Inverse spectral problem for non-self-adjoint Jacobi operators

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Based on:
A. Pushnitski, F. Štampach: An inverse spectral problem for non-self-adjoint Jacobi matrices
arXiv:2305.19608

Contents

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(Operators $J(a,b)$ and $J(|a|,b)$ have the same spectral measure.)

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- A distinguished vector:

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Remark:
OG polynomials: Define sequence of polynomials $p \equiv (p_0, p_1, \dots)^T$ recursively by

$$Jp(x) = xp(x), \quad \text{i.e.} \quad a_{n-1}p_{n-1}(x) + b_n p_n(x) + a_n p_{n+1}(x) = 0, \quad (a_{-1} := 0)$$

and $p_0(x) = 1$.

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and $p_0(x) = 1$. Theorem ↑ a.k.a. Favard's theorem $\Rightarrow \mu$ is a measure of orthogonality:

$$\int_\mathbb{R} p_n(x) p_m(x) \mathrm{d}\mu(x) = \delta_{m,n}.$$

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$$\mu \quad \rightsquigarrow \quad \text{moments / m-function} \quad \rightsquigarrow \quad b_n, a_n^2 \quad \overset{a_n>0}{\rightsquigarrow} \quad b_n, a_n$$

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Remark 2:
OG polynomials: Given μ, an application of the Gram–Schmidt to $1, x, x^2, \ldots$ in $L^2(\mu)$ produces the sequence of monic polynomials P_n satisfying

$$P_{n+1}(x) = (x - b_n)P_n(x) - a_{n-1}^2 P_{n-1}(x), \quad P_0(x) = 1$$

From the recurrence we can reconstruct b_n and a_n^2.

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- What are the implications for orthogonal polynomials?

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Inessential assumptions:

- Normalization $a_n > 0$
  (can be replace by $a_n \in \mathbb{C}$ with $\arg a_n$ prescribed)

The 1st component of spectral data - the spectral measure ν

Spectral measure:

$$\nu(\Delta) := \langle \chi_\Delta(|J|)\delta_0, \delta_0 \rangle, \quad \Delta \in \mathcal{B}_\mathbb{R},$$

where $|J| := \sqrt{J^* J}$.

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The multiplicity of spectrum of $|J|$ is ≤ 2 and vector δ_0 is of maximal type for $|J|$, i.e

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The 2nd component of spectral data - the phase function ψ

Recall

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- The theorem only uses $CJC = J^*$ and $C\delta_0 = \delta_0$. It can be deduced from the refined polar decomposition for C-symmetric operators [Garcia-Putinar, 2007].

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In particular, if $J = J^* \geq 0$, then $\mu = \nu$ and $\psi \equiv 1$.

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The integrand expanded:

$$(1 + \Re\psi(s))q_m(s)\overline{q}_n(s) + (1 - \Re\psi(s))q_m(-s)\overline{q}_n(-s) \\ + \mathrm{i}\Im\psi(s)\left[q_m(s)\overline{q}_n(-s) - q_m(-s)\overline{q}_n(s)\right]$$

Thank you!