On non-self-adjoint Toeplitz matrices with purely real spectrum

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Based on: B. Shapiro, F. Štampach: `Non-self-adjoint Toeplitz matrices whose principal submatrices have real spectrum`, arXiv:1702.00741 [math.CA]

Contents

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Remark:

If a is analytic in $\mathbb{C} \setminus \{0\}$ (especially, if a is a Laurent polynomial), then the assumption 1 can be omitted.

The case of banded Toeplitz matrices

- Question: If $\Lambda(a) \subset \mathbb{R}$, can the set $\Lambda(a)$ be approached from the complex plane? That is, can $\mathrm{spec}(T_n(a))$ contain non-real eigenvalues for some n?

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Remark:
It is a very surprising feature of banded Toeplitz matrices that the asymptotic reality of the eigenvalues (claim 1) forces all eigenvalues of all $T_n(b)$ to be real (claim 3). Hence, if, for instance, the 2×2 matrix $T_2(b)$ has a non-real eigenvalue, there is no chance for the limiting set $\Lambda(b)$ to be real!

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- $$\Lambda(b) \subset \mathbb{R} \quad \Longleftrightarrow \quad a^3 \geq 27b^2 > 0.$$

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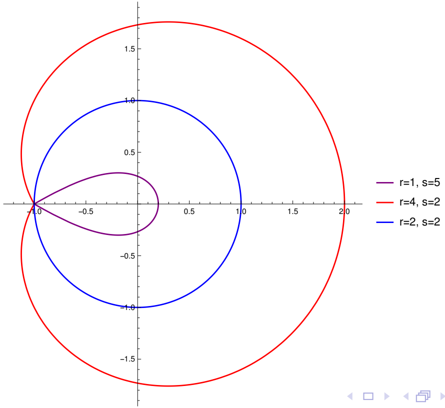

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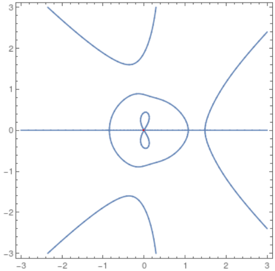

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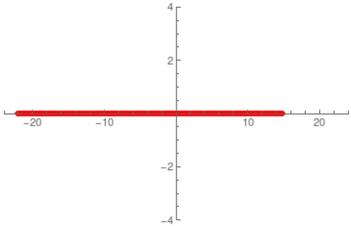

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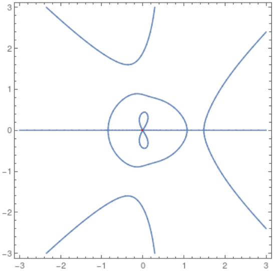

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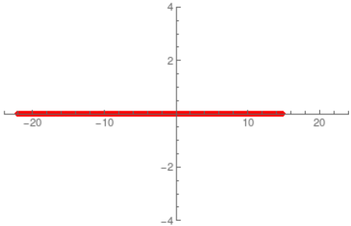

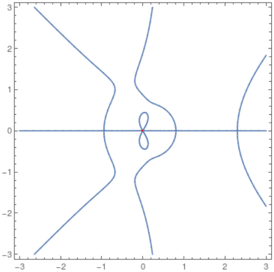

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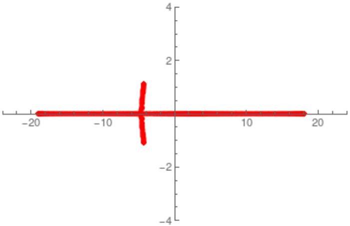

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- The density of the limiting measure can be also expressed in terms of the zeros $z_1(\lambda)$, $z_2(\lambda)$, $\ldots z_{r+s}(\lambda)$, [Hirschman Jr., 1967].

The limiting measure and the Jordan curve without critical points

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Let $b'(\gamma(t)) \neq 0$ for all $t \in (0,\pi)$. Then $b \circ \gamma$ restricted to $(0,\pi)$ is either strictly increasing or decreasing; the limiting measure μ is supported on the interval $[\alpha,\beta] := b(\gamma([0,\pi]))$ and its density satisfies

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(x) = \pm\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}(b\circ\gamma)^{-1}(x),$$

for $x \in (\alpha,\beta)$, where the $+$ sign is used when $b\circ\gamma$ increases on $(0,\pi)$, and the $-$ sign is used otherwise.

