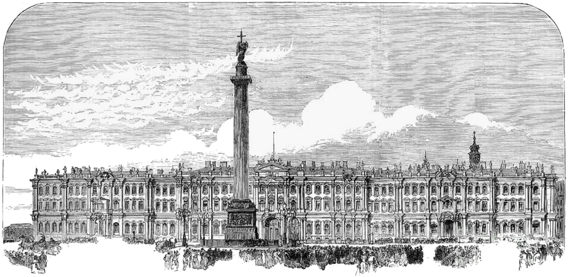
Spectral analysis of some doubly infinite Jacobi matrices via characteristic function

František Štampach



THE WINTER PALACE, ST. PETERSBURG, WHERE THE LATE EMPEROR DIED ON SUNDAY LAST.

OTAMP 2016 – tribute to Boris Pavlov & Michael Solomyak

August 3, 2016

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1 Jacobi operator

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4 Diagonals admitting global regularization and examples

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where $\lambda_n \in \mathbb{C}$ and $w_n \in \mathbb{C} \setminus \{0\}$, we associate two operators $J_{\min}$ and $J_{\max}$ acting on $\ell^2(\mathbb{Z})$.

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$$J_{\max}^* = C J_{\min} C \quad \text{and} \quad J_{\min}^* = C J_{\max} C$$

where C is the complex conjugation operator, $(Cx)_n = \overline{x_n}$.

Proper case and spectrum of Jacobi operator

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- Let $J_{\min} = J_{\max} =: J$. Then
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As a consequence,
$$\sigma_r(J) = \emptyset$$
and so
$$\sigma(J) = \sigma_p(J) \cup \sigma_c(J).$$

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Function $\mathfrak{F}$

Definition:

We define $\mathfrak{F} : \operatorname{Dom}\mathfrak{F} \to \mathbb{C}$ by

$$\mathfrak{F}\left(\{x_k\}_{k=-\infty}^{\infty}\right) := 1 + \sum_{m=1}^{\infty}(-1)^m \sum_{k_1=-\infty}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} \prod_{j=1}^{m} x_{k_j} x_{k_j+1}$$

where

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In addition, for $\{x_n\}_{n=N_1}^{N_2}$, $N_1, N_2 \in \mathbb{Z} \cup \{-\infty, +\infty\}$, $N_1 \leq N_2$, such that

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we put $x_n := 0$ for $n < N_1$ and $n > N_2$ and define

$$\mathfrak{F}\left(\{x_k\}_{k=N_1}^{N_2}\right) := \mathfrak{F}\left(\{x_k\}_{k=-\infty}^{\infty}\right).$$

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If you are interested:

1. F. Š. and P. Šťovíček, *Linear Alg. Appl.* (2011), **arXiv:1011.1241**.
2. F. Š. and P. Šťovíček: *Linear Alg. Appl.* (2013), **arXiv:1201.1743**.
3. F. Š. and P. Šťovíček: *J. Math. Anal. Appl.* (2014), **arXiv:1301.2125**.
4. F. Š. and P. Šťovíček: *Linear Alg. Appl.* (2015), **arXiv:1403.8083**.

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Characteristic function

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- This assumption determines the class of matrices $\mathcal{J}$ form which we can define the **characteristic function**:

$$F_{\mathcal{J}}(z) := \mathfrak{F}\left(\left\{ \frac{\gamma_n^2}{z - \lambda_n} \right\}_{n=-\infty}^{\infty} \right), \quad \forall z \in \mathbb{C}_0^{\lambda},$$

where $\{\gamma_n\}$ is any sequence satisfying the difference equation $\gamma_n \gamma_{n+1} = w_n$, $\forall n \in \mathbb{Z}$.

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  where $\{\gamma_n\}$ is any sequence satisfying the difference equation $\gamma_n \gamma_{n+1} = w_n$, $\forall n \in \mathbb{Z}$.
- Function $F_{\mathcal{J}}$ is well define and entire on $\mathbb{C}_0^\lambda$. Further, $F_{\mathcal{J}}$ meromorphic on $\mathbb{C} \setminus \mathrm{der}(\lambda)$ having poles of finite order (or removable singularities) at points $z = \lambda_n$.

