Optimal spectral enclosures for 1D discrete Schrödinger operators with complex potentials

Frantisek Štampach

Joint with: O. O. Ibrogimov, D. Krejčiřík, A. Laptev

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Contents

Motivation: the (continuous) Schrödinger operator on the line

Theorem (Abramov, Aslanyan, Davies [JPA, 2001])

For a complex valued $V \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, one has

$$\sigma_p\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V\right) \setminus [0, \infty) \subset \left\{\lambda \in \mathbb{C} \mid 4|\lambda| \leq \|V\|^2_{L^1(\mathbb{R})}\right\}.$$

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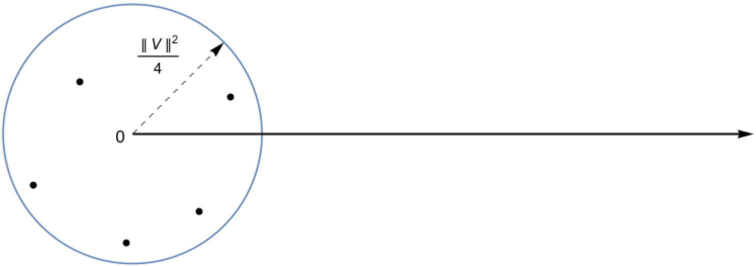

The discrete Schrödinger operator on $\mathbb{Z}$

- Difference operators on $\ell^2(\mathbb{Z})$:

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$$\boxed{H_0 + V} = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & 1 & v_{-1} & 1 & & & \\ & & 1 & v_0 & 1 & & \\ & & & 1 & v_1 & 1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

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- The resolvent of H_0:
$$\boxed{(H_0 - \lambda)^{-1}_{m,n} = \frac{k^{|m-n|}}{k - k^{-1}}} \qquad m, n \in \mathbb{Z},$$
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- The Joukowski map:
$$\lambda(k) = k^{-1} + k$$
is 1–1 mapping of the punctured unit disk $0 < |k| < 1$ onto $\mathbb{C} \setminus [-2, 2]$.

The spectral enclosure for ℓ^1-potentials

Theorem (Ibrogimov, F. Š. [IEOT, 2019])

Let $v \in \ell^1(\mathbb{Z})$. Then

$$\sigma_{\mathrm{p}}(H_0 + V) \subset \left\{ \lambda \in \mathbb{C} \setminus (-2, 2) \;\middle|\; |\lambda^2 - 4| \leq \|v\|^2_{\ell^1(\mathbb{Z})} \right\}.$$

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In addition, the estimate is optimal in the following sense:

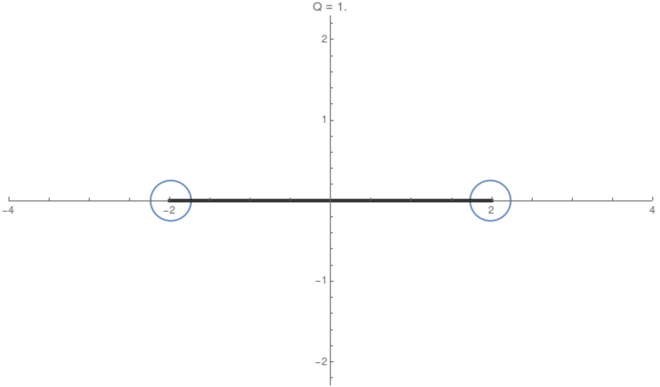
To any boundary point of the spectral enclosure which does not belong to $(-2,2)$, there exists an ℓ^1–potential V so that this boundary point is an eigenvalue of the corresponding discrete Schrödinger operator $H_0 + V$.

Geometry of the spectral enclosure

The boundary curve for $Q := \|v\|_{\ell^1(\mathbb{Z})}$:

$$|\lambda^2 - 4| = Q^2.$$

...it is the Cassini oval with two foci at ± 2.



