On the Eigenvalue Problem for a Certain Class of Jacobi Matrices

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Definition

Let me define $\mathfrak{F} : D \to \mathbb{C}$ by relation

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1},$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables let me identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$.

- $\mathfrak{F}$ is well defined on D due to estimation

$$|\mathfrak{F}(x)| \leq \exp\left(\sum_{k=1}^{\infty} |x_k x_{k+1}| \right).$$

- Note that the domain D is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$.

- For all $x \in D$ and $k = 1, 2, \dots$ one has

Recursive relation

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \dots, x_k)\, \mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \dots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(T^{k+1} x)$$

where T denotes the truncation operator from the left defined on the space of all sequences:

$$T(\{x_k\}_{k=1}^{\infty}) = \{x_{k+1}\}_{k=1}^{\infty}.$$

- Especially for $k = 1$, one gets the simple relation

$$\mathfrak{F}(x) = \mathfrak{F}(Tx) - x_1 x_2\, \mathfrak{F}(T^2 x).$$

- Moreover, for x finite the relation has the form

$$\mathfrak{F}(x_1, x_2, x_3, \dots, x_n) = \mathfrak{F}(x_2, x_3, \dots, x_n) - x_1 x_2\, \mathfrak{F}(x_3, \dots, x_n).$$

- Functions $\mathfrak{F}$ restricted on $\ell^2(\mathbb{N})$ is a continuous functional on $\ell^2(\mathbb{N})$. Further, for $x \in D$, it holds

$$\lim_{n \to \infty} \mathfrak{F}(x_1, x_2, \dots, x_n) = \mathfrak{F}(x) \qquad \text{and} \qquad \lim_{n \to \infty} \mathfrak{F}(T^n x) = 1.$$

- Initial values $\mathfrak{F}(\emptyset) = \mathfrak{F}(x_1) = 1$ together with relation

$$\mathfrak{F}(x_1, \dots, x_{n-1}, x_n) = \mathfrak{F}(x_1, \dots, x_{n-2}, x_{n-1}) - x_{n-1}x_n\,\mathfrak{F}(x_1, \dots, x_{n-3}, x_{n-2})$$

  determine recursively and unambiguously $\mathfrak{F}(x_1, \dots, x_n)$ for any finite number of variables.

- Other equivalent definitions of $\mathfrak{F}(x_1, x_2, \dots, x_n)$ were found:

$$\mathfrak{F}(x_1, x_2, \dots, x_n) = \det X_n = \det \begin{pmatrix} 1 & x_1 & & & \\ x_2 & 1 & x_2 & & \\ & \ddots & \ddots & \ddots & \\ & & x_{n-1} & 1 & x_{n-1} \\ & & & x_n & 1 \end{pmatrix}$$

- If $\mathfrak{F}(x_1, x_2, \dots, x_k) \neq 0$ for $k = 1, 2, \dots, n-1$ then it holds

$$\mathfrak{F}(x_1, x_2, \dots, x_n) = \prod_{k=1}^{n} (e_k, X_k^{-1} e_k)^{-1}.$$

1. The case of geometric sequence:

Let $t, w \in \mathbb{C}$, $|t| < 1$, then it holds

$$\mathfrak{F}\left(\left\{t^{k-1}w\right\}_{k=1}^{\infty}\right) = 1 + \sum_{m=1}^{\infty}(-1)^m \frac{t^{m(2m-1)}w^{2m}}{(1-t^2)(1-t^4)\ldots(1-t^{2m})}.$$

The function on the RHS can be identified with a q-hypergeometric series ${}_0\phi_1(;0;t^2,-tw^2)$ [Gasper&Rahman04].

