The Nevanlinna parametrization and orthognality relations for q-Lommel polynomials in the indeterminate case

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Contents

Hahn-Exton q-Bessel function

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$$\begin{aligned} J_\nu(z;q) &= z^\nu \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \, {}_1\phi_1\left(0; q^{\nu+1}; q, qz^2\right) \\ &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n \, q^{n(n+1)/2}}{(q;q)_n \, (q^{\nu+1};q)_n} z^{2n+\nu}. \end{aligned}$$

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- The Hahn-Exton q-Bessel function has been intensively studied in past (difference eq., orthogonality properties, asymptotic formulas, zeros, etc.), for instance by Koelink, Swarttouw, Ismail, Annaby, etal.

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$$J_{\nu+n}(z) = R_{n,\nu}(z)J_\nu(z) - R_{n-1,\nu+1}(z)J_{\nu-1}(z), \text{ for } n \in \mathbb{Z}_+,$$

where $R_{n,\nu}(z)$ are polynomials in z^{-1} known today as Lommel polynomials.

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- q-Lommel polynomials have been intensively studied in 90's by Koelink, Van Aschee, Swarttouw, and others.

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- q-Lommel polynomials satisfy the three-term recurrence

$$h_{n-1,\nu}(w;q) - (w^{-1} + w(1-q^{\nu+n}))h_{n,\nu}(w;q) + h_{n+1,\nu}(w;q) = 0,$$

with initial conditions $h_{-1,\nu}(w;q) = 0$ and $h_{0,\nu}(w;q) = 1$.

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- Thus, by the Favard's theorem, sequence $h_{n,\nu}(w;q)$ forms the orthogonal polynomial sequence in q^{ν}. We use slightly different parametrization: $x := q^{\nu}$ and $a := w^{-2}$.

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- Thus, by the Favard's theorem, sequence $h_{n,\nu}(w;q)$ forms the orthogonal polynomial sequence in q^{ν}. We use slightly different parametrization: $x := q^{\nu}$ and $a := w^{-2}$.
- Hence the monic version of q-Lommel polynomials $F_n(a;q,x)$ is determined by the recurrence

$$u_{n+1} = (x - (a+1)q^{-n})u_n - aq^{-2n+1}u_{n-1}$$

  with initial setting $F_{-1}(a;q,x) = 0$ and $F_0(a;q,x) = 1$.

Proposition

The Hamburger as well as the Stieltjes moment problem associated with polynomials $F_n(a; q, x)$ is indeterminate if and only if $q < a < 1/q$.

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i) *Hamburger indeterminacy:* Based on the fact that values of $F_n(a; q, 0)$ as well as the value of the first associate polynomials at $x = 0$ can be expressed explicitly. Then one verifies the corresponding orthogonal polynomials $P_n(a; q, 0)$ and $Q_n(a; q, 0)$ of the first and second kind are both square summable iff $q < a < 1/q$.

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Orthogonality relation [Koelink99]

For $a > 0$ and $m, n \in \mathbb{Z}_+$, it holds

$$\sum_{k=1}^{\infty} \frac{{}_1\phi_1(0; aq; q, q\xi_k)}{\partial_x|_{x=\xi_k}\, {}_1\phi_1(0; aq; q, x)} F_n(a; q, \xi_k) F_m(a; q, \xi_k) = -a^n q^{-n^2} \delta_{mn}.$$

Contents

Generating function

For $|t| < \min(1, a^{-1})$, it holds

$$\sum_{n=0}^{\infty} q^{\binom{n}{2}} F_n(a; q, x)(-t)^n = \frac{1}{(1-t)(1-at)}\, {}_2\phi_2(0, q; qt, qat; q, xt).$$

Formula for the generating function

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Sketch of the proof:
By denoting the LHS of the above formula $V(t)$, one finds V fulfills first order q-difference equation

$$(1-t)(1-at)V(t) = 1 - xtV(qt)$$

with initial condition $V(0) = 1$. This can be solved explicitly by iteration and one arrives at the function on RHS of the generating formula (at least formally).

- Recall Nevanlinna functions A, B, C, and D defined by

$$A(z) = z\sum_{n=0}^{\infty} Q_n(0)Q_n(z), \qquad B(z) = -1 + z\sum_{n=0}^{\infty} Q_n(0)P_n(z),$$

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where P_n and Q_n are orthogonal polynomials of the first and second kind, respectively.