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- For $\lambda\notin[-2,2]\equiv\sigma(H_0)$, the proof relies on the Birman–Schwinger principle (one implication):

$$\lambda\in\sigma_p(H_0+V)\quad\Longrightarrow\quad -1\in\sigma_p(K(\lambda)),$$

for

$$K(\lambda):=|V|^{1/2}(H_0-\lambda)^{-1}V_{1/2},$$

and

$$|V|^{1/2}e_n=\sqrt{|v_n|}e_n\quad\text{and}\quad V_{1/2}e_n=\operatorname{sgn}(v_n)\sqrt{|v_n|}e_n$$

with the complex signum function $\operatorname{sgn} z=z/|z|$, if $z\neq0$, and $\operatorname{sgn}0=0$.

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- In particular,

$$\boxed{\lambda \in \sigma_p(H_0+V) \quad\Longrightarrow\quad \|K(\lambda)\| \geq 1.}$$

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- For any $\psi \in \ell^2(\mathbb{Z})$, we estimate

$$\|K(\lambda)\psi\|^2_{\ell^2(\mathbb{Z})} \leq \sum_{m\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} \sqrt{|v_m|} \left|(H_0 - \lambda)^{-1}_{m,n}\right| \sqrt{|v_n|} |\psi_n| \right)^2$$
$$\leq \frac{\|v\|_{\ell^1(\mathbb{Z})}}{|\lambda^2 - 4|} \left(\sum_{m\in\mathbb{Z}} \sqrt{|v_n|} |\psi_n| \right)^2 \leq \frac{\|v\|^2_{\ell^1(\mathbb{Z})}}{|\lambda^2 - 4|} \|\psi\|^2_{\ell^2(\mathbb{Z})}.$$

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- In other words,

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- Thus, if $\lambda \in \sigma_p(H_0+V)$, then $\boxed{|\lambda^2-4| \leq \|v\|^2_{\ell^1(\mathbb{Z})}.}$

The optimality

- Delta potential:

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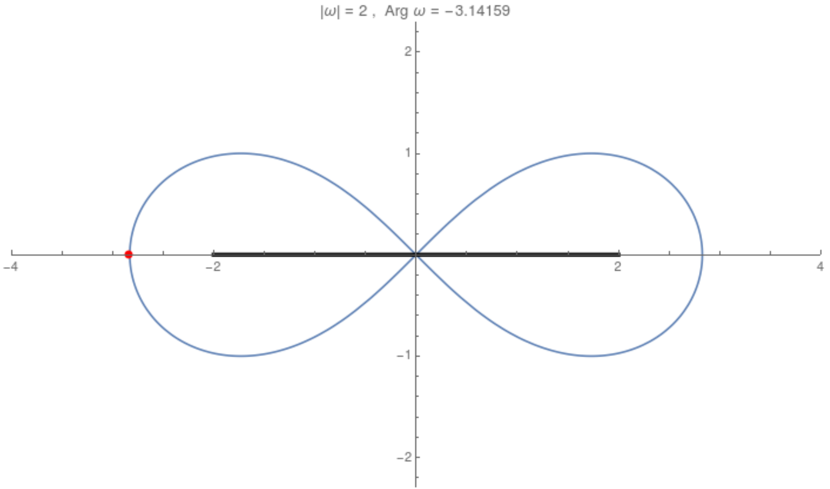
- Moreover, for any $Q > 0$, one has

$$\{\lambda_\omega \mid \omega = Qe^{\mathrm{i}\theta}, -\pi < \theta \leq \pi\} = \{\lambda \in \mathbb{C} \mid |\lambda^2 - 4| = Q^2\}.$$

Numerical illustration: the delta potential demonstrates optimality



The spectral enclosure the best possible

Corollary

One has

$$\bigcup_{\|v\|_{\ell^1(\mathbb{Z})}\leq Q} \sigma(H_0+V) = [-2,2] \cup \{\lambda\in\mathbb{C} \mid |\lambda^2-4| \leq Q^2\}.$$

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On the other hand, we do not know whether

$$\bigcup_{\|v\|_{\ell^1(\mathbb{Z})}=Q} \sigma(H_0+V) \stackrel{?}{=} [-2,2] \cup \{\lambda \in \mathbb{C} \mid |\lambda^2-4| \leq Q^2\}.$$

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Let $V \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ be complex valued and $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ be the Schrödinger operator acting on $L^2(\mathbb{R}_+)$ with *Dirichlet* boundary condition at 0. Then

$$\boxed{\sigma_d\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V\right) \subset \left\{\lambda \in \mathbb{C} \;\middle|\; 4|\lambda| \leq h\left(\cot\frac{\arg\lambda}{2}\right)\|V\|^2_{L^1(\mathbb{R}_+)}\right\},}$$

