The Moment Problem

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Seminar talk - analysis group

November 2, 2016

Outline

Contents

Motivation

- Chebychev's question (1874, most likely inspired by work of I. J. Bienaymé, 1853): *If for some positive function f,*

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- Answer: In general, *no*.

Contents

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Let $I \subset \mathbb{R}$ be a closed interval. For a positive measure μ on I the nth moment is defined as

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One can restrict oneself to cases:

- $I = \mathbb{R}$ - *Hamburger* moment problem ($\mathcal{M}_H$ = set of solutions)
- $I = [0, +\infty)$ - *Stieltjes* moment problem ($\mathcal{M}_S$ = set of solutions)
- $I = [0, 1]$ - *Hausdorff* moment problem

Hausdorff moment problem

Theorem (Hausdorff, 1923)

The moment problem has a solution on $[0,1]$ iff sequence $\{s_n\}_{n\geq 0}$ is *completely monotonic*, i.e.,

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for all $k, n \in \mathbb{Z}_+$, where $(\Delta s)_n = s_{n+1} - s_n$.

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Further, we will discuss the Stieltjes and Hamburger moment problem only...

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- For $\{s_n\}_{n\geq 0}$, we denote $H_N(s)$ the $N \times N$ Hankel matrix with entries
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- Define two sesquilinear forms H_N and S_N on $\mathbb{C}^N$ by
$$H_N(x,y) := \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \overline{x_i} y_j s_{i+j} \quad \text{and} \quad S_N(x,y) := \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \overline{x_i} y_j s_{i+j+1}.$$

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- Let $\mu \in \mathcal{M}_H$ or $\mu \in \mathcal{M}_S$ with infinite support. By observing that
$$H_N(y,y) = \int \left| \sum_{i=0}^{N-1} y_i x^i \right|^2 d\mu(x) \quad \text{and} \quad S_N(y,y) = \int x \left| \sum_{i=0}^{N-1} y_i x^i \right|^2 d\mu(x),$$
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Necessary condition for the existence:

A necessary condition for the Hamburger moment problem to have a solution (with infinite support) is the sesquilinear form H_N is PD for all $N \in \mathbb{Z}_+$. A necessary condition for the Stieltjes moment problem to have a solution (with infinite support) is both sesquilinear forms H_N and S_N are PD for all $N \in \mathbb{Z}_+$.

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$$P(x) = \sum_{k=0}^{N-1} a_k x^k, \quad \text{and} \quad Q(x) = \sum_{k=0}^{N-1} b_k x^k,$$

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- In particular,

$$\langle 1, A^n 1 \rangle = \langle 1, x^n \rangle = s_n, \quad n \in \mathbb{N}_0.$$

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- Especially, for $f(x) = x^n$, one finds

$$s_n = \langle 1, A^n 1\rangle = \langle 1, (A')^n 1\rangle = \int_{\mathbb{R}} x^n d\mu(x).$$

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Theorem (existence):

i) A necessary and sufficient condition for $\mathcal{M}_H \neq \emptyset$ (with infinite support) is

$$\det H_N(s) > 0 \quad \forall N \in \mathbb{N}.$$

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- Historically, this result has not been obtained by using the spectral theorem that was invented later.

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- In one direction, it is not clear that distinct self-adjoint extensions A_1' and A_2' give rise to distinct measures μ_1 and μ_2.

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- It is not easy to prove the theorem.
- In one direction, it is not clear that distinct self-adjoint extensions A_1' and A_2' give rise to distinct measures μ_1 and μ_2.
- The other direction is even less clear. For not only is it not obvious, it is **false** that every solution of the moment problem arise from some measure given by spectral measure of some self-adjoint extension.

Uniqueness

- In view of the connection of the moment problem and self-adjoint extensions, the following result is reasonable.

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- A solution of the moment problem which comes from a self-adjoint extension of A is called *N-extremal* solution (von Neumann [Simon], extremal [Shohat–Tamarkin]).

Contents

Jacobi matrix and Orthogonal Polynomials

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- By construction, P_n is a polynomial of degree n with real coefficients and

$$\langle P_m, P_n \rangle = \delta_{mn}, \quad \forall m, n \in \mathbb{N}_0.$$

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- $\{P_n\}_{n=0}^{\infty}$ are determined by moment sequence $\{s_n\}_{s=0}^{\infty}$,

$$P_n(x) = \frac{1}{\sqrt{\det[H_{n+1}(s)H_n(s)]}} \begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \dots & s_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}.$$

- Since

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- And A has, in the basis $\{P_n\}_{n=0}^{\infty}$, a symmetric tridiagonal matrix representation.