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$$b(z) = z^{-3} - z^{-2} + 7z^{-1} + 9z - 2z^2 + 2z^3 - z^4,$$

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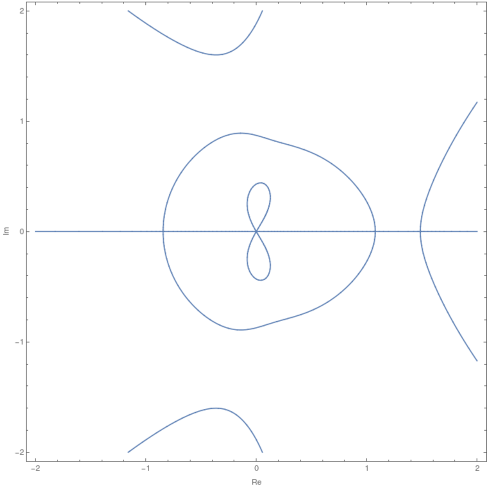

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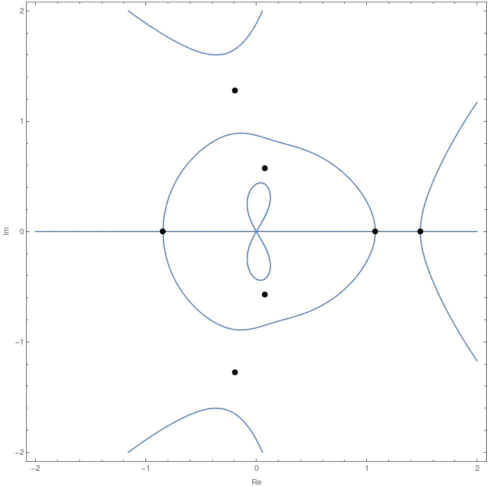

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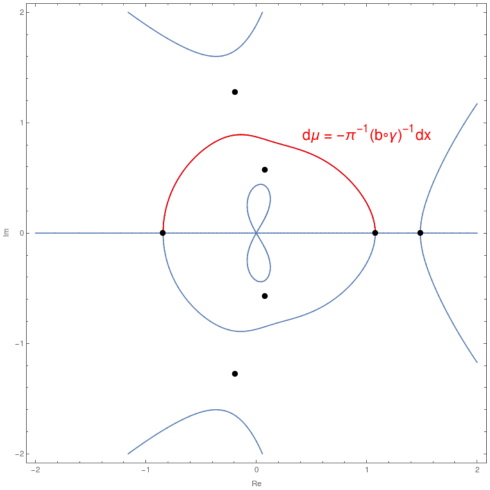

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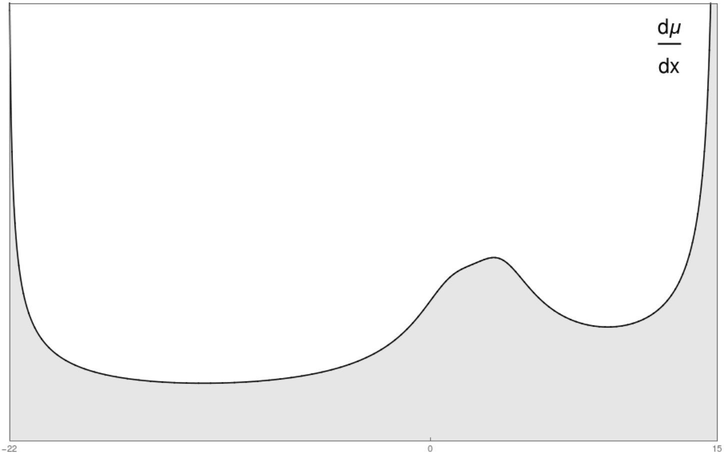

