An open problem on the discrete Rellich inequality

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Mathematical aspects of the physics with non-self-adjoint operators
Banff, Canada
July 11-15, 2022

OP appears in: An improved discrete Rellich inequality on the half-line arXiv:2206.11007

Contents

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- The classical Hardy inequality on $L^2(0,\infty)$:

$$\boxed{\int_0^\infty |u'(x)|^2 \,\mathrm{d}x \geq \frac{1}{4}\int_0^\infty \frac{|u(x)|^2}{x^2}\mathrm{d}x,} \qquad u \in H^1(0,\infty),\ u(0)=0.$$

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- The improved Hardy inequality on $\ell^2(\mathbb{N}_0)$ [Keller-Pinchover-Pogorzelski, 2018]:

$$\boxed{\sum_{n=1}^{\infty} |u_n - u_{n-1}|^2 \geq \sum_{n=1}^{\infty}\rho_n|u_n|^2,} \qquad u \in \ell^2(\mathbb{N}_0),\ u_0 = 0.$$

where

$$\rho_n = 2 - \sqrt{\frac{n-1}{n}} - \sqrt{\frac{n+1}{n}} > \frac{1}{4n^2}, \qquad \forall n \in \mathbb{N}.$$

An equivalent formulation of the discrete Hardy inequalities

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- Difference operators on $\ell^2(\mathbb{N})$:

$$(Du)_n := \begin{cases} u_{n-1} - u_n, & n > 1, \\ -u_1, & n = 1, \end{cases} \qquad (D^* u)_n = u_{n+1} - u_n, \qquad n \in \mathbb{N}.$$

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- The discrete Laplacian on $\ell^2(\mathbb{N})$:

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- The discrete Hardy inequality on $\ell^2(\mathbb{N})$:

$$\boxed{-\Delta \geq \rho.}$$

An application: Spectral stability of the discrete Schrödiner operators

Theorem [Krejčiřík-Laptev-Š.,2022]

If the complex potential $V = \mathrm{diag}(v_1, v_2, \dots)$ satisfies the abstract Hardy inequality

$$\boxed{-\Delta \geq |V|,}$$

then

$$\boxed{\sigma_d(-\Delta + V) = \emptyset,} \quad \text{which means} \quad \boxed{\sigma(-\Delta + V) = \sigma(-\Delta) = [-2, 2].}$$

(In fact, one has $\sigma_c(-\Delta + V) \subset [-2, 2]$, $\sigma_p(-\Delta + V) \subset \{\pm 2\}$, and $\sigma_r(-\Delta + V) = \emptyset$.)

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Corollary (Concrete criteria)

If the complex potential $V = \mathrm{diag}(v_1, v_2, \dots)$ satisfies

$$|v_n| \leq 2 - \sqrt{\frac{n-1}{n}} - \sqrt{\frac{n+1}{n}},$$

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$$\boxed{\int_0^\infty \left|u''(x)\right|^2 \mathrm{d}x \geq \frac{9}{16}\int_0^\infty \frac{|u(x)|^2}{x^4}\mathrm{d}x,} \qquad u \in H^2(0,\infty),\ u(0)=u'(0)=0.$$

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$$\boxed{\sum_{n=1}^\infty |(-\Delta u)_n|^2 \geq \sum_{n=1}^\infty \rho_n |u_n|^2,} \qquad u \in \ell^2(\mathbb{N}_0),\ u_0=u_1=0,$$

where

$$\rho_n = 6 - 4\left(\frac{n+1}{n}\right)^{3/2} - 4\left(\frac{n-1}{n}\right)^{3/2} + \left(\frac{n+2}{n}\right)^{3/2} + \left(\frac{n-2}{n}\right)^{3/2} > \frac{9}{16n^4}.$$

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Remark: While the requirement $u_0=0$ is reasonable for the singularity of the Rellich weight at $n=0$, the condition $u_1=0$ seems reasonable when compared to the continuous setting. On the other hand, Rellich inequalities for the discrete bi-Laplacian Δ^2 on $\ell^2(\mathbb{N})$ is of great interest, too!

The open problem: A discrete Rellich inequality on $\ell^2(\mathbb{N})$

Consider the discrete bi-Laplacian on $\ell^2(\mathbb{N})$:

$$\Delta^2 = \begin{pmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & 1 & -4 & 6 & -4 & 1 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

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1. Is there $c > 0$ such that the inequality

$$\boxed{\Delta^2 \geq \rho(c) \ \text{ on } \ell^2(\mathbb{N}),} \quad \text{where } \ \rho_n(c) := \frac{c}{n^4} \ ?$$

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3. Is there an improved or even critical weight ρ such that $\Delta^2 \geq \rho$ on $\ell^2(\mathbb{N})$?

Thank you!