On Lieb–Thirring inequlities for non-self-adjoint Jacobi matrices and Schrödinger operators

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Mathematical aspects of the physics with non-self-adjoint operators:
10 years after,
actually, 11 years after

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~~December 7-11, 2020~~
February 1-5, 2021 (*Hurray!*)

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THE JACOBI OPERATOR

- Let $\{a_n\}_{n\in\mathbb{Z}}$, $\{b_n\}_{n\in\mathbb{Z}}$, $\{c_n\}_{n\in\mathbb{Z}}$ be given bounded complex sequences.

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- We denote

$$d_n := \max\{|a_{n-1}-1|, |a_n-1|, |b_n|, |c_{n-1}-1|, |c_n-1|\}, \quad n \in \mathbb{Z}.$$

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$$d_n := \max\{|a_{n-1}-1|, |a_n-1|, |b_n|, |c_{n-1}-1|, |c_n-1|\}, \quad n \in \mathbb{Z}.$$
- If $\lim_{n\to\pm\infty} d_n = 0$, then
$$\sigma_{ess}(J) = [-2,2] \quad \text{and} \quad \sigma(J) = [-2,2] \cup \sigma_d(J).$$

LIEB–THIRRING INEQUALITIES FOR S.A. JACOBI OPERATORS

Theorem (Hundertmark–Simon, JAT'02)

Suppose $a_n = c_n > 0$ and $b_n \in \mathbb{R}$.

Lieb–Thirring inequalities for s.a. Jacobi operators

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Suppose $a_n = c_n > 0$ and $b_n \in \mathbb{R}$. If $d \in \ell^p(\mathbb{Z})$, for $p \geq 1$, then

$$\sum_{\lambda \in \sigma_d(J) \cap (-\infty, -2)} |\lambda + 2|^{p-1/2} + \sum_{\lambda \in \sigma_d(J) \cap (2, \infty)} |\lambda - 2|^{p-1/2} \leq C_p \|d\|_{\ell^p}^p,$$

where C_p is an explicit constant independent of J.

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- Equivalently,

$$\sum_{\lambda \in \sigma_d(J)} (\mathrm{dist}(\lambda, [-2,2]))^{p-1/2} \leq C_p \|d\|_{\ell^p}^p.$$

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Question 1

Does the above inequality hold true for general (possibly n.s.a.) Jacobi operators with $d \in \ell^p(\mathbb{Z})$?

The conjecture of Hansmann and Katriel

Conjecture (Hansmann–Katriel, CAOT'11)

No.

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The Lieb-Thirring inequality

$$\sum_{\lambda\in\sigma_d(J)} (\mathrm{dist}(\lambda,[-2,2]))^{p-1/2} \le C_p \|d\|_{\ell^p}^p$$

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Answer 1

True.

FURTHER ATTEMPTS TO FIND AN ADMISSIBLE EXTENSION

- Recall the H.–S. result for s.a. Jacobi operators:

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- Using the observation

$$\frac{\operatorname{dist}(\lambda,[-2,2])^p}{|\lambda^2-4|^{1/2}} \leq \frac{1}{2}\begin{cases} |\lambda-2|^{p-1/2}, & \text{if } \lambda>2, \\ |\lambda+2|^{p-1/2}, & \text{if } \lambda<-2, \end{cases}$$

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the H.–S. result implies

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- This is already very close to what was proven by Hansmann and Katriel for general Jacobi operators...

LIEB–THIRRING INEQ. FOR N.S.A. JACOBI OPERATORS

Theorem (Hansmann–Katriel, CAOT'11)

Suppose $\tau \in (0,1)$ and $d \in \ell^p(\mathbb{Z})$ with $p \geq 1$. Then

$$\sum_{\lambda \in \sigma_d(J)} \frac{(\text{dist}(\lambda, [-2,2]))^{p+\tau}}{|\lambda^2 - 4|^{1/2}} \leq C_{p,\tau} \|d\|_{\ell^p}^p, \quad \text{if } p > 1,$$

and

$$\sum_{\lambda \in \sigma_d(J)} \frac{(\text{dist}(\lambda, [-2,2]))^{1+\tau}}{|\lambda^2 - 4|^{1/2+\tau/4}} \leq C_{\tau} \|d\|_{\ell^1}, \quad \text{if } p = 1.$$

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- In the s.a. case, the above inequalities hold true also if $\tau = 0$.

Lieb–Thirring ineq. for n.s.a. Jacobi operators

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Question 2

Does the above inequalities remain valid for $\tau = 0$ and general Jacobi operators with $d \in \ell^p(\mathbb{Z})$?