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$$h(a) := \sup_{y\geq 0}\left|e^{\mathrm{i}ay} - e^{-y}\right|^2$$

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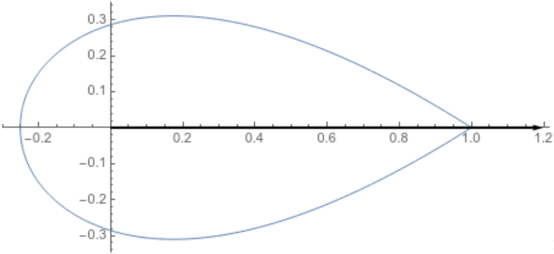

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- The discrete *Dirichlet* and *Neumann* Laplacians on $\mathbb{N}$:

$$D^*D = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad DD^* = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

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- We denote $J_0 := 2 - D^*D$ and $J_1 := 2 - DD^*$. Then

$$J_0 = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad J_1 := \begin{pmatrix} 1 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

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- For $a \in \mathbb{C}$, we put

$$J_a := \begin{pmatrix} a & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

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- The discrete Robin Schrödinger operator on $\mathbb{N}$:

$$\boxed{J_a + V} = \begin{pmatrix} a+v_1 & 1 & & & \\ 1 & v_2 & 1 & & \\ & 1 & v_3 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Spectral properties of J_a

- Spectrum of J_a and its parts:

$$\sigma_{\mathrm{c}}(J_a) = [-2, 2], \quad \sigma_{\mathrm{r}}(J_a) = \emptyset, \quad \text{and} \quad \sigma_{\mathrm{p}}(J_a) = \begin{cases} \emptyset, & \text{if } |a| \leq 1, \\ \{a + a^{-1}\}, & \text{if } |a| > 1. \end{cases}$$

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- If $v_n \to 0$, as $n \to \infty$, then V is compact and

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Spectral properties of J_a

- Spectrum of J_a and its parts:

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- The resolvent of J_a:

$$\boxed{(J_a - \lambda)^{-1}_{m,n} = \frac{(k-a)k^{m+n-1} - (k^{-1}-a)k^{|n-m|+1}}{(1-ak)(k^{-1}-k)}} \qquad m,n \in \mathbb{N},$$

where $\lambda = k^{-1} + k \notin \sigma(J_a)$ for $0 < |k| < 1$.

The spectral enclosure for ℓ^1-potentials

Theorem

Let $a \in \mathbb{C}$ and $v \in \ell^1(\mathbb{N})$. Then

$$\sigma_{\mathrm{p}}(J_a + V) \subset \left\{ \lambda \in \mathbb{C} \setminus (-2,2) \;\middle|\; |\lambda^2 - 4| \leq g_a(\lambda)\, \|v\|^2_{\ell^1(\mathbb{N})} \right\},$$

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$$g_a\left(k + k^{-1}\right) := \sup_{n \in \mathbb{N}} \left| 1 - \frac{k-a}{1-ak}\, k^{2n-1} \right|.$$

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To any non-real boundary point of the spectral enclosure, there exists an ℓ^1–potential V so that this boundary point is an eigenvalue of the corresponding discrete Schrödinger operator $J_a + V$.

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Remark:

Dirichlet: $g_0(\lambda) = \sup_{n \in \mathbb{N}} \left|1 - k^{2n}\right|$ Neumann: $g_1(\lambda) = \sup_{n \in \mathbb{N}} \left|1 + k^{2n-1}\right|$