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- Consequently, we obtained the following correspondences:

$$\begin{array}{ccc} \text{moment sequence} & \leftrightarrow & \text{Jacobi matrix} \\ \updownarrow & & \updownarrow \\ \text{Orthogonal Polynomials} & \leftrightarrow & \text{three-term recurrence} \end{array}$$

Contents

Sufficient conditions for determinacy - moment sequence

It is desirable to be able to decide whether the moment problem is determinate (or indeterminate) just by looking at the moment sequence $\{s_n\}_{n=0}^{\infty}$, or the Jacobi matrix (seq. $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$), or orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$.

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Theorem (Carleman, 1922, 1926):

If

$$1)\quad \sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{|s_{2n}|}} = \infty \qquad \text{or} \qquad 2)\quad \sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$$

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- Hence, e.g., if $\{a_n\}_{n=0}^{\infty}$ is bounded or there are $R, C > 0$ such that

  $$|s_n| \leq CR^n n!,$$

  for all n sufficiently large, we have determinate Hamburger m.p. If

  $$|s_n| \leq CR^n (2n)!,$$

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Sufficient conditions for determinacy - Jacobi matrix

Theorem (Chihara, 1989):

Let

$$\lim_{n\to\infty} b_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{a_n^2}{b_n b_{n+1}} = L < \frac{1}{4}.$$

then the Hamburger moment problem is determinate if

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- Chihara uses totally different approach to the problem - concept of chain sequences.

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- Recall $\{P_n\}_{n=0}^{\infty}$ are determined by the three-term recurrence

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The solutions of the Hamburger moment problem in the indeterminate case are parametrized via homeomorphism $\phi \mapsto \mu_\phi$ of $\mathcal{P} \cup \{\infty\}$ onto $\mathcal{M}_H$ given by

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- A, B, C, D are called *Nevanlinna functions* and $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ the *Nevanlinna matrix*.

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$$\int_{\mathbb{R}} \frac{d\mu_\phi(x)}{x-z} = -\frac{A(z)\phi(z) - C(z)}{B(z)\phi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where A, B, C, D are certain entire function determined by the problem (i.e., the moment sequence, or orthogonal polynomials, ...).

- A, B, C, D are called *Nevanlinna functions* and $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ the *Nevanlinna matrix*.
- The solution μ_ϕ can be then expressed by using Stiltjes-Perron inversion formula.

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An example due to Stieltjes

- Note first that, for $k \in \mathbb{Z}_+$,

$$\int_0^\infty u^k u^{-\ln u} \sin(2\pi \ln u) du = 0.$$

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- Hence polynomials are not dense in $L^2(d\mu_\vartheta)$. This is a typical situation for solutions of indeterminate moment problems which are not N-extremal.

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More on A,B,C,D:

- A,B,C,D are entire functions of order ≤ 1, if the order is 1, the exponential type is 0 [Riesz, 1923]
- A,B,C,D have the same order, type and Phragmén-Lindenlöf indicator function [Berg and Pedersen, 1994]

Important solutions 1/2

- If $\phi(z) = t \in \mathbb{R} \cup \{\infty\}$ then $\phi \in \mathcal{P} \cup \{\infty\}$ and μ_t is a discrete measure of the form

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- N-extremal solutions are indeed extreme points in $\mathcal{M}_H$ - but not the only ones.

Important solutions 2/2

- If we set

$$\phi(z) = \begin{cases} \beta + i\gamma, & \Im z > 0, \\ \beta - i\gamma, & \Im z < 0, \end{cases}$$

for $\beta \in \mathbb{R}$ and $\gamma > 0$, then $\phi \in \mathcal{P}$ and $\mu_{\beta,\gamma}$ is absolutely continuous with density

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- The solution $\mu_{0,1}$ is the one that maximizes certain entropy integral, (see Krein's condition). More general and additional information are provided in [Gabardo, 1992].

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- Suppose $\{s_n\}_{n=0}^{\infty}$ is a sequence of Stieltjes moments such that the moment problem is indeterminate in the sense of Hamburger.

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- For the indeterminate Stieljes moment problem there is a slightly more elegant way how to describe $\mathcal{M}_S$ known as *Krein parametrization*, [Krein, 1967].

References:

1. J. A. Shohat, J. D. Tamarkin, *The Problem of Moments*, Math. Surveys, vol. 1, AMS, New York, 1943.
2. N. I. Akhiezer: *The Classical Moment Problem and Some Related Questions in Analysis*, Oliver & Boyd, Edinburgh, 1965.
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