On discrete Hardy inequalities

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Contents

The classical Hardy inequalities

The classical p-Hardy inequalities ($p > 1$):

1) Discrete:

$$\forall v \in C_c(\mathbb{N}) \qquad \boxed{\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} v_k \right|^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} |v_n|^p}$$

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$$\forall \varphi \in C_c^{\infty}(0,\infty) \qquad \boxed{\int_0^{\infty}\left|\frac{1}{x}\int_0^{x}\varphi(t)\mathrm{d}t\right|^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_0^{\infty}|\varphi(x)|^p \mathrm{d}x}$$

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- Credited to G. H. Hardy.
- But many other mathematicians contributed (E. Landau, G. Pólya, I. Schur, M. Riesz,...).

The (pre)history of Hardy inequalities

- Hardy's original motivation was to find a "simple" proof of the *weak* form the ℓ^2-Hilbert inequality:

$$\forall v \in \ell^2(\mathbb{N}) \qquad \boxed{\sum_{m,n=1}^{\infty} \frac{\overline{v_m} v_n}{m+n} \leq \pi \sum_{n=1}^{\infty} |v_n|^2}$$

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- A fifth simple proof introduced by Hardy in 1920 using the *weak* form of the ℓ^2-Hardy inequality

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} v_k \right|^2 \leq 4 \sum_{n=1}^{\infty} |v_n|^2$$

  appearing (almost, implicitly) in his paper in 1919 (the convergence already in 1915).

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...from now on, we consider only $p = 2$.

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- Thus, assuming the discrete Hardy inequality:

$$\sum_{n=1}^{\infty}(v_n - v_{n-1})^2 \geq \frac{1}{4}\sum_{n=1}^{\infty}\frac{v_n^2}{n^2},$$

we deduce the continuous one.

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- The situation on the discrete side is different!

The optimal discrete Hardy weight

Theorem [Keller-Pinchover-Pogorzelski, 2018]

For all $u \in C_c(\mathbb{N})$, $u_0 := 0$, one has

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}|^2 \geq \sum_{n=1}^{\infty} w_n^{\text{KPP}} |u_n|^2$$

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- The claim is a particular case of a more abstract result formulated in the graph setting [Keller-Pinchover-Pogorzelski, 2018].

The remainder term in the optimal discrete Hardy inequality

Theorem [Krejčiřík-Š., 2022]

For all $u \in C_c(\mathbb{N})$, $u_0 = 0$, we have the identity

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- Solution:

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$$\sum_{n=1}^{\infty} |u_n - u_{n-1}|^2 \geq \sum_{n=1}^{\infty} w_n |u_n|^2$$
and subtract
$$\sum_{n=1}^{\infty} |u_{n-1} - u_n|^2 = \sum_{n=1}^{\infty} w_n^{\text{KPP}} |u_n|^2 + \sum_{n=1}^{\infty} \left| \sqrt[4]{\frac{n+1}{n}} u_n - \sqrt[4]{\frac{n}{n+1}} u_{n+1} \right|^2.$$
- It yields the inequality
$$0 \geq \sum_{n=1}^{\infty} (w_n - w_n^{\text{KPP}}) |u_n|^2 - \sum_{n=1}^{\infty} \left| \sqrt[4]{\frac{n+1}{n}} u_n - \sqrt[4]{\frac{n}{n+1}} u_{n+1} \right|^2.$$
- By setting $u_n := \sqrt{n}$, we get
$$0 \geq \sum_{n=1}^{\infty} (w_n - w_n^{\text{KPP}}) n.$$
- If moreover $w \geq w^{\text{KPP}}$, it follows $w = w^{\text{KPP}}$.
- But beware the cheating $u \notin C_c(\mathbb{N})$! $\rightsquigarrow$ A suitable regularization solves the issue.

A generalization of the identity

Theorem [Laptev-Krejčiřík-Š., 2022]

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If $g > 0$ is a sequence such that $-\Delta g \geq 0$, then, for any $u \in C_c(\mathbb{N})$ with $u_0 = 0$, we have

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Moreover, if there exists $\xi^N \in C_c(\mathbb{N})$ such that $\xi^N \leq \xi^{N+1}$, $\xi^N \to 1$ as $N \to \infty$, and

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Remark: The identity can be further generalized in at least two respects [Huang-Ye].

Application: more critical Hardy weights

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- If $g_n := n^q$, for $0 < q \leq 1/2$, in the previous theorem, we get the inequality

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}|^2 \geq \sum_{n=1}^{\infty} w_n(q)|u_n|^2$$

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(But $w(1/2)$ has the heaviest tail - is optimal at infinity - which $w(q)$ are not.)