2. The case of Bessel functions:

Let $w \in \mathbb{C}$ and $\nu \notin -\mathbb{N}$, then it holds

$$J_\nu(2w) = \frac{w^\nu}{\Gamma(\nu+1)}\mathfrak{F}\left(\left\{\frac{w}{\nu+k}\right\}_{k=1}^{\infty}\right).$$

Recursive relation for $\mathfrak{F}$ written in this special case has the form

$$wJ_{\nu-1}(2w) - \nu J_\nu(2w) + wJ_{\nu+1}(2w) = 0.$$

The function $\mathfrak{F}$ is related to various fields of mathematics:

- **the theory of Orthogonal Polynomials** [Akhiezer, Chihara, Ismail]
- the theory of Continued Fractions
- the eigenvalue problem for certain class of Jacobi matrices

For $\lambda_n \in \mathbb{R}$ and $w_n > 0$, OPs can be defined recursively by

$$w_{n-1}y_{n-1}(x) + \lambda_n y_n(x) + w_n y_{n+1}(x) = xy_n(x), \quad n = 1, 2, \ldots$$

and OPs of the first kind $P_n(x)$ satisfy initial conditions $P_0(x) = 1$, $P_1(x) = (x - \lambda_1)/w_1$, while OPs of the second kind $Q_n(x)$ satisfy $Q_0(x) = 0$, $Q_1(x) = 1/w_1$. OPs are related to $\mathfrak{F}$ through identities

$$P_n(z) = \prod_{k=1}^{n} \left(\frac{z - \lambda_k}{w_k} \right) \mathfrak{F} \left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=1}^{n} \right), \quad n = 0, 1 \ldots,$$

$$Q_n(z) = \frac{1}{w_1} \prod_{k=2}^{n} \left(\frac{z - \lambda_k}{w_k} \right) \mathfrak{F} \left(\left\{ \frac{\gamma_l^2}{\lambda_l - z} \right\}_{l=2}^{n} \right), \quad n = 0, 1 \ldots$$

where the sequence $\{\gamma_n\}$ can be defined recursively as $\gamma_1 = 1$, $\gamma_{k+1} = w_k/\gamma_k$.

The function $\mathfrak{F}$ is concerned with various fields of mathematics:

- the theory of Orthogonal Polynomials
- **the theory of Continued Fractions** [Teschl, Ifantis, Stieltjes]
- the eigenvalue problem for certain class of Jacobi matrices

Function $\mathfrak{F}$ is related to a continued fraction. For a given $x \in D$ such that $\mathfrak{F}(x) \neq 0$, it holds

$$\frac{\mathfrak{F}(Tx)}{\mathfrak{F}(x)} = \cfrac{1}{1 - \cfrac{x_1 x_2}{1 - \cfrac{x_2 x_3}{1 - \cfrac{x_3 x_4}{1 - \dots}}}}.$$

Example:

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \cfrac{z}{2(\nu+1) - \cfrac{z^2}{2(\nu+2) - \cfrac{z^2}{2(\nu+3) - \cfrac{z^2}{2(\nu+4) - \dots}}}}.$$

The function $\mathfrak{F}$ is concerned with various fields of mathematics:

- the theory of Orthogonal Polynomials
- the theory of Continued Fractions
- **the eigenvalue problem for certain class of Jacobi matrices**

- Let us denote

$$\mathcal{J} := \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where $w \equiv \{w_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$ and $\lambda \equiv \{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$.

- Let $J_0 x := \mathcal{J} x$ with $x \in \operatorname{span}\{e_1, e_2, \dots\} =: \operatorname{Dom}(J_0)$ and $J := \overline{J_0}$.
- Let J_n be the n-th truncation of $\mathcal{J}$,

$$J_n = \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & \ddots & \ddots & \ddots & \\ & & w_{n-2} & \lambda_{n-1} & w_{n-1} \\ & & & w_{n-1} & \lambda_n \end{pmatrix}.$$

The characteristic function of a finite complex Jacobi matrix can be expressed in terms of $\mathfrak{F}$:

Proposition

Let $n \in \mathbb{N}$ a $z \in \mathbb{C}$, then it holds

$$\det(J_n - zI_n) = \left(\prod_{k=1}^{n}(\lambda_n - z)\right)\mathfrak{F}\left(\frac{\gamma_1^2}{\lambda_1 - z}, \frac{\gamma_2^2}{\lambda_2 - z}, \ldots, \frac{\gamma_n^2}{\lambda_n - z}\right)$$

where the sequence $\{\gamma_n\}$ can be defined recursively as $\gamma_1 = 1$, $\gamma_{k+1} = w_k/\gamma_k$.