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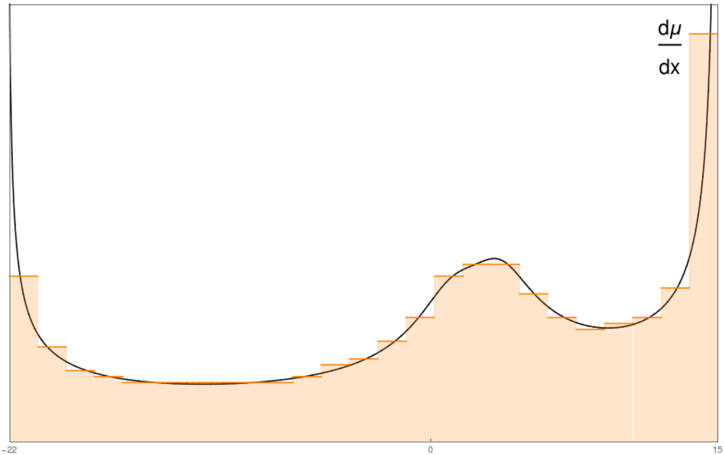

The limiting measure and the Jordan curve with critical points

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$$\frac{\mathrm{d}\mu_i}{\mathrm{d}x}(x) = \pm\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}(b\circ\gamma)^{-1}(x)$$

for all $x \in (\alpha_i, \beta_i)$ and all $i \in \{1, 2, \ldots, \ell+1\}$. The $+$ sign is used when $b \circ \gamma$ increases on (α_i, β_i), and the $-$ sign is used otherwise.

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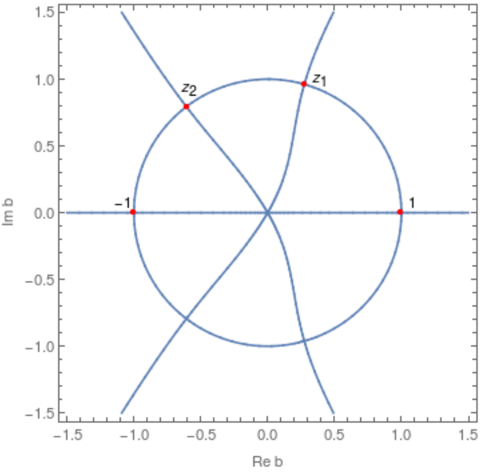

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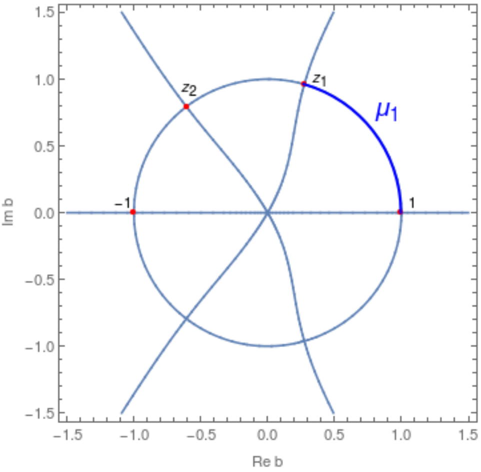

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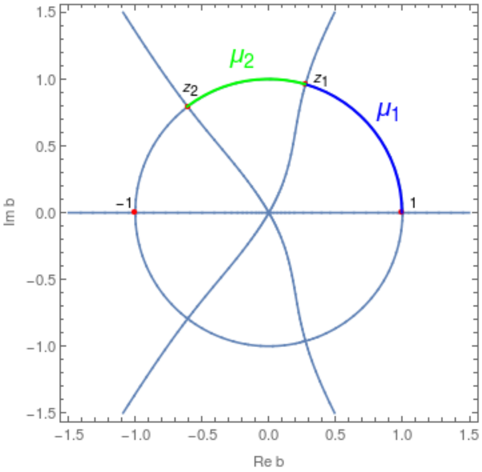