The zero set of characteristic function

- It is not clear whether the assumption

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- Assuming, additionally, that

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Theorem:

Under the above mentioned assumptions, one has

$$\sigma(J) \cap \mathbb{C}_0^\lambda = \sigma_p(J) \cap \mathbb{C}_0^\lambda = \{z \in \mathbb{C}_0^\lambda \mid F_{\mathcal{J}}(z) = 0\}.$$

On eigenvectors and multiplicities

- For $z \in \mathbb{C}_0^{\lambda}$, put

$$f_n(z) := \left(\prod_{k=1}^{n}{}^{*} \frac{w_{k-1}}{z - \lambda_k} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=n+1}^{\infty} \right)$$

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Theorem:

i) All the eigenvalues of J have geometric multiplicity equal to one with $f(z)$ the eigenvector corresponding to the eigenvalue $z \in \mathbb{C}_0^\lambda$.
ii) Suppose, in addition, that $\mathbb{C} \setminus \operatorname{der}(\lambda)$ is connected. Then $\sigma_p(J)$ has no accumulation point in $\mathbb{C} \setminus \operatorname{der}(\lambda)$ and the algebraic multiplicity of an eigenvalue $z \in \mathbb{C}_0^\lambda$ of J coincides with the order of z as a root of $F_{\mathcal{J}}$. In this case, the space of generalized eigenvectors is spanned by

$$f(z), f'(z), \dots, f^{(m-1)}(z)$$

where m is the algebraic multiplicity of z.

Local regularization - from $\mathbb{C}_0^\lambda$ to $\mathbb{C} \setminus \operatorname{der}(\lambda)$

- All the spectral results have been restricted to the set $\mathbb{C}_0^\lambda$. Recall, for example,

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- However, using the method based on the characteristic function, it is not possible to decide whether points from $\operatorname{der}(\lambda)$ belong to $\sigma(J)$ or not.
- For the sake of brevity, we do not explain the details of this local regularization procedure in this talk. Rather we describe a global regularization of the characteristic function in 3 different cases and provide illustrating examples ...

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$$\Phi_p^+(z) := \prod_{n=1}^{\infty} \left(1 - \frac{\lambda_n}{z}\right) \exp\left(\sum_{j=1}^{p-1} \frac{1}{j} \left(\frac{\lambda_n}{z}\right)^j\right)$$
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$$\lambda \in \ell^p(\mathbb{Z}) \text{ for some } p \in \mathbb{N} \quad \text{and} \quad w \in \ell^2(\mathbb{Z}),$$

  the two main assumptions are automatically satisfied. In addition, we may introduce the functions

$$\Phi_p^+(z) := \prod_{n=1}^{\infty} \left(1 - \frac{\lambda_n}{z}\right) \exp\left(\sum_{j=1}^{p-1} \frac{1}{j} \left(\frac{\lambda_n}{z}\right)^j\right)$$

  and

$$\Phi_p(z) := \prod_{n=-\infty}^{\infty} \left(1 - \frac{\lambda_n}{z}\right) \exp\left(\sum_{j=1}^{p-1} \frac{1}{j} \left(\frac{\lambda_n}{z}\right)^j\right).$$

- Then one can get rid of all the nonzero singularities of $F_{\mathcal{J}}$ and f by putting

$$\tilde{F}_{\mathcal{J}}(z) := \Phi_p(z) F_{\mathcal{J}}(z) \quad \text{and} \quad \tilde{f}(z) := \Phi_p^+(z) f(z).$$

  Functions $\tilde{F}_{\mathcal{J}}$ and $\tilde{f}$ are entire on $\mathbb{C} \setminus \{0\}$.

I. Compact case - spectral results

Proposition:

If

$$w \in \ell^2(\mathbb{Z}) \quad \text{and} \quad \lambda \in \ell^p(\mathbb{Z}) \quad \text{for some} \quad p \geq 1,$$

then

$$\sigma(J) = \sigma_p(J) \cup \{0\} = \{z \in \mathbb{C} \setminus \{0\} \mid \tilde{F}_{\mathcal{J}}(z) = 0\} \cup \{0\}$$

and $\tilde{f}(z)$ is the eigenvector of J corresponding to a nonzero eigenvalue z. In addition, the algebraic multiplicity of a non-zero eigenvalue z of J coincides with the order of z as a zero of $\tilde{F}_{\mathcal{J}}$.

