# Non-self-adjoint Toeplitz matrices with purely real spectrum and related problems

František Štampach

Symmetries of Discrete Systems and Processes

Děčín

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Based on: B. Shapiro, F. Štampach: `Non-self-adjoint Toeplitz matrices whose principal submatrices have real spectrum`, arXiv:1702.00741 [math.CA]

# Contents


1. Toeplitz matrices with real spectrum
2. The asymptotic eigenvalue distribution
3. Connections to the Hamburger Moment Problem and Orthogonal Polynomials

# Toeplitz matrix

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**Remark:**
If $a$ is analytic in $\mathbb{C} \setminus \{0\}$ (especially, if $a$ is a Laurent polynomial), then the assumption 1 can be omitted.

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**Remark:**

It is a very surprising feature of banded Toeplitz matrices that the asymptotic reality of the eigenvalues (claim 1) forces all eigenvalues of all $T_n(b)$ to be real (claim 3). Hence, if, for instance, the $2 \times 2$ matrix $T_2(b)$ has a non-real eigenvalue, there is no chance for the limiting set $\Lambda(b)$ to be real!

# Examples

1. Tridiagonal Toeplitz matrix:

$$b(z) = z^{-1} + az, \qquad (a \in \mathbb{C} \setminus \{0\}).$$

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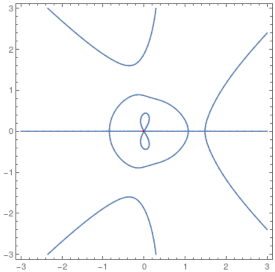


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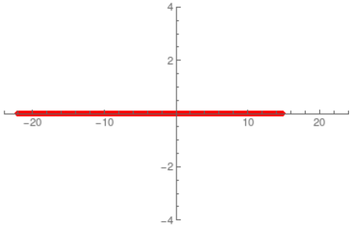

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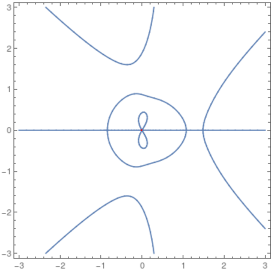


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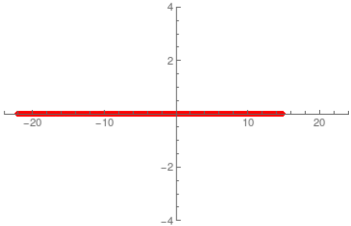


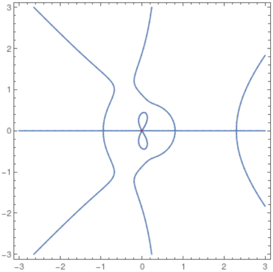


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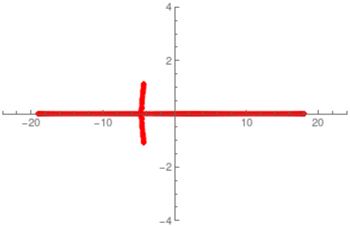

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- The set $\Lambda(b)$ can be described in terms of zeros of the polynomial $z \mapsto z^r(b(z) - \lambda)$ [Schmidt and Spitzer, 1960].
- The weak limit of the eigenvalue-counting measures of $T_n(b)$:

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_k^{(n)}}$$

exists, as $n \to \infty$, and is absolutely continuous measure $\mu$ supported on $\Lambda(b)$ whose density can be expressed in terms of zeros of $z \mapsto z^r(b(z) - \lambda)$ [Hirschman Jr., 1967].

# The limiting measure and the Jordan curve without critical points

1. Let $T_n(b)$ be a banded Toeplitz matrix with **real** elements:

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**Theorem:**

Let $b'(\gamma(t)) \neq 0$ for all $t \in (0, \pi)$. Then $b \circ \gamma$ restricted to $(0, \pi)$ is either strictly increasing or decreasing; the limiting measure $\mu$ is supported on the interval $[\alpha, \beta] := b(\gamma([0, \pi]))$ and its density satisfies

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(x) = \pm\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}(b \circ \gamma)^{-1}(x),$$

for $x \in (\alpha, \beta)$, where the $+$ sign is used when $b \circ \gamma$ increases on $(0, \pi)$, and the $-$ sign is used otherwise.