Answer 2

No.

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The Lieb–Thirring inequality

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does not extend to general Jacobi operators with $d \in \ell^p(\mathbb{Z})$.

The discrete Schrödinger operator

- Counterexamples are found among discrete Schrödinger operators

$$T(b) = J_0 + b$$

with complex potential $b \in \ell^p(\mathbb{Z})$, i.e., Jacobi operators J with $a_n = c_n = 1, \forall n \in \mathbb{Z}$.

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Theorem

For any $p \geq 0$ and $\omega < p$, one has
$$\sup_{0 \neq b \in \ell^p(\mathbb{Z})} \frac{1}{\|b\|_{\ell^p}^p} \sum_{\lambda \in \sigma_d(T(b))} (\mathrm{dist}(\lambda, [-2,2]))^\omega = \infty.$$

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- In particular, for $\omega = p - 1/2$, the theorem confirms the conjecture of Hansmann and Katriel.
- On the other hand, if $\omega \geq p$, the inequality

$$\sum_{\lambda \in \sigma_d(J)} (\mathrm{dist}(\lambda, [-2,2]))^\omega \leq C_p \|d\|_{\ell^p}^p$$

holds for any (possibly n.s.a.) Jacobi operator J (Hansmann, LMP'11).

The DSO with rectangular barrier potential and complex coupling

- For $n \in \mathbb{N}$ and $h > 0$, define

$$T_{h,n} := J_0 + \mathrm{i}hP_n,$$

where P_n is OG projection onto $\operatorname{span}\{e_1, \dots e_n\}$.

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- By the Birman–Schwinger principle, $\lambda \notin [-2, 2]$ is an eigenvalue of $T_{n,h}$ iff

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$$\det\left(1 + \mathrm{i}hP_n(J_0 - \lambda)^{-1}P_n\right) = 0.$$
- Write $\lambda \notin [-2, 2]$ as $\lambda = k + k^{-1}$, where $0 < |k| < 1$. Then
$$(J_0 - \lambda)^{-1} = \frac{k}{k^2 - 1}Q(k),$$
where $Q(k)$ is the Laurent operator with entries $Q_{i,j}(k) = k^{|j-i|}$.

- We are led to the characteristic equation

$$\det\left(1+\frac{\mathrm{i}kh}{k^2-1}Q_n(k)\right)=0,$$

for $Q_n(k) := P_n Q(k) P_n \restriction \operatorname{Ran} P_n$ (Kac–Murdock–Szegö matrix).

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- Introducing a new parameter z by equation

$$\mathrm{i}h = k + k^{-1} - z - z^{-1}$$

the characteristic functions takes a fully explicit form

$$\det\left(1+\frac{\mathrm{i}kh}{k^2-1}Q_n(k)\right)=\frac{k^{2n}}{1-k^2}\,\frac{\mathrm{i}^n h^n}{(z-k)^n(1-kz)^n}\,\frac{z^{2n}(z-k)^2-(1-kz)^2}{z^2-1}.$$

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- Solving $z^{2n}(z-k)^2-(1-kz)^2=0$ for $k=k(z)$ yields

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- Plugging back...

- ...we arrive at two equations:

$$\mathrm{i}h\left(z^{n+1}-1\right)\left(z^{n-1}-1\right)-z^{n-2}\left(z^{2}-1\right)^{2}=0, \quad (*)$$
$$\mathrm{i}h\left(z^{n+1}+1\right)\left(z^{n-1}+1\right)+z^{n-2}\left(z^{2}-1\right)^{2}=0, \quad (**)$$

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- In summary, we obtain:

Proposition

One has

$$\lambda \in \sigma_d(T_{h,n}) \quad \Longleftrightarrow \quad \lambda = \mathrm{i}h + z + z^{-1},$$

for $z \in \mathbb{C}$, $|z|<1$, $\operatorname{Im} z > 0$, which is either a solution of $(*)$ or $(**)$ satisfying the constraint $|z^{n+1}-1|<|z^{n}-z|$ or $|z^{n+1}+1|<|z^{n}+z|$, respectively.

