathrm{d}\nu(s),$$
where
$$p^e(s) := \frac{p(s) + p(-s)}{2}, \qquad p^o(s) := \frac{p(s) - p(-s)}{2}$$
and (ν, ψ) are spectral data of B.
- The non-degenaracy of $[\cdot, \cdot]$ is non-trivial.

Sketch of the proof

- Apply Gram–Schmidt process to monomials $1, s, s^2, \ldots$ w.r.t. the inner product

$$[p,q] := \langle p(B)\delta, q(B)\delta\rangle = \int_0^\infty \left\langle \begin{pmatrix} \frac{1}{\psi(s)} & \psi(s) \\ 1 \end{pmatrix} \begin{pmatrix} p^e(s) \\ p^o(s) \end{pmatrix}, \begin{pmatrix} q^e(s) \\ q^o(s) \end{pmatrix} \right\rangle \mathrm{d}\nu(s),$$

  where
$$p^e(s) := \frac{p(s)+p(-s)}{2}, \qquad p^o(s) := \frac{p(s)-p(-s)}{2}$$
  and (ν, ψ) are spectral data of B.
- The non-degenaracy of $[\cdot,\cdot]$ is non-trivial.
- The resulting sequence $\{q_n\}_{n=0}^\infty$ of anti-orthonormal polynomials satisfies the *3-term recurrence*:

$$\boxed{a_{n-1}q_{n-1}(s) + b_n q_n(s) + a_n q_{n+1}(s) = s\overline{q_n}(s), \quad n \geq 0,}$$

  with a convention $a_{-1} = q_{-1} = 0$ and normalised by conditions $a_n > 0$ and $q_0 = 1$. Here $b_n \in \mathbb{C}$!

Sketch of the proof

- Apply Gram–Schmidt process to monomials $1, s, s^2, \dots$ w.r.t. the inner product

$$[p,q] := \langle p(B)\delta, q(B)\delta \rangle = \int_0^\infty \left\langle \begin{pmatrix} 1 & \psi(s) \\ \overline{\psi(s)} & 1 \end{pmatrix} \begin{pmatrix} p^e(s) \\ p^o(s) \end{pmatrix}, \begin{pmatrix} q^e(s) \\ q^o(s) \end{pmatrix} \right\rangle \mathrm{d}\nu(s),$$

where
$$p^e(s) := \frac{p(s)+p(-s)}{2}, \qquad p^o(s) := \frac{p(s)-p(-s)}{2}$$
and (ν, ψ) are spectral data of B.

- The non-degenaracy of $[\cdot,\cdot]$ is non-trivial.
- The resulting sequence $\{q_n\}_{n=0}^\infty$ of anti-orthonormal polynomials satisfies the *3-term recurrence*:

$$\boxed{a_{n-1}q_{n-1}(s) + b_n q_n(s) + a_n q_{n+1}(s) = s\overline{q_n}(s), \quad n \geq 0,}$$

with a convention $a_{-1} = q_{-1} = 0$ and normalised by conditions $a_n > 0$ and $q_0 = 1$. Here $b_n \in \mathbb{C}$!

- From the 3-term recurrence, we read off a_n and b_n, and define $J_{\mathcal{C}}$.

Sketch of the proof

- Apply Gram–Schmidt process to monomials $1, s, s^2, \dots$ w.r.t. the inner product

$$[p,q] := \langle p(B)\delta, q(B)\delta\rangle = \int_0^\infty \left\langle \begin{pmatrix} 1 & \psi(s) \\ \overline{\psi(s)} & 1 \end{pmatrix} \begin{pmatrix} p^e(s) \\ p^o(s) \end{pmatrix}, \begin{pmatrix} q^e(s) \\ q^o(s) \end{pmatrix} \right\rangle \mathrm{d}\nu(s),$$

  where

$$p^e(s) := \frac{p(s)+p(-s)}{2}, \qquad p^o(s) := \frac{p(s)-p(-s)}{2}$$

  and (ν, ψ) are spectral data of B.
- The non-degenaracy of $[\cdot,\cdot]$ is non-trivial.
- The resulting sequence $\{q_n\}_{n=0}^\infty$ of anti-orthonormal polynomials satisfies the *3-term recurrence*:

$$\boxed{a_{n-1}q_{n-1}(s) + b_n q_n(s) + a_n q_{n+1}(s) = s\overline{q_n}(s), \quad n \geq 0,}$$

  with a convention $a_{-1} = q_{-1} = 0$ and normalised by conditions $a_n > 0$ and $q_0 = 1$. Here $b_n \in \mathbb{C}$!
- From the 3-term recurrence, we read off a_n and b_n, and define $J_{\mathcal{C}}$.
- $\{p_n(B)\delta\}_{n=0}^\infty$ is ONB of $\mathcal{H}$, and the unitary map $V : \mathcal{H} \to \ell^2(\mathbb{N}_0)$ is defined by

$$V : p_n(B)\delta \mapsto \delta_n, \quad n \geq 0.$$

Sketch of the proof

- Apply Gram–Schmidt process to monomials $1, s, s^2, \ldots$ w.r.t. the inner product
$$[p,q] := \langle p(B)\delta, q(B)\delta\rangle = \int_0^\infty \left\langle \begin{pmatrix} 1 & \psi(s) \\ \overline{\psi(s)} & 1 \end{pmatrix} \begin{pmatrix} p^e(s) \\ p^o(s) \end{pmatrix}, \begin{pmatrix} q^e(s) \\ q^o(s) \end{pmatrix} \right\rangle \mathrm{d}\nu(s),$$
where
$$p^e(s) := \frac{p(s)+p(-s)}{2}, \qquad p^o(s) := \frac{p(s)-p(-s)}{2}$$
and (ν, ψ) are spectral data of B.
- The non-degenaracy of $[\cdot,\cdot]$ is non-trivial.
- The resulting sequence $\{q_n\}_{n=0}^\infty$ of anti-orthonormal polynomials satisfies the *3-term recurrence*:
$$\boxed{a_{n-1}q_{n-1}(s) + b_n q_n(s) + a_n q_{n+1}(s) = s\overline{q_n}(s), \quad n \geq 0,}$$
with a convention $a_{-1} = q_{-1} = 0$ and normalised by conditions $a_n > 0$ and $q_0 = 1$. Here $b_n \in \mathbb{C}$!
- From the 3-term recurrence, we read off a_n and b_n, and define $J_\mathcal{C}$.
- $\{p_n(B)\delta\}_{n=0}^\infty$ is ONB of $\mathcal{H}$, and the unitary map $V : \mathcal{H} \to \ell^2(\mathbb{N}_0)$ is defined by
$$V : p_n(B)\delta \mapsto \delta_n, \quad n \geq 0.$$
- Then one can show that $V\delta = \delta_0$ and $VBV^{-1} = J_\mathcal{C}$ as well as the uniqueness.

Motivated by:

- P. Gérard, S. Grellier: *The cubic Szegő equation and Hankel operators*, Astérisque **389** (2017).
- A. Pushnitski, F. Š.: *An inverse spectral problem for non-self-adjoint Jacobi matrices*, Int. Math. Res. Not. **2024** (2024).

Based on:

- A. Pushnitski, F. Š.: *A functional model and tridiagonalisation for symmetric anti-linear operators*, preprint (2024), `arXiv:2402.01237`.

Obrigado!