# The spectral pair for Schrödinger operators with complex potentials on the half-line

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# Contents

# Self-adjoint Schrödinger operator on the half-line

- For $q : \mathbb{R}_+ \to \mathbb{R}$ a measurable *bounded* function, we define

$$\boxed{H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q} \qquad \text{in } L^2(\mathbb{R}_+)$$

with

$$\operatorname{Dom} H = \{f \in W^{2,2}(\mathbb{R}_+) \mid f'(0) + \alpha f(0) = 0\},$$

where $\alpha \in \mathbb{R} \cup \{\infty\}$ (if $\alpha = \infty$, $f(0) = 0$).

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  where $\alpha \in \mathbb{R} \cup \{\infty\}$ (if $\alpha = \infty$, $f(0) = 0$).
- For $\lambda \in \mathbb{C}$, denote by $\varphi$ and $\theta$ the solutions of

$$-f'' + qf = \lambda f$$

  satisfying the Cauchy data:

$$\begin{aligned} \varphi(0,\lambda) &= \sin\gamma, & \theta(0,\lambda) &= \cos\gamma, \\ \varphi'(0,\lambda) &= -\cos\gamma, & \theta'(0,\lambda) &= \sin\gamma, \end{aligned}$$

  where $\gamma = \gamma(\alpha) \in [0, \pi)$.

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$$\varphi'(0,\lambda)=-\cos\gamma, \qquad \theta'(0,\lambda)=\sin\gamma,$$

  where $\gamma=\gamma(\alpha)\in[0,\pi)$.
- $q$ bounded $\Rightarrow$ for every $\lambda\in\mathbb{C}\setminus\mathbb{R}$ there exists unique $m_\alpha(\lambda)\in\mathbb{C}$ such that

$$\theta(\cdot,\lambda)-\varphi(\cdot,\lambda)m_\alpha(\lambda)\in L^2(\mathbb{R}_+).$$

# The spectral measure of $H$

- $H = H^* \;\Rightarrow\; m_\alpha : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ is Herglotz–Nevanlinna, i.e.

$$m_\alpha \text{ is analytic}; \qquad \operatorname{Im} m_\alpha(\lambda) \geq 0, \text{ if } \operatorname{Im}\lambda > 0; \qquad \overline{m_\alpha(\lambda)} = m_\alpha(\overline{\lambda}).$$

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- Herglotz–Nevanlinna integral representation:

$$m_\alpha(\lambda) = \operatorname{Re} m_\alpha(\mathrm{i}) + \int_{-\infty}^{\infty} \left( \frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) \mathrm{d}\sigma(t).$$

The measure $\sigma$ is the spectral measure of $H$. ($H$ is unitarily equivalent to the operator of multiplication by the independent variable in $L^2_\sigma(\mathbb{R})$.)

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- The inverse spectral theory of $H$ $\simeq$ properties of the spectral map:

$$\boxed{(\alpha, q) \mapsto \sigma.}$$

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- Some limitations: Marchenko's asymptotic formulas (1952): as $r \to \infty$,

$$\sigma((-\infty, r]) = \begin{cases} c_\alpha r^{1/2} + o(r^{1/2}), & \text{if } \alpha \neq \infty, \\ c_\infty r^{3/2} + o\left(r^{3/2}\right), & \text{if } \alpha = \infty. \end{cases}$$

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- Complete characterisation:
  - Gelfand–Levitan, Krein (1951-3-5); more regularity on $q$
  - Remling (2002); locally integrable $q$
  - Contributions by many (Gesztesy, Simon,...)

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- After identifying $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$ with $L^2(\mathbb{R}_+; \mathbb{C}^2)$, we get

$$\boxed{\mathbf{H} = -\epsilon \frac{\mathrm{d}^2}{\mathrm{d}x^2} + Q, \quad \text{with } \epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix},}$$

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$$\boxed{\operatorname{Dom}\mathbf{H}=\{F\in W^{2,2}(\mathbb{R}_+;\mathbb{C}^2)\mid F'(0)+AF(0)=0\}, \quad \text{with } A=\begin{pmatrix}\overline{\alpha} & 0\\ 0 & \alpha\end{pmatrix}.}$$

# The spectral measure of **H**

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$$-\epsilon F'' + QF = \lambda F, \qquad \text{on } \mathbb{R}_+$$

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- For every $\lambda \in \mathbb{C} \setminus \mathbb{R}$, there is unique $M_\alpha(\lambda) \in \mathbb{C}^{2,2}$ such that

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  The $\mathbb{C}^{2,2}$-valued measure $\Sigma$ is called the spectral measure of **H**.