- By the Nevanlinna theorem, all measures of orthogonality μ_φ for which

$$\int_{\mathbb{R}} P_n(x)P_m(x)d\mu_\varphi(x) = \delta_{mn}, \quad m,n \in \mathbb{Z}_+,$$

are parametrized according to

$$\int_{\mathbb{R}} \frac{d\mu_\varphi(x)}{z-x} = \frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where $\varphi \in \mathcal{P} \cup \{\infty\}$ and $\mathcal{P}$ is the space of Pick functions.

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$$A(a; q, z) = \frac{zq}{1-a} \sum_{n=1}^{\infty} (-1)^{n+1} (a^{-n} - 1) q^{\binom{n}{2}} F_{n-1}(a; q, qz)$$

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- By this way (and using simple identity for q-hypergeometric series) one arrives at the formula

$$A(a; q, z) = \frac{1}{1-a} \left[{}_1\phi_1(0; aq; q, qz) - {}_1\phi_1(0; a^{-1}q; q, a^{-1}qz) \right].$$

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- Similar computation leads to formulas for B, C, and D, and the result is . . .

An explicit form of the Nevanlinna functions

There are two special functions arising naturally in the formulas for Nevanlinna functions:

$$\varphi_a(z) = {}_1\phi_1(0; qa; q, z) \quad \text{and} \quad \psi_a(z) = {}_1\phi_1(0; qa^{-1}; q, a^{-1}z)$$

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Theorem

Let $1 \neq a \in (q, q^{-1})$. Then the entire functions from the Nevanlinna parametrization are as follows:

$$A(a, q; z) = \frac{1}{1-a}\left[\varphi_a(qz) - \psi_a(qz)\right], \quad B(a, q; z) = \frac{1}{1-a}\left[a\psi_a(z) - \varphi_a(z)\right],$$

$$C(a, q; z) = \frac{1}{1-a}\left[\psi_a(qz) - a\varphi_a(qz)\right], \quad D(a, q; z) = \frac{a}{1-a}\left[\varphi_a(z) - \psi_a(z)\right].$$

An explicit form of the Nevanlinna functions in the case $a = 1$

For the sake of completeness we present Nevanlinna functions in the special case with $a = 1$ in terms of functions:

$$\varphi_1(z) = {}_1\phi_1(0; q; q, z) \quad \text{and} \quad \chi_1(z) = \frac{\partial}{\partial p}\bigg|_{p=q} {}_1\phi_1(0; p; q, z)$$

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Theorem

For $a = 1$, Nevanlinna functions take the form

$$A(1, q; z) = -2q\,\chi_1(qz) - z\,\frac{\partial}{\partial z}\,\varphi_1(qz), \quad B(1, q; z) = 2q\,\chi_1(z) + z^2\frac{\partial}{\partial z}\left(z^{-1}\varphi_1(z)\right),$$

$$C(1, q; z) = 2q\,\chi_1(qz) + \frac{\partial}{\partial z}\left(z\varphi_1(qz)\right), \quad D(1; q, z) = -2q\,\chi_1(z) - z\frac{\partial}{\partial z}\,\varphi_1(z).$$

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- Recall N-extremal measures μ_t correspond to the choice
$$\varphi = t, \quad t \in \mathbb{R} \cup \{\infty\},$$
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Theorem

Let $1 \neq a \in (q, q^{-1})$. Then all N-extremal measures $\mu_t = \mu_t(a, q)$ are of the form

$$\mu_t = \sum_{x \in \mathfrak{Z}_t} \rho(x)\, \delta_x \ \text{ where } \ \frac{1}{\rho(x)} = \frac{a}{1-a}\big(\psi_a(x)\varphi_a'(x) - \varphi_a(x)\psi_a'(x)\big),$$

$$\mathfrak{Z}_t = \mathfrak{Z}_t(a, q) = \{x \in \mathbb{R} \mid a(t+1)\psi_a(x) - (t+a)\varphi_a(x) = 0\},$$

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- By using identity (which is $AD - BC = 1$)

$$\varphi_a(z)\psi_a(qz) - a\psi_a(z)\varphi_a(qz) = 1 - a, \qquad a \neq 1,$$

one finds the measure derived by Koelink is μ_{-1}, and the orthogonality relation reads

$$\sum_{k=1}^{\infty} \frac{\varphi_a(q\xi_k)}{\varphi_a'(\xi_k)} F_n(a; q, \xi_k) F_m(a; q, \xi_k) = -a^n q^{-n^2} \delta_{mn}$$

where $\{\xi_k \mid k \in \mathbb{N}\}$ are all zeros of the function φ_a.