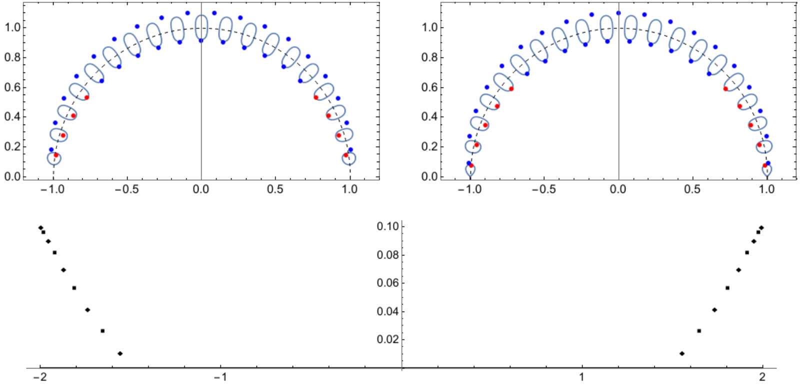

Figure: A numerical illustration of spectrum of $T_{h,n}$ for $h = 1/10$ and $n = 39$.

TOWARDS THE PROOF OF H.–K. CONJECTURE

- Next, we put $h = h_n := n^{-2/3}$ and consider the sequence $T_n := T_{h_n,n}$.
- Fix $0 < \epsilon < 1/2$.

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- Fix $0 < \epsilon < 1/2$.
- Then we can show that, for n sufficiently large, there are $(1-2\epsilon)n/2$ solutions z_j of the algebraic equations $(*)$ and $(**)$ located in the sector

$$\epsilon\pi < \arg z_j < (1-\epsilon)\pi,$$

and each z_j gives rise to an eigenvalue λ_j of T_n.

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- Moreover, these eigenvalues have the asymptotic behavior

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- Moreover, these eigenvalues have the asymptotic behavior

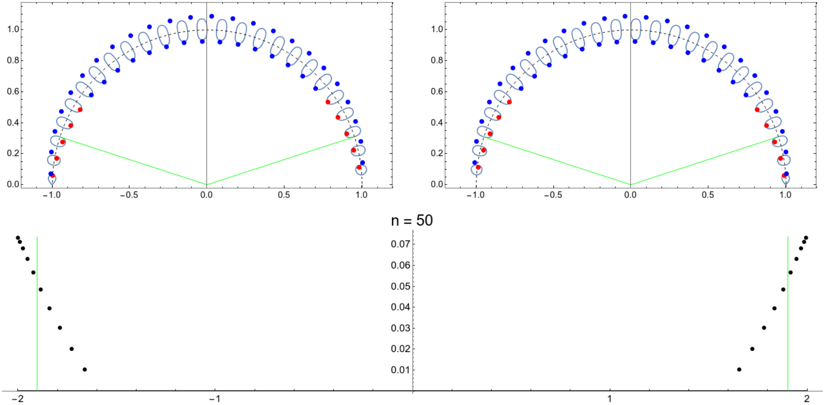
$$\lambda_j = 2\cos\phi_j + \mathrm{i}n^{-2/3} + O\left(\frac{\log n}{n}\right), \quad n \to \infty.$$

- It follows

$$\operatorname{dist}(\lambda_j, [-2,2]) = n^{-2/3} + O\left(\frac{\log n}{n}\right), \quad n \to \infty,$$

uniformly in j.

A NUMERICAL ILLUSTRATION



- For $\omega < p$ and $\epsilon < 1/4$, we have

$$\sum_{\lambda \in \sigma_d(T_n)} (\mathrm{dist}(\lambda, [-2,2]))^\omega \geq \frac{n}{4}\left(n^{-2/3} + O\left(\frac{\log n}{n}\right)\right)^\omega$$
$$= \frac{n^{1-2\omega/3}}{4}\left(1 + O\left(\frac{\log n}{n^{1/3}}\right)\right), \quad n \to \infty.$$

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$$= \frac{n^{1-2\omega/3}}{4}\left(1 + O\left(\frac{\log n}{n^{1/3}}\right)\right), \quad n \to \infty.$$

- In total, for n sufficiently large, we get

$$\frac{1}{n^{1-2p/3}} \sum_{\lambda\in\sigma_d(T_n)} (\text{dist}(\lambda, [-2,2]))^\omega \geq \frac{1}{8} n^{2(p-\omega)/3},$$

which implies:

- For $\omega < p$ and $\epsilon < 1/4$, we have

$$\sum_{\lambda\in\sigma_d(T_n)} (\mathrm{dist}(\lambda,[-2,2]))^{\omega} \geq \frac{n}{4}\left(n^{-2/3} + O\left(\frac{\log n}{n}\right)\right)^{\omega}$$
$$= \frac{n^{1-2\omega/3}}{4}\left(1 + O\left(\frac{\log n}{n^{1/3}}\right)\right), \quad n \to \infty.$$

- In total, for n sufficiently large, we get

$$\frac{1}{n^{1-2p/3}} \sum_{\lambda\in\sigma_d(T_n)} (\mathrm{dist}(\lambda,[-2,2]))^{\omega} \geq \frac{1}{8} n^{2(p-\omega)/3},$$

which implies:

For any $p \geq 0$ and $\omega < p$, one has

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...and the H.–K. conjecture follows (for $\omega = p - 1/2$).