- The proof is based on the decomposition

$$J_n = G_n \tilde{J}_n G_n$$

where $G_n = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a diagonal matrix and

$$\tilde{J}_n = \begin{pmatrix} \tilde{\lambda}_1 & 1 & & & \\ 1 & \tilde{\lambda}_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \tilde{\lambda}_{n-1} & 1 \\ & & & 1 & \tilde{\lambda}_n \end{pmatrix}$$

with $\tilde{\lambda}_k = \lambda_k/\gamma_k^2$.

In the rest, let me suppose:

- the set of all accumulation points $\operatorname{der}(\lambda)$ of the sequence $\lambda \equiv \{\lambda_n\}$ is finite.
- Let for at least one $z \in \mathbb{C} \setminus \overline{\lambda}$ it holds

$$\sum_{n=1}^{\infty} \left| \frac{w_n^2}{(\lambda_n - z)(\lambda_{n+1} - z)} \right| < \infty.$$

(Then it holds for all $z \in \mathbb{C} \setminus \overline{\lambda}$ and the convergence of the sum is local uniform on $\mathbb{C} \setminus \overline{\lambda}$.)

Under these assumptions, the function

$$F_J(z) := \mathfrak{F}\left(\left\{ \frac{\gamma_n^2}{\lambda_n - z} \right\}_{n=1}^{\infty}\right)$$

is well defined on $\mathbb{C} \setminus \overline{\lambda}$ and is an analytic function on $\mathbb{C} \setminus \overline{\lambda}$ with poles in points $z \in \lambda \setminus \operatorname{der}(\lambda)$ of finite order less or equal to $r_z = \sum_{k=1}^{\infty} \delta_{(z,\lambda_k)}$.

- Further, we slightly extend the definition of $F_J(z)$. For $\xi \in \mathbb{C} \setminus \mathrm{der}(\lambda)$ and $k \in \mathbb{N}_0$ let us define

$$F^{\xi}_{J,k}(z) := \begin{cases} (z-\xi)^{r_\xi}\,\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - z}\right\}_{n=k+1}^{\infty}\right), & \text{if } z \neq \xi \\ \lim_{z\to\xi}(z-\xi)^{r_\xi}\,\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - z}\right\}_{n=k+1}^{\infty}\right), & \text{if } z = \xi \end{cases}$$

where

$$r_\xi = \sum_{k=1}^{\infty} \delta_{(\lambda_k,\xi)} \in \mathbb{N}_0.$$

- If $k = 0$ we write $F^{\xi}_J(z)$ instead of $F^{\xi}_{J,0}(z)$.
- If $\xi \notin \lambda$ then $F^{\xi}_J(z) \equiv F_J(z)$.
- Function $F^{z}_J(z)$ is well defined on $\mathbb{C} \setminus \mathrm{der}(\lambda)$.
- Let us denote

$$\mathfrak{Z}(\mathcal{J}) := \{z \in \mathbb{C} : F^{z}_J(z) = 0\}.$$

Proposition

If $F_J^z(z) = 0$ for some $z \in \mathbb{C} \setminus \operatorname{der}(\lambda)$, then z is an eigenvalue of J and vector $\xi(z) \equiv \{\xi_k(z)\}_{k=1}^{\infty}$, where

$$\xi_k(z) := \prod_{l=1}^{k} \left(\frac{w_{l-1}}{z - \lambda_l} \right) F_{J,k}^z(z), \qquad (w_0 := 1)$$

is the respective eigenvector.