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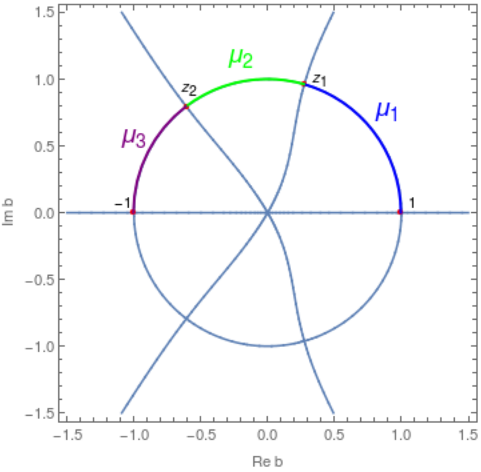

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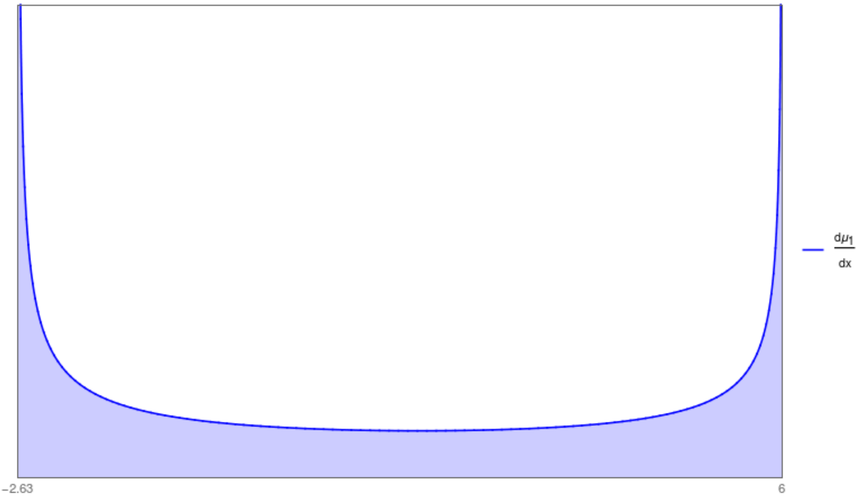

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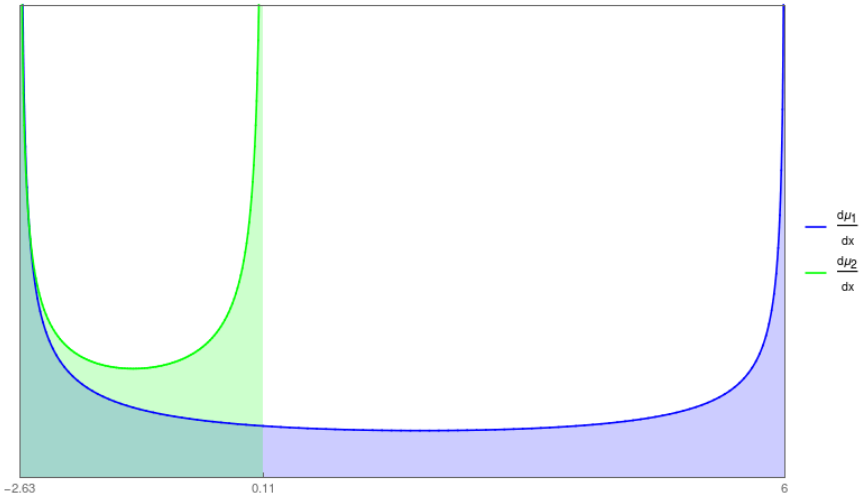