TOWARDS ANSWER 2

- Recall the 2nd open problem: Does the inequality

$$\sum_{\lambda \in \sigma_d(J)} \frac{(\mathrm{dist}(\lambda, [-2,2]))^p}{|\lambda^2 - 4|^{1/2}} \leq C_p \|d\|_{\ell^p}^p.$$

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For any $p \geq 1$ and $\sigma \geq 1/2$, one has

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- The same sequence T_n as before can be used here but the analysis is more delicate...

- Asymptotic analysis yields

$$\lambda_j = 2\cos\phi_j + \mathrm{i}n^{-2/3} + O\left(\frac{\log n}{n}\right), \quad n \to \infty,$$

with

$$\phi_j = \frac{\pi(4j-1)}{2n} + O\left(\frac{1}{n^{3/2}}\right), \quad n \to \infty,$$

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- Particularly, it follows that

$$\mathrm{dist}(\lambda_j, [-2,2]) \geq \frac{1}{2}n^{-2/3}$$

for all j and n sufficiently large.

- Then

$$\liminf_{n\to\infty} n^{2p/3-1} \sum_{\lambda\in\sigma_d(T_n)} \frac{(\mathrm{dist}(\lambda,[-2,2]))^p}{|\lambda^2-4|^\sigma} \geq \liminf_{n\to\infty} \frac{1}{n}\sum_j \frac{1}{|\lambda_j^2-4|^\sigma}$$

$$= C_\sigma \int_{\epsilon\pi}^{(1-\epsilon)\pi} \frac{\mathrm{d}x}{(4-4\cos^2 x)^\sigma},$$

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and $\epsilon > 0$ can be arbitrarily small, one finally gets

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for all $\sigma \geq 1/2$.

LIEB-THIRRING INEQUALITIES FOR S.A. SCHRÖDINGER OPERATORS

- Consider the Schrödinger operator $H := -\Delta + V$ on $L^2(\mathbb{R}^d)$ with $V \in L^p(\mathbb{R}^d)$, where

$$\begin{aligned} p \geq 1, &\quad \text{if } \ d = 1, \\ p > 1, &\quad \text{if } \ d = 2, \\ p \geq d/2, &\quad \text{if } \ d \geq 3. \end{aligned}$$

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Theorem (Lieb–Thirring)

Suppose V is real-valued and satisfies the above conditions. Then

$$\sum_{\lambda \in \sigma_d(H)} |\lambda|^{p-\frac{d}{2}} \leq C_{p,d} \|V\|_{L^p}^p.$$

- Equivalent formulation:

$$\sum_{\lambda \in \sigma_d(H)} \frac{(\mathrm{dist}(\lambda, [0, \infty)))^p}{|\lambda|^{d/2}} \leq C_{p,d} \|V\|_{L^p}^p.$$

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Does the above inequality remain valid for complex-valued potentials $V \in L^p(\mathbb{R}^d)$?

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(Partial) Answer 3

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(Partial) Answer 3

1. If $d = 1$, then NO.
2. If $d \geq 2$, then UNKNOWN - *work in progress.*

Answer 3 in dimension 1

Theorem

Let $d = 1$. For all $p \geq 1$ and $\sigma \geq 1/2$, one has

$$\sup_{0 \neq V \in L^p(\mathbb{R})} \frac{1}{\|V\|_{L^p}^p} \sum_{\lambda \in \sigma_d(H)} \frac{(\mathrm{dist}(\lambda, [0, \infty)))^p}{|\lambda|^\sigma} = \infty.$$

ANSWER 3 IN DIMENSION 1

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- In analogy to the discrete case, the operator family that demonstrates the theorem is

$$H_h := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{i}}{h}\chi_{[-h,h]}, \quad h > 0.$$

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- The problem can be reduced to a study of asymptotic properties of discrete eigenvalues of

$$\tilde{H}_h := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}h\chi_{[-1,1]},$$

for $h \to \infty$.

Based on:

S. Bögli, F. Š.: *On Lieb-Thirring inequalities for one-dimensional non-self-adjoint Jacobi and Schrödinger operators*, J. Spectr. Theory (to appear), arXiv:2004.09794.

Thank you!