- Hence the inclusion

$$\mathfrak{Z}(\mathcal{J}) \setminus \operatorname{der}(\lambda) \subset \operatorname{spec}_p(J) \setminus \operatorname{der}(\lambda)$$

  holds.
- Moreover, for $z \notin \operatorname{der}(\lambda)$, vector $\xi(z)$ satisfies the formula

$$\sum_{k=1}^{\infty} (\xi_k(z))^2 = \xi_0'(z)\xi_1(z) - \xi_0(z)\xi_1'(z).$$

- Consequently, if λ and w are real sequences and $z \in \operatorname{spec}_p(J) \setminus \operatorname{der}(\lambda)$ then

$$\|\xi(z)\|^2 = \xi_0'(z)\xi_1(z).$$

Proposition

Let $\rho(J) \neq \emptyset$. If $F_J^z(z) \neq 0$, for some $z \notin \mathrm{der}(\lambda)$, then $z \in \rho(J)$.

- Consequently, one has

$$\mathrm{spec}_p(J) \setminus \mathrm{der}(\lambda) = \mathrm{spec}(J) \setminus \mathrm{der}(\lambda) = \mathfrak{Z}(\mathcal{J}) \setminus \mathrm{der}(\lambda).$$

Corollary

Let $\rho(J) \neq \emptyset$. Then $\mathrm{spec}_p(J) \setminus \mathrm{der}(\lambda)$ contains only simple isolated eigenvalues.

The Green function $G(m,n;z) := (e_m,(J-z)^{-1}e_n)$, $m,n\in\mathbb{N}$, and especially the Weyl m-function $m(z) := G(1,1;z)$ is also expressible in terms of $\mathfrak{F}$.

- Let $\rho(J)\neq\emptyset$ then, for $i,j\in\mathbb{N}$, one has

$$G(i,j;z) = -\frac{1}{w_M}\prod_{l=m}^{M}\left(\frac{w_l}{z-\lambda_l}\right)\frac{\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=1}^{m-1}\right)\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=M+1}^{\infty}\right)}{\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-z}\right\}_{l=1}^{\infty}\right)}$$

  where $m := \min(i,j)$ and $M := \max(i,j)$.
- Especially, for the Weyl m-function, one gets the relation

$$m(z) = \frac{\mathfrak{F}\left(\left\{\frac{\gamma_j^2}{\lambda_j-z}\right\}_{j=2}^{\infty}\right)}{(\lambda_1-z)\mathfrak{F}\left(\left\{\frac{\gamma_j^2}{\lambda_j-z}\right\}_{j=1}^{\infty}\right)} = \cfrac{1}{z-\lambda_1-\cfrac{w_1^2}{z-\lambda_2-\cfrac{w_2^2}{z-\lambda_3-\cdots}}}.$$

Example 1 (unbounded operator)

- Let $\lambda_n = \alpha n$, $\alpha \neq 0$ and $w_n = w \neq 0$, $n = 1, 2, \ldots$. With this choice one has

$$J = \begin{pmatrix} \alpha & w & & & \\ w & 2\alpha & w & & \\ & w & 3\alpha & w & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \gamma_n = \begin{cases} 1, & \text{if } n \text{ odd} \\ w, & \text{if } n \text{ even.} \end{cases}$$

- The characteristic function can be expressed as

$$F_J(z) = \left(\frac{w}{\alpha}\right)^{\frac{z}{\alpha}} \Gamma\left(1 - \frac{z}{\alpha}\right) J_{-\frac{z}{\alpha}}\left(\frac{2w}{\alpha}\right).$$

- Since the term $(w/\alpha)^{\frac{z}{\alpha}}\Gamma(1 - z/\alpha)$ does not effect zeros of $F_J(z)$ and, moreover, the term $\Gamma(1 - z/\alpha)$ causes singularities in $z = \alpha, 2\alpha, \ldots$, one arrives at the following expression

$$\operatorname{spec}(J) = \{z \in \mathbb{C};\ J_{-\frac{z}{\alpha}}\left(\frac{2w}{\alpha}\right) = 0\},$$

and since

$$\xi_k(z) = \frac{(-1)^k}{w}\left(\frac{w}{\alpha}\right)^{\frac{z}{\alpha}} \Gamma\left(1 - \frac{z}{\alpha}\right) J_{k-\frac{z}{\alpha}}\left(\frac{2w}{\alpha}\right), \qquad (\text{for } z \notin \alpha\mathbb{N}),$$