Geometry of the optimal spectral enclosures

The boundary curve for $Q := \|v\|_{\ell^1(\mathbb{N})}$:

$$\boxed{|\lambda^2 - 4| = g_a(\lambda)\, Q^2.}$$

Geometry of the optimal spectral enclosures

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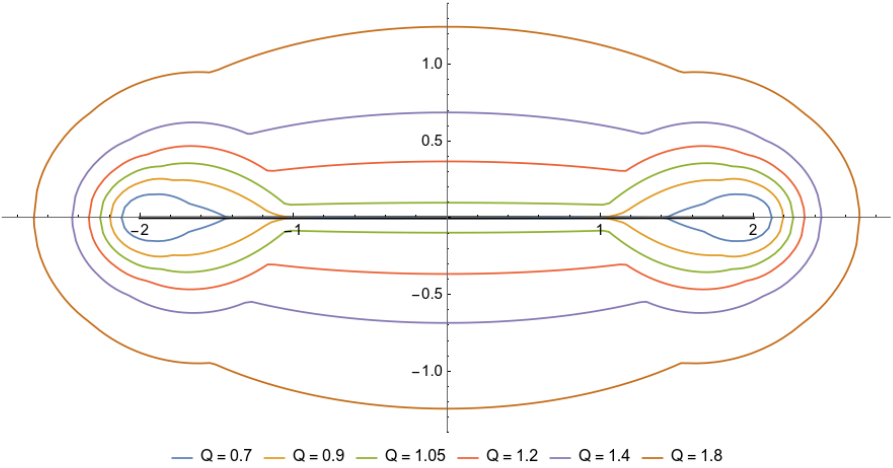

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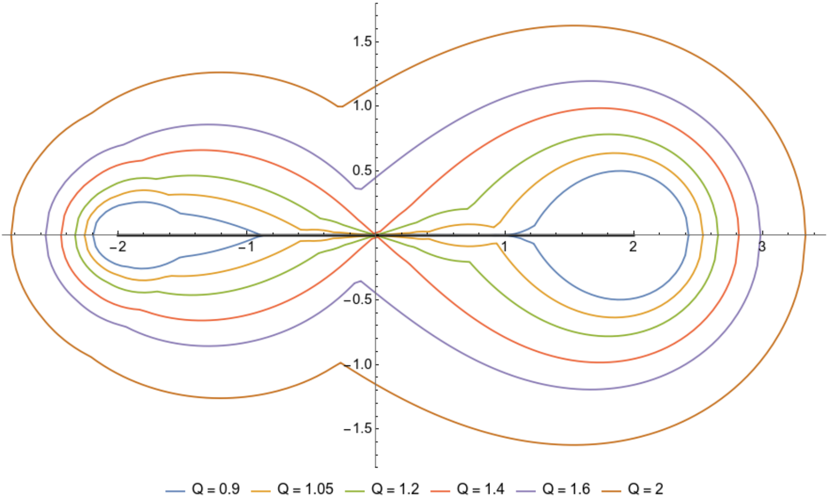

Geometry of the optimal spectral enclosures

The boundary curve for $Q := \|v\|_{\ell^1(\mathbb{N})}$:

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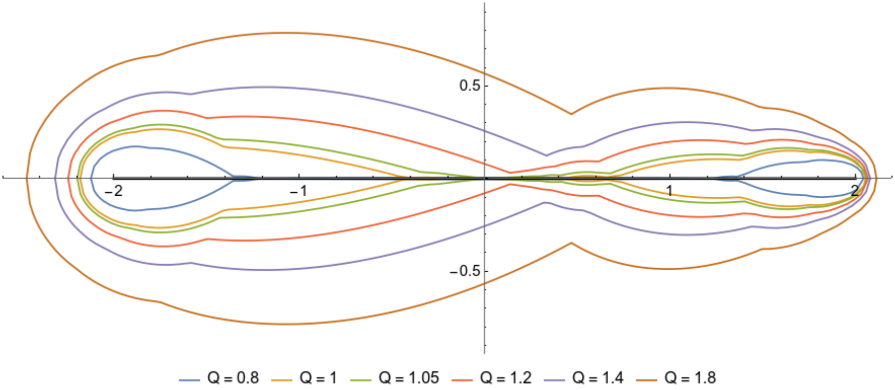
Robin