Application: spectral stability of the discrete Laplacian on $\mathbb{N}$

Recall

$$\sigma(-\Delta) = \sigma_{\mathrm{ess}}(-\Delta) = \sigma_{\mathrm{ac}}(-\Delta) = [0, 4].$$

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To get concrete conditions on the potential v, we can, for example, chose

$$w_n = 2 - \left(1 - \frac{1}{n}\right)^q - \left(1 + \frac{1}{n}\right)^q, \quad 0 < q \leq \frac{1}{2}.$$

Contents

The improved discrete Rellich inequality on $\mathbb{N}$

The continuous Rellich inequality [Rellich, 1954-56]:

$$\int_0^\infty \left|\varphi''(x)\right|^2 \mathrm{d}x \geq \frac{9}{16}\int_0^\infty \frac{|\varphi(x)|^2}{x^4}\mathrm{d}x,$$

for $\varphi \in C_c^\infty(0,\infty)$ (or $\varphi \in H^2(0,\infty)$, $\varphi(0) = \varphi'(0) = 0$). Constant $\frac{9}{16}$ is the best possible.

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Further remarks to the improved discrete Rellich inequality

- In the proof, the factorization method is employed to derive the identity

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- Recently improved by [Huang-Ye]:

$$w_n^{\mathrm{HY}}=\frac{1}{4n^2}\left[1+\left(1-\frac{1}{n}\right)^{-2}-\left(1-\frac{1}{n}\right)^{-1/2}-\left(1+\frac{1}{n}\right)^{3/2}\right]=\frac{9}{16n^4}+\frac{15}{16n^5}+\dots.$$

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What is a critical discrete Rellich weight?

Open Problem

Find a critical discrete Rellich weight.

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The inequality [Birman 1961, Glazman 1965, Owen 1999, Gesztesy-etal. 2018]

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Contents

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1) If $d = 1, 2$, there is no Hardy inequality for $-\Delta$, i.e.,

$$-\Delta \geq V \geq 0 \quad \Longrightarrow \quad V = 0.$$

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Classical (continuous) Hardy inequalities on $\mathbb{R}^d$

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Remark: In $\mathbb{R}^2$, we have the inequality

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Discrete Laplacian on $\mathbb{Z}^d$:

$$(\Delta u)_n := \sum_{j=1}^{d} (D_j^* D_j u)_n = \sum_{j=1}^{d} (u_{n-e_j} - 2u_n + u_{n+e_j})$$

where $u \in \ell^2(\mathbb{Z}^d)$, $n \in \mathbb{Z}^d$, e_j is the jth vector of standard basis of $\mathbb{R}^d$, and $(D_j u)_n := u_{n-e_j} - u_n$.

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for all $u \in C_c(\{|n| \geq 2\})$. It was recently shown [Huang-Ye] that

$$\sum_{n\in\mathbb{Z}^2} \overline{u}_n(-\Delta u)_n \geq \sum_{n\in\mathbb{Z}^2} w_n|u_n|^2,$$

with

$$w_n = \frac{1}{4|n|^2 \log^2 |n|} + \frac{48(n_1^4 + n_2^4)}{|n|^4} - \frac{36}{|n|^4|log|n|} + O\left(\frac{1}{|n|^4 \log^2 |n|}\right), \quad |n| \to \infty.$$

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- Lower bound [Kapitanski-Laptev, 2016]:

$$C_d \geq \frac{4(d-2)}{\pi^2} \frac{d - 2\pi\sqrt{2d-4}}{d^2 - 6d + 16 - 4\sqrt{2d-4}}.$$

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Open Problem

$$C_d = ?$$

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Theorem [Keller-Pinchover-Pogorzelski, 2018]

Let $d \geq 3$. There exists a critical (even optimal) Hardy weight w on $\mathbb{Z}^d$ such that

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$$w_n = \frac{(d-2)^2}{4} \frac{1}{|n|^2} + O\left(\frac{1}{|n|^3}\right), \quad |n| \to \infty.$$

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- Weight w is not known explicitly but can be constructed from the Green kernel of $-\Delta$. (There is even a generalization to a graph setting [Keller-Pinchover-Pogorzelski, 2018].)
- An alternative proof (not using the Green kernel), a weighted generalization of the result, and more detailed asymptotics due to [Huang-Ye],

$$w_n = \frac{(d-2)^2}{4}\frac{1}{|n|^2} + A_d\frac{1}{|n|^4} + O\left(\frac{1}{|n|^6}\right), \quad |n| \to \infty.$$

  with an explicit constant A_d.

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(Even more mysteries in: weighted generalizations, graph setting, ℓ^p-variants, etc.)

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Thank you!