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Remark: In this case for $p \geq 2$, one can show that J is a compact operator from the Schatten–von Neumann class $\mathcal{S}_p$ and

$$\tilde{F}_{\mathcal{J}}(1/z) = \det_p(1 - zJ), \quad \forall z \in \mathbb{C}.$$

So the above statement may be deduced from results of the theory of regularized determinants.

I. Compact case - example

- For $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ and $\beta \in \mathbb{C} \setminus \{0\}$ put

$$\lambda_n = \frac{1}{n-1+\alpha} \quad \text{and} \quad w_n = \frac{\beta}{\sqrt{(n-1+\alpha)(n+\alpha)}}.$$

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- The regularized characteristic function is

$$\tilde{F}_{\mathcal{J}}(z) = \frac{\sin \pi(\alpha - z^{-1})}{\sin \pi\alpha} e^{z^{-1}\pi \cot \pi\alpha}, \quad \forall z \neq 0,$$

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- Note that $\sigma(J)$ does not depend on β. Hence, for any $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, J is a non-self-adjoint operator with purely real spectrum.

I. Compact resolvent case - regularization

- If we assume $\lambda_n \neq 0$, $\forall n \in \mathbb{Z}$,

$$\sum_{n=-\infty}^{\infty} \left| \frac{w_n^2}{\lambda_n \lambda_{n+1}} \right| < \infty \quad \text{and} \quad \sum_{n=-\infty}^{\infty} \frac{1}{|\lambda_n|^p} < \infty, \quad \text{for some } p \in \mathbb{N},$$

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- The global regularization of $F_{\mathcal{J}}$ and f is done by putting

$$\tilde{F}_{\mathcal{J}}(z) := \Psi_p(z) F_{\mathcal{J}}(z) \quad \text{and} \quad \tilde{f}(z) := \Psi_p^+(z) f(z).$$

Functions $\tilde{F}_{\mathcal{J}}$ and $\tilde{f}$ are entire.

II. Compact resolvent case - spectral results

Proposition:

Let $\lambda_n \neq 0$ and $w_n \neq 0$ be such that

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Further, let $\tilde{F}_{\mathcal{J}}$ does not vanish identically on $\mathbb{C}$. Then $J_{\min} = J_{\max} =: J$,

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Remark: For $z \in \mathbb{C}$, we may introduce the operator $A(z)$ determined by equalities

$$A(z)e_n = \frac{w_{n-1}}{\sqrt{\lambda_{n-1}\lambda_n}} e_{n-1} - \frac{z}{\lambda_n} e_n + \frac{w_n}{\sqrt{\lambda_n \lambda_{n+1}}} e_{n+1}, \quad n \in \mathbb{Z},$$

which is, if $p \geq 2$, in $\mathcal{S}_p$. In addition, it can be shown that

$$z \in \sigma(J) \iff -1 \in \sigma((A(z))$$

and we have

$$\det_p (1 + A(z)) = \tilde{F}_{\mathcal{J}}(z), \quad z \in \mathbb{C}.$$

II. Compact resolvent case - example

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$$\lambda_n = n \quad \text{and} \quad w_n = w \neq 0.$$

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- The n-th element of the eigenvector to the eigenvalue $N \in \mathbb{Z}$ reads

$$v_n(N) = (-1)^n J_{n-N}(2w), \quad n \in \mathbb{Z}.$$

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- If we assume $\lambda_n \neq 0$, for $n \leq 0$,

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are to be used to regularize $F_{\mathcal{J}}$ and f:

$$\tilde{F}_{\mathcal{J}}(z) := \Phi_p^+(z) \Psi_p^-(z) F_{\mathcal{J}}(z) \quad \text{and} \quad \tilde{f}(z) := \Phi_p^+(z) f(z), \quad \text{for } z \neq 0.$$

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- For $q, \beta \in \mathbb{C}$, $0 < |q| < 1$ and $\beta \neq 0$, put

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- The n-th element of the eigenvector to the eigenvalue $z \in \sigma_p(J)$ reads

$$\tilde{f}_n(z) = z^{-n} \beta^n q^{n(n-1)/4} \, {}_0\tilde{\phi}_1 \left(-; z^{-1} q^{n+1}; q, -q^{n+1} z^{-2} \beta^2\right).$$

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- Whenever $q \in (-1, 1)$ and β is purely imaginary, then J is a non-self-adjoint operator with purely real spectrum. In this case, J is diagonalizable if and only if $\beta \notin \mathrm{i}q^{\mathbb{Z}/2}$.

спасибо