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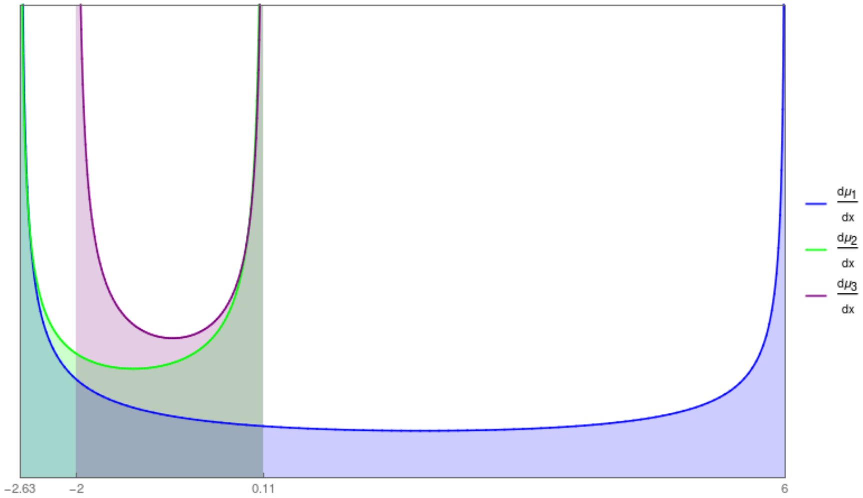

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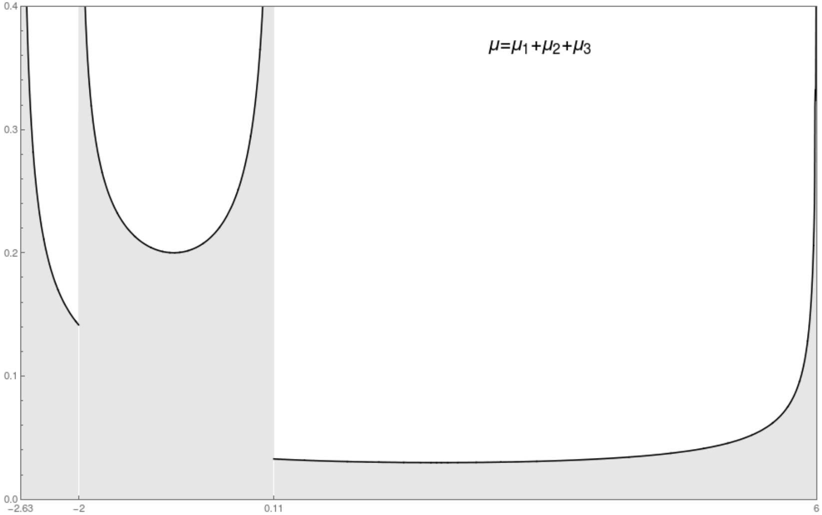

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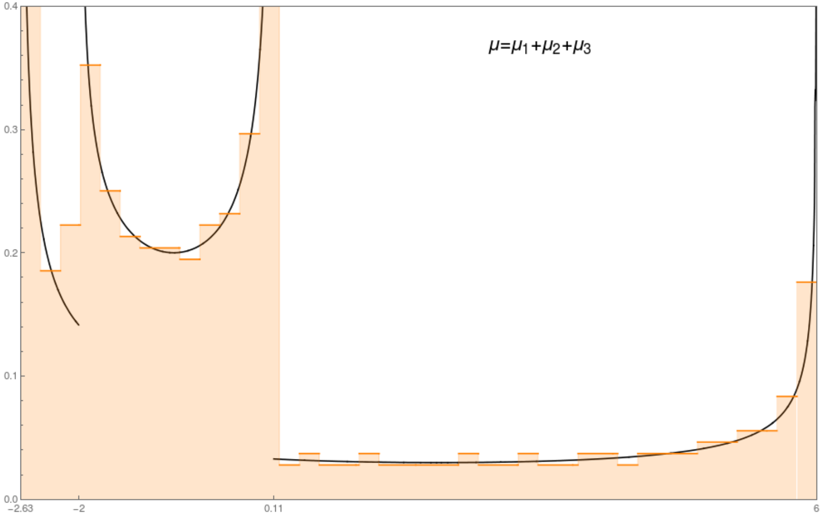

