An improved discrete Rellich inequality

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1. In 1915, when trying to find a simple proof of the Hilbert theorem G. Hardy proved

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where the constant is the best possible.

The discrete Hardy inequality

- An equivalent formulation of the discrete Hardy inequality

$$\sum_{n=1}^{\infty}\left|\frac{a_1+\cdots+a_n}{n}\right|^2 \leq 4\sum_{n=1}^{\infty}|a_n|^2$$

is obtained by setting $u_n := a_1 + \cdots + a_n$ and reads

$$\boxed{\sum_{n=1}^{\infty}|u_n - u_{n-1}|^2 \geq \frac{1}{4}\sum_{n=1}^{\infty}\frac{|u_n|^2}{n^2},} \qquad (u_0 := 0).$$

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Remark: The weight $1/(4x^2)$ in the continuous Hardy inequality

$$\int_0^{\infty}|u'(x)|^2\,\mathrm{d}x \geq \frac{1}{4}\int_0^{\infty}\frac{|u(x)|^2}{x^2}\mathrm{d}x, \qquad u \in H^1(0,\infty),\ u(0)=0,$$

cannot be further improved, i.e., for any Hardy weight ρ, it holds

$$\rho(x) \geq \frac{1}{4x^2} \quad \text{a.e. } x>0 \quad \Longrightarrow \quad \rho(x) = \frac{1}{4x^2} \quad \text{a.e. } x>0.$$

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$$\boxed{\sum_{n=1}^{\infty} |u_n - u_{n-1}|^2 \geq \sum_{n=1}^{\infty} \rho_n^{(1)} |u_n|^2,} \qquad (u_0 := 0),$$

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Remark: Alternatively, the inequality can be written as $\boxed{-\Delta \geq \rho^{(1)}}$ in the sense of quadratic forms on the space $H_0^1(\mathbb{N}_0) := \{u \in \ell^2(\mathbb{N}_0) \mid u_0 = 0\}$, where

$$(-\Delta u)_0 := 2u_0 - u_1, \qquad (-\Delta u)_n := -u_{n-1}+2u_n-u_{n+1}, \quad \text{if } n \geq 1,$$

is the discrete (Dirichlet) Laplacian on $\ell^2(\mathbb{N}_0)$.

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where

$$(R_1 u)_n := a_n u_n - \frac{1}{a_n} u_{n+1}, \quad n \in \mathbb{N},$$

and $\{a_n\} \subset \mathbb{R} \setminus \{0\}$ is an unknown sequence we seek.

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The remainder term in the improved discrete Hardy inequality

Theorem [Krejčiřík-Š., 2022]

For all $u \in H_0^1(\mathbb{N}_0)$, we have the identity

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- $$\rho_n^{(1)} = 2 - \sqrt{\frac{n-1}{n}} - \sqrt{\frac{n+1}{n}} = \frac{-\Delta g_n^{(1)}}{g_n^{(1)}}, \quad \text{where} \;\; g_n^{(1)} = \sqrt{n}.$$

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- The explicit form of R_1 and the fact $R_1 g^{(1)} = 0$ yields a simple proof of

the criticality of $\rho^{(1)}$: If ρ is a Hardy weight and $\rho \geq \rho^{(1)} \implies \rho = \rho^{(1)}$.

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- But $g^{(1)} \notin H_0^1(\mathbb{N}_0)$! A suitable regularization of $g^{(1)}$ $\rightsquigarrow$ a rigorous proof.

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3. $\rho^{(1)}$ is optimal by infinity: For all $M \in \mathbb{N}$, one has

$$\inf_{u \in H_0^M(\mathbb{N}_0) \setminus \{0\}} \frac{\sum_{n=1}^{\infty} |u_{n-1} - u_n|^2}{\sum_{n=1}^{\infty} \rho_n^{(1)} |u_n|^2} = 1,$$

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- The optimality of $\rho^{(1)}$ was first proven by [Keller-Pinchover-Pogorzelski, 2018]; an elementary proof also exists [Gerhat-Krejčiřík-Š., prep. 2022].

Contents

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$$\int_0^\infty \left|u''(x)\right|^2 \mathrm{d}x \geq \frac{9}{16}\int_0^\infty \frac{|u(x)|^2}{x^4}\mathrm{d}x,$$

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The goal: To adapt the method which works in the Hardy case to deduce an improved discrete Rellich inequality.

An improved discrete Rellich weight

Theorem [Gerhat-Krejčiřík-Š. prep. 2022]

For all $u \in H_0^2(\mathbb{N}_0)$, the discrete Rellich inequality

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holds with

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$$\rho_n^{(2)} = 6 - 4\left(\frac{n+1}{n}\right)^{3/2} - 4\left(\frac{n-1}{n}\right)^{3/2} + \left(\frac{n+2}{n}\right)^{3/2} + \left(\frac{n-2}{n}\right)^{3/2}.$$

An improved discrete Rellich weight

Theorem [Gerhat-Krejčiřík-Š. prep. 2022]

For all $u \in H_0^2(\mathbb{N}_0)$, the discrete Rellich inequality

$$\sum_{n=1}^{\infty} |(-\Delta u)_n|^2 \geq \sum_{n=2}^{\infty} \rho_n^{(2)} |u_n|^2$$

holds with

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- One has

$$\rho_n^{(2)} > \frac{9}{16n^4} \quad \text{and} \quad \rho_n^{(2)} = \frac{9}{16n^4} + O\left(\frac{1}{n^6}\right), \quad n \to \infty.$$

Comments on the proof

- Actually, we prove the identity

$$\boxed{\sum_{n=1}^{\infty}\left|(-\Delta u)_n\right|^2=\sum_{n=2}^{\infty}\rho_n^{(2)}|u_n|^2+\sum_{n=1}^{\infty}\left|(R_2 u)_n\right|^2,}\qquad u\in H_0^2(\mathbb{N}_0),$$

where the remainder has the form

$$(R_2 u)_n := c_n u_n - b_n u_{n+1} + \frac{1}{c_n}u_{n+2}$$

depending on two unknown sequences $\{b_n\}\subset\mathbb{R}$ and $\{c_n\}\subset\mathbb{R}\setminus\{0\}$.

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- Unlike the Hardy case, the remainder R_2 is not found explicitly. But the existence of positive solutions $\{b_n\}$ and $\{c_n\}$ has been proven.

On the optimality of $\rho^{(2)}$

- The criticality of $\rho^{(2)}$ has not been established:

$$\text{If } \rho \text{ is a Rellich weight such that } \rho \geq \rho^{(2)} \quad \stackrel{???}{\Longrightarrow} \quad \rho = \rho^{(2)}.$$

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Proposition

If ρ is a Rellich weight such that $\rho \geq \rho^{(2)}$, then

$$\sum_{n=2}^{\infty} n^3 \left(\rho_n - \rho_n^{(2)} \right) \leq 8\sqrt{2} - 3\sqrt{3}.$$

Contents

Higher order Hardy-like inequalities

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Obrigado!