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## Theorem

There exists a unique **even** positive measure $\nu$ on $\mathbb{R}$ and a unique **odd** complex-valued function $\psi \in L^\infty(\nu)$ satisfying $\|\psi\|_\infty \leq 1$ such that

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2. Moreover, the spectrum of $|H|$ has:
   - multiplicity one on $\{s > 0 \mid |\psi(s)| = 1\}$,
   - multiplicity two on $\{s > 0 \mid |\psi(s)| < 1\}$.

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**Remark**: In fact, we prove the injectivity of $(\alpha, Q) \mapsto \Sigma$ for any $Q = Q^*$.
We apply a neat argument of C. Bennewitz [CMP, 2001].

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$$\lim_{r\to\infty} \frac{\nu([0,r])}{r^{1/2}} = \lim_{r\to\infty} \frac{1}{r^{1/2}} \int_0^r \psi(s)\mathrm{d}\nu(s) = \frac{1+|\alpha|^2}{\pi}, \qquad \text{if } \alpha \neq \infty$$

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Let $M$ and $M^0$ be Herglotz–Nevanlinna matrix-valued functions with measures $\Sigma$ and $\Sigma^0$, respectively.

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for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$. If 0 and 1 are not point masses of $\Sigma^0$, then

$$\Sigma([0,r]) = rg(r)\left(\Sigma^0([0,1]) + o(1)\right), \qquad \text{as } r \to \infty.$$

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Furthermore, $H = H^* \geq 0 \iff \psi(s) = 1$ for $\nu$-a.e. $s > 0$.

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Furthermore, $H = H^* \geq 0 \iff \psi(s) = 1$ for $\nu$-a.e. $s > 0$.

- **Remark**: There is a generalisation of the last claim describing spectral pairs for normal $H$.

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**Lemma (the distinguished solution)**

Let $\lambda > 0$ be a simple eigenvalue of $|H|$. Then there exists a unique, up to multiplication by $\pm 1$, normalised function $e \in L^2(\mathbb{R}_+)$, which satisfies

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- For $f \in \operatorname{Dom} H$, put

$$\ell_\alpha(f) := \frac{\overline{\alpha} f'(0) - f(0)}{\sqrt{1 + |\alpha|^2}}.$$

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## Theorem

Let $\lambda > 0$ be a simple eigenvalue of $|H|$ and let $e$ be the distinguished solution from above. Then $\ell_\alpha(e) \neq 0$ and

$$\boxed{\nu(\{\lambda\}) = \frac{1}{2}|\ell_\alpha(e)|^2, \quad \psi(\lambda) = \frac{\overline{\ell_\alpha(e)^2}}{|\ell_\alpha(e)|^2}.}$$

# Related literature

Hankel operators:

- P. Gérard, S. Grellier: *The cubic Szegő equation and Hankel operators*, Astérisque **389** (2017).
- P. Gérard, A. Pushnitski, S. Treil: *An inverse spectral problem for non-compact Hankel operators with simple spectrum*, J. Anal. Math. **154** (2024).

Jacobi operators:

- A. Pushnitski, F. Š.: *An inverse spectral problem for non-self-adjoint Jacobi matrices*, Int. Math. Res. Not. **2024** (2024).
- A. Pushnitski, F. Š.: *A functional model and tridiagonalisation for symmetric anti-linear operators*, preprint (2024), `arXiv:2402.01237`.

Schrödinger operators:

- A. Pushnitski, F. Š.: *The Borg–Marchenko uniqueness theorem for complex potentials*, preprint (2025), `arXiv:2503.03248`.

# THANK YOU!


[Picture kindly provided by the professional photographer Jani Virtanen, 7/7/25.]