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- Fully explicit expressions for the limiting measures are available for

$$(r,s) \in \{(1,1),(1,2),(2,2)\}.$$

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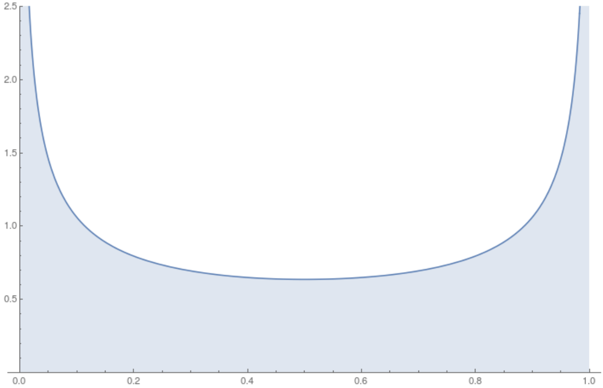

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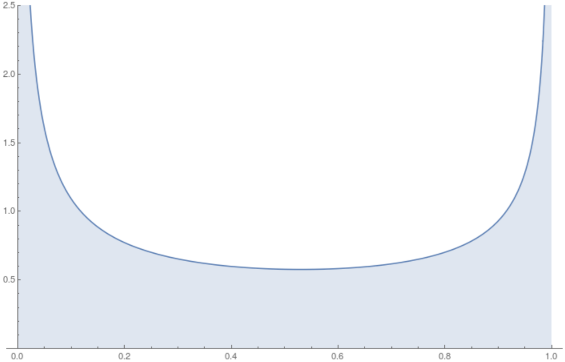

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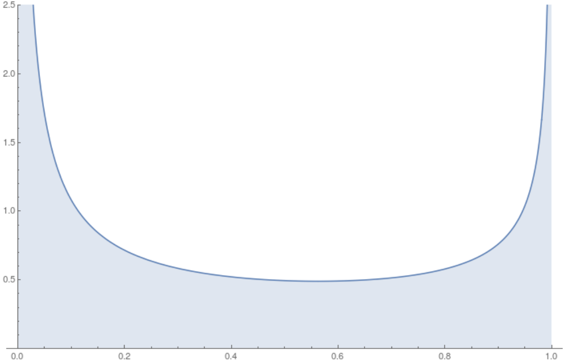

Contents

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- It does not seem that there is a simple condition in terms of $a_{-r}, \dots, a_s$.
- But one possible condition might follow from the connection to HMP.

Limiting measure as a solution to the HMP

- Recall that the HMP asks for the determination of a real supported measure μ such that

$$\int_{\mathbb{R}} x^m \mathrm{d}\mu(x) = h_m, \quad \forall m \in \mathbb{N}_0,$$

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Proposition:

Let $b^{-1}(\mathbb{R})$ contains a Jordan curve. Then the limiting measure μ coincides with the unique solution of the determinate HMP with moments

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Corollary:

If $b^{-1}(\mathbb{R})$ contains a Jordan curve, then the moment Hankel matrix

$$H_n := (h_{i+j})_{i,j=0}^{n-1}$$

is positive-definite for all $n \in \mathbb{N}_0$.

An open problem

- The Hankel matrices H_n are positive-definite iff $\det H_n > 0$ for all $n \in \mathbb{N}_0$.

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- In principle, the inequality $\det H_n > 0$ can be expressed a in terms of $a_{-r}, \dots, a_s$.
- Namely, if

$$p_n(a_{-r}, \dots, a_s) := \text{ the constant term of } \prod_{1 \leq i < j \leq n} \left(b\left(z_j\right) - b\left(z_i\right)\right)^2,$$

the condition reads

$$p_n(a_{-r}, \dots, a_s) > 0, \quad \forall n \in \mathbb{N}.$$

An open problem

- The Hankel matrices H_n are positive-definite iff $\det H_n > 0$ for all $n \in \mathbb{N}_0$.
- In principle, the inequality $\det H_n > 0$ can be expressed a in terms of $a_{-r}, \dots, a_s$.
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- It has been proved only with an additional condition:

$$\begin{array}{c} p_n(a_{-r}, \dots, a_s) > 0, \quad \forall n \in \mathbb{N}, \\ \text{and } \mathbb{C} \setminus \Lambda(b) \text{ is connected} \end{array} \quad \Longrightarrow \quad \Lambda(b) \subset \mathbb{R}.$$

Thank you!