the formula for the kth entry of the respective eigenvector is

$$v_k(z) = (-1)^k J_{k-\frac{z}{\alpha}}\left(\frac{2w}{\alpha}\right).$$

- Let $\lambda_n = 1/n$ and $w_n = 1/\sqrt{n(n+1)}$, $n = 1, 2, \ldots$. Then matrix J has the form

$$J = \begin{pmatrix} 1 & 1/\sqrt{2} & & & \\ 1/\sqrt{2} & 1/2 & 1/\sqrt{6} & & \\ & 1/\sqrt{6} & 1/3 & 1/\sqrt{12} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

- In this case one has

$$F_J(z) = \sum_{s=0}^{\infty} \frac{1}{z^s} \frac{1}{s!} \prod_{j=1}^{s} \frac{1}{1 - jz} = z^{-\frac{1}{z}} \Gamma\left(1 - \frac{1}{z}\right) J_{-\frac{1}{z}}\left(\frac{2}{z}\right).$$

By the main result, one gets

$$\mathrm{spec}(J) = \left\{ \frac{1}{z} \in \mathbb{R} : \; J_{-z}(2z) = 0 \right\} \cup \{0\}$$

and the kth entry of the respective eigenvector has the form

$$v_k(z) = \sqrt{k} J_{k-\frac{1}{z}}\left(\frac{2}{z}\right).$$

- Let $q \in (0,1)$, $\lambda_n = q^{n-1}$ and $w_n = (\sqrt{q})^{n-1}$, $n = 1, 2, \dots$. Then matrix J has the form

$$J = \begin{pmatrix} 1 & 1 & & & \\ 1 & q & \sqrt{q} & & \\ & \sqrt{q} & q^2 & q & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

- The characteristic function $F_J(z)$ can be identified with a basic hypergeometric series ${}_0\phi_1(;1/z;q,1/z^2)$ where

$${}_0\phi_1(;b;q,z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{(q;q)_k (b;q)_k} z^k.$$

and

$$(\alpha;q)_k = \prod_{j=0}^{k-1} \left(1 - \alpha q^j\right), \; k = 0, 1, 2, \dots.$$

- Hence

$$\mathrm{spec}(J) = \left\{ \frac{1}{z} \in \mathbb{R};\; {}_0\phi_1(;z;q,z^2) = 0 \right\} \cup \{0\}.$$

Example 3 (compact operator with zero diagonal)

Jacobi matrices J with zero diagonal, more precisely, matrices with $\lambda_n = 0$, $n \in \mathbb{N}$ and $w \in \ell^2(\mathbb{N})$, can be investigated in more detail. This is a special case of compact Jacobi matrices and we have

$$\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{z}\right\}_{n=1}^{\infty}\right) = \sum_{m=0}^{\infty} \frac{(-1)^m}{z^{2m}} \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} w_{k_1}^2 w_{k_2}^2 \ldots w_{k_m}^2,$$

which is the Laurent series for the function we are interested in. In the previous part we have proved

$$\operatorname{spec}(J) = \left\{z \in \mathbb{R} : \mathfrak{F}\left(\left\{\frac{\gamma_n^2}{z}\right\}_{n=1}^{\infty}\right) = 0\right\} \cup \{0\}.$$

Since the function is an even function in z the spectrum of J is symmetric with respect to 0.

- Let $\lambda_n = 0$, $w_n = \beta/\sqrt{(n+\alpha)(n+\alpha+1)}$, $\alpha > -1$, $\beta > 0$, $n = 1, 2, \ldots$. Then the results are

$$\operatorname{spec}(J) = \left\{\frac{2\beta}{z} \in \mathbb{R} : J_\alpha(z) = 0\right\} \cup \{0\},$$

$$v_k(z) = \sqrt{\alpha + k} J_{\alpha+k}\left(\frac{2\beta}{z}\right).$$

Thank you!