On the spectral pair for Schrödinger operators with complex potentials on the half-line

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joint with Alexander Pushnitski
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Workshop in honor of Christiane Tretter
Mathematical Institute at the University of Bern
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Happy birthday, Christiane!

`A. Pushnitski, F. Štampach: The Borg-Marchenko uniqueness theorem for complex potentials, arXiv:2503.03248, (2025).`

Contents

Self-adjoint Schrödinger operator on the half-line

- For $q : \mathbb{R}_+ \to \mathbb{R}$ a measurable *bounded* function, we define

$$\boxed{H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q} \qquad \text{in } L^2(\mathbb{R}_+)$$

with

$$\mathrm{Dom}\, H = \{f \in W^{2,2}(\mathbb{R}_+) \mid f'(0) + \alpha f(0) = 0\},$$

where $\alpha \in \mathbb{R} \cup \{\infty\}$ (if $\alpha = \infty$, $f(0) = 0$).

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  where $\alpha \in \mathbb{R} \cup \{\infty\}$ (if $\alpha = \infty$, $f(0) = 0$).
- For $\lambda \in \mathbb{C}$, denote by φ and θ the solutions of

$$-f'' + qf = \lambda f$$

  satisfying the Cauchy data:

$$\begin{aligned} \varphi(0,\lambda) &= \sin\gamma, & \theta(0,\lambda) &= \cos\gamma, \\ \varphi'(0,\lambda) &= -\cos\gamma, & \theta'(0,\lambda) &= \sin\gamma, \end{aligned}$$

  where $\gamma = \gamma(\alpha) \in [0,\pi)$.

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  where $\gamma = \gamma(\alpha) \in [0,\pi)$.
- q bounded $\Rightarrow$ for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists unique $m_\alpha(\lambda) \in \mathbb{C}$ such that

$$\theta(\cdot,\lambda) - \varphi(\cdot,\lambda) m_\alpha(\lambda) \in L^2(\mathbb{R}_+).$$

The spectral measure of H

- $H = H^* \;\Rightarrow\; m_\alpha : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ is Herglotz–Nevanlinna, i.e.

$$m_\alpha \text{ is analytic}; \qquad \operatorname{Im} m_\alpha(\lambda) \geq 0, \text{ if } \operatorname{Im} \lambda > 0; \qquad \overline{m_\alpha(\lambda)} = m_\alpha(\overline{\lambda}).$$

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- Herglotz–Nevanlinna integral representation:

$$m_\alpha(\lambda) = \operatorname{Re} m_\alpha(\mathrm{i}) + \int_{-\infty}^{\infty} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) \mathrm{d}\sigma(t).$$

The measure σ is the spectral measure of H. (H is unitarily equivalent to the operator of multiplication by the independent variable in $L^2_\sigma(\mathbb{R})$.)

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- The inverse spectral theory of H $\simeq$ properties of the spectral map:

$$\boxed{(\alpha, q) \mapsto \sigma.}$$

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- Some limitations: Marchenko's asymptotic formulas (1952): as $r \to \infty$,

$$\sigma((-\infty, r]) = \begin{cases} c_\alpha r^{1/2} + o(r^{1/2}), & \text{if } \alpha \neq \infty, \\ c_\infty r^{3/2} + o\left(r^{3/2}\right), & \text{if } \alpha = \infty. \end{cases}$$

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- Complete characterisation:
  - Gelfand–Levitan, Krein (1951-3-5); more regularity on q
  - Remling (2002); locally integrable q
  - Contributions by many (Gesztesy, Simon,...)

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- Let Φ, Θ be the 2×2 matrix-valued solutions (the fundamental system) of
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The $\mathbb{C}^{2,2}$-valued measure Σ is called the spectral measure of **H**.

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There exists a unique **even** positive measure ν on $\mathbb{R}$ and a unique **odd** complex-valued function $\psi \in L^{\infty}(\nu)$ satisfying $\|\psi\|_{\infty} \leq 1$ such that

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2. Moreover, the spectrum of $|H|$ has:
   - multiplicity one on $\{s > 0 \mid |\psi(s)| = 1\}$,
   - multiplicity two on $\{s > 0 \mid |\psi(s)| < 1\}$.

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Remark: In fact, we prove the injectivity of $(\alpha, Q) \mapsto \Sigma$ for any $Q = Q^*$.

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The spectral pair (ν, ψ) of H satisfies:

$$\lim_{r\to\infty} \frac{\nu([0,r])}{r^{1/2}} = \lim_{r\to\infty} \frac{1}{r^{1/2}} \int_0^r \psi(s)\mathrm{d}\nu(s) = \frac{1+|\alpha|^2}{\pi}, \qquad \text{if } \alpha \neq \infty$$

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for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

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for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$. If 0 and 1 are not point masses of Σ^0, then

$$\Sigma([0,r]) = rg(r)\left(\Sigma^0([0,1]) + o(1)\right), \qquad \text{as } r \to \infty.$$

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Furthermore, $H = H^* \geq 0 \iff \psi(s) = 1$ for ν-a.e. $s > 0$.

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- **Remark**: There is a generalisation of the last claim describing spectral pairs for normal H.

Spectral pair at a simple singular value of H

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Lemma (the distinguished solution)

Let $\lambda > 0$ be a simple eigenvalue of $|H|$. Then there exists a unique, up to multiplication by ± 1, normalised function $e \in L^2(\mathbb{R}_+)$, which satisfies

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Theorem

Let $\lambda > 0$ be a simple eigenvalue of $|H|$ and let e be the distinguished solution from above. Then $\ell_\alpha(e) \neq 0$ and

$$\boxed{\nu(\{\lambda\}) = \frac{1}{2}|\ell_\alpha(e)|^2, \quad \psi(\lambda) = \frac{\overline{\ell_\alpha(e)^2}}{|\ell_\alpha(e)|^2}.}$$

Related literature

Hankel operators:

- P. Gérard, S. Grellier: *The cubic Szegő equation and Hankel operators*, Astérisque **389** (2017).
- P. Gérard, A. Pushnitski, S. Treil: *An inverse spectral problem for non-compact Hankel operators with simple spectrum*, J. Anal. Math. **154** (2024).

Jacobi operators:

- A. Pushnitski, F. Š.: *An inverse spectral problem for non-self-adjoint Jacobi matrices*, Int. Math. Res. Not. **2024** (2024).
- A. Pushnitski, F. Š.: *A functional model and tridiagonalisation for symmetric anti-linear operators*, preprint (2024), `arXiv:2402.01237`.

Schrödinger operators:

- A. Pushnitski, F. Š.: *The Borg–Marchenko uniqueness theorem for complex potentials*, preprint (2025), `arXiv:2503.03248`.

Related literature

Hankel operators:

- P. Gérard, S. Grellier: *The cubic Szegő equation and Hankel operators*, Astérisque **389** (2017).
- P. Gérard, A. Pushnitski, S. Treil: *An inverse spectral problem for non-compact Hankel operators with simple spectrum*, J. Anal. Math. **154** (2024).

Jacobi operators:

- A. Pushnitski, F. Š.: *An inverse spectral problem for non-self-adjoint Jacobi matrices*, Int. Math. Res. Not. **2024** (2024).
- A. Pushnitski, F. Š.: *A functional model and tridiagonalisation for symmetric anti-linear operators*, preprint (2024), `arXiv:2402.01237`.

Schrödinger operators:

- A. Pushnitski, F. Š.: *The Borg–Marchenko uniqueness theorem for complex potentials*, preprint (2025), `arXiv:2503.03248`.

Thank you!