Contents

- Coefficients from the three-term recurrence for q-Lommel polynomials define two-parameter family of real symmetric Jacobi matrices

$$\mathcal{T} \equiv \mathcal{T}(a;q) = \begin{pmatrix} \beta_0 & \alpha_0 & & & \\ \alpha_0 & \beta_1 & \alpha_1 & & \\ & \alpha_1 & \beta_2 & \alpha_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where

$$\alpha_n = \sqrt{a}q^{-n-1/2}, \quad \beta_n = (1+a)q^{-n}$$

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- The operator $T_{\min}$ is self-adjoint if and only if $0 < a \notin (q, q^{-1})$.

Jacobi matrix related with q-Lommel polynomials

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- With $\mathcal{T}$ we associate the pair of unbounded Jacobi operators $T_{\min}$ and $T_{\max}$ (by usual construction).
- The operator $T_{\min}$ is self-adjoint if and only if $0 < a \notin (q, q^{-1})$.
- If $a \in (q, q^{1})$ then $T_{\min}$ has deficiency indices $(1, 1)$. All mutually different self-adjoint extensions of $T_{\min}$ are parametrized by $\kappa \in P^1(\mathbb{R}) \equiv \mathbb{R} \cup \{\infty\}$.

- Coefficients from the three-term recurrence for q-Lommel polynomials define two-parameter family of real symmetric Jacobi matrices

$$\mathcal{T} \equiv \mathcal{T}(a;q) = \begin{pmatrix} \beta_0 & \alpha_0 & & & \\ \alpha_0 & \beta_1 & \alpha_1 & & \\ & \alpha_1 & \beta_2 & \alpha_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

  where

$$\alpha_n = \sqrt{a}q^{-n-1/2}, \quad \beta_n = (1+a)q^{-n}$$

- With $\mathcal{T}$ we associate the pair of unbounded Jacobi operators $T_{\min}$ and $T_{\max}$ (by usual construction).
- The operator $T_{\min}$ is self-adjoint if and only if $0 < a \notin (q, q^{-1})$.
- If $a \in (q, q^{1})$ then $T_{\min}$ has deficiency indices $(1,1)$. All mutually different self-adjoint extensions of $T_{\min}$ are parametrized by $\kappa \in P^1(\mathbb{R}) \equiv \mathbb{R} \cup \{\infty\}$.
- Previous results provide us with an explicit description of spectral properties of Jacobi operators associated with $\mathcal{T}$ (in terms of special functions).

Recall the previous notation:

$$\varphi_a(z) = {}_1\phi_1(0; qa; q, z) \quad \text{and} \quad \psi_a(z) = {}_1\phi_1(0; qa^{-1}; q, a^{-1}z).$$

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Theorem

The set of zeros of φ_a coincide with the spectrum of T, provided $a \notin (q, q^{-1})$, or T^F provided $1 \neq a \in (q, q^{-1})$. The components of a corresponding eigenvector can be chosen as

$$u_k(x) = a^{k/2}\varphi_a(q^{k+1}x), \quad k \in \mathbb{Z}_+.$$

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$$u_k(\kappa, x) = q^{k/2}\left(\kappa a^{k/2}\varphi_a(q^{k+1}x) + a^{-k/2}\psi_a(q^{k+1}x)\right), \quad k \in \mathbb{Z}_+.$$

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An immediate consequence is the orthogonality relation [Koelink&Swarttouw94]

$$\sum_{k=0}^{\infty} q^k J_\nu(q^{(k+1)/2}w_m; q)\, J_\nu(q^{(k+1)/2}w_n; q) = -\frac{q^{-1+\nu/2}}{2w_n} J_\nu(q^{1/2}w_n; q)\, \frac{\partial J_\nu(w_n; q)}{\partial z}\, \delta_{m,n}$$

where $0 < w_1 < w_2 < w_3 < \ldots$ are positive zeros of $J_\nu(z; q)$, $\nu > -1$, $m, n \in \mathbb{N}$.

References:

- F. Štampach, P. Šťovíček: *The Hahn-Exton q-Bessel function as the characteristic function of a Jacobi matrix*, `arXiv:1404.7647 [math.SP]`.
- F. Štampach, P. Šťovíček: *The Nevanlinna parametrization for q-Lommel polynomials in the indeterminate case*, `arXiv:1407.0217 [math.SP]`.

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Thank you!