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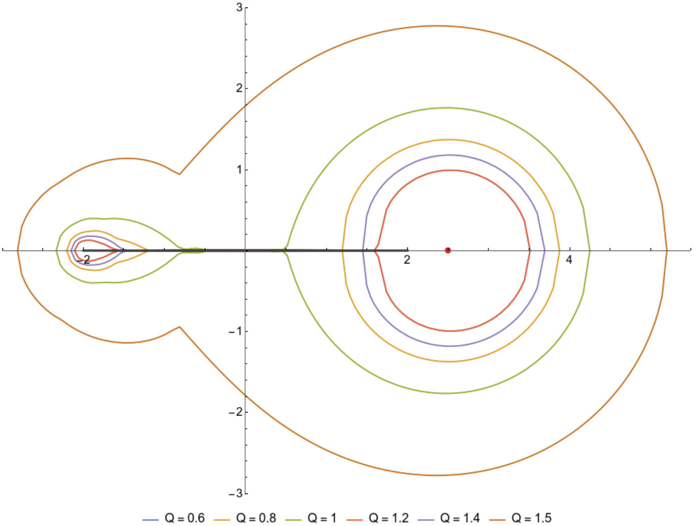

Contents

Spectral stability: perturbations not producing eigenvalues

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- Recall that

$$\{v_n\}_{n=1}^{\infty} \subset \mathbb{C} \quad \text{and} \quad V = \mathrm{diag}(v_1, v_2, \dots).$$

Theorem

If the matrix K with elements

$$K_{m,n} = \sqrt{|v_n|}\min(m,n)\sqrt{|v_m|}, \quad m,n \in \mathbb{N},$$

satisfies $\|K\| < 1$, then $\boxed{\sigma(J_0 + V) = \sigma_c(J_0 + V) = \sigma(J_0) = [-2, 2].}$

Equivalently, if there exists $c < 1$ such that for all $\psi \in \ell^2(\mathbb{N})$ it holds

$$\sum_{n=1}^{\infty} |v_n||\psi_n|^2 \leq c \sum_{n=1}^{\infty} |\psi_n - \psi_{n-1}|^2 \qquad (\textit{Hardy})$$

where $\psi_0 := 0$, then $\boxed{\sigma(J_0 + V) = \sigma_c(J_0 + V) = \sigma(J_0) = [-2, 2].}$

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Remark: No similar stability for H_0 or J_1 (H_0, J_1 are critical).

Discrete Hardy inequalities

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The classical discrete Hardy inequality [Hardy, Landau, 1921]

For all $\psi \in \ell^2(\mathbb{N})$, one has

$$\sum_{n=1}^{\infty} |\psi_n - \psi_{n-1}|^2 \geq \sum_{n=1}^{\infty} \frac{1}{4n^2} |\psi_n|^2.$$

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The improved discrete Hardy inequality [Keller, Pinchover, Pogorzelski, CMP2018]

For all $\psi \in \ell^2(\mathbb{N})$, one has

$$\sum_{n=1}^{\infty} |\psi_n - \psi_{n-1}|^2 \geq \sum_{n=1}^{\infty} w_n |\psi_n|^2,$$

where

$$w_n = 2 - \sqrt{1 - \frac{1}{n}} - \sqrt{1 + \frac{1}{n}} = \frac{1}{4n^2} + \frac{5}{64n^4} + \frac{21}{512n^6} + \dots > \frac{1}{4n^2}.$$

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Moreover, the improved weight is optimal ... [next slide].

Discrete Hardy inequalities

The improved weight

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Opt3) $(\forall \epsilon > 0)(\forall m \in \mathbb{N})(\exists \psi$ supported on $\mathbb{N}_{\geq m})(\sum_n |\psi_n - \psi_{n-1}|^2 < (1+\epsilon) \sum_n w_n |\psi_n|^2)$. (*optimality near infinity*)

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Even more explicit Hardy weights:

For all $\psi \in \ell^2(\mathbb{N})$ and $q \in (0, 1/2]$, one has

$$\boxed{\sum_{n=1}^{\infty} |\psi_n - \psi_{n-1}|^2 \geq \sum_{n=1}^{\infty} \left[2 - \left(1 - \frac{1}{n}\right)^q - \left(1 + \frac{1}{n}\right)^q\right] |\psi_n|^2.}$$

For $q \in (0, 1/2)$, (Opt1) holds.

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$$|v_n| \leq c \left[2 - \left(1 - \frac{1}{n}\right)^q - \left(1 + \frac{1}{n}\right)^q \right],$$

for all $n \in \mathbb{N}$ and some $c < 1$ and $q \in (0, 1/2]$, then

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$$\boxed{\sigma_d(J_0 + V) = \emptyset.}$$

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Thank you!