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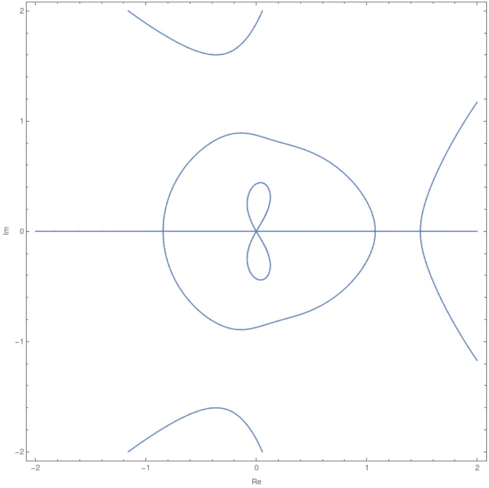

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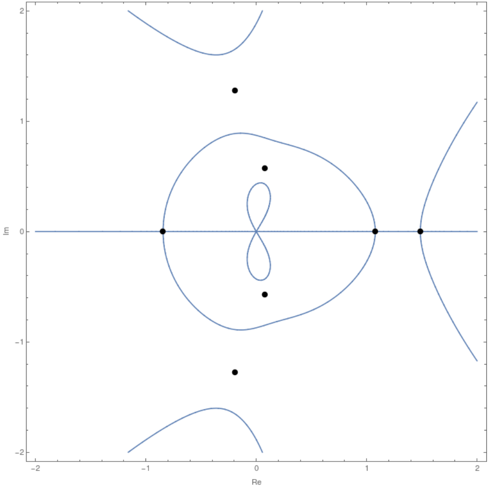

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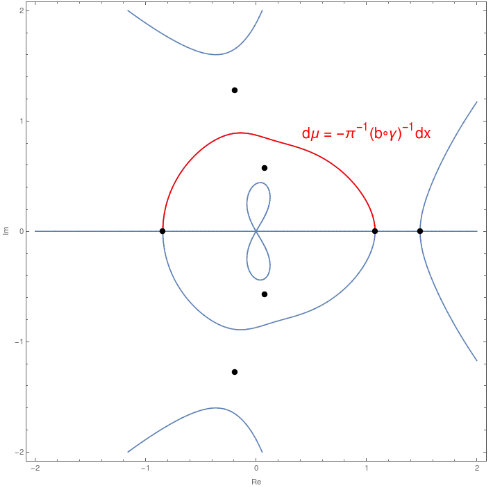

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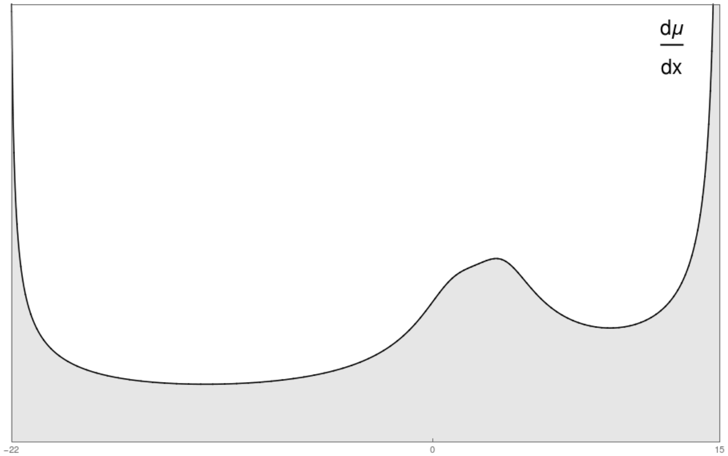

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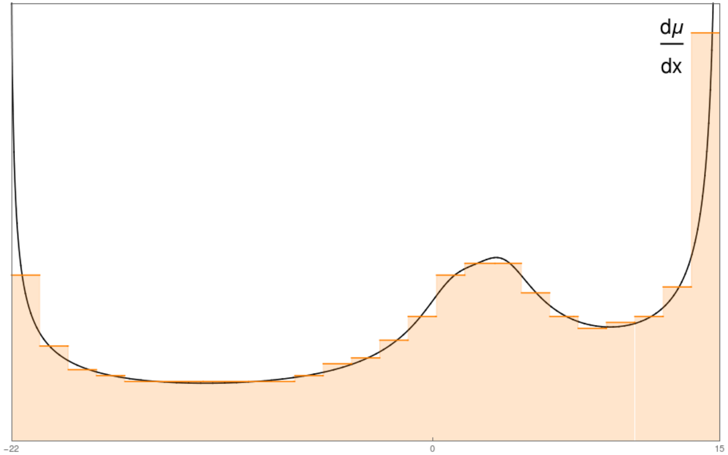

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$$\frac{\mathrm{d}\mu_i}{\mathrm{d}x}(x) = \pm \frac{1}{\pi} \frac{\mathrm{d}}{\mathrm{d}x} (b \circ \gamma)^{-1}(x)$$

for all $x \in (\alpha_i, \beta_i)$ and all $i \in \{1, 2, \ldots, \ell + 1\}$. The $+$ sign is used when $b \circ \gamma$ increases on $(\alpha_i, \beta_i)$, and the $-$ sign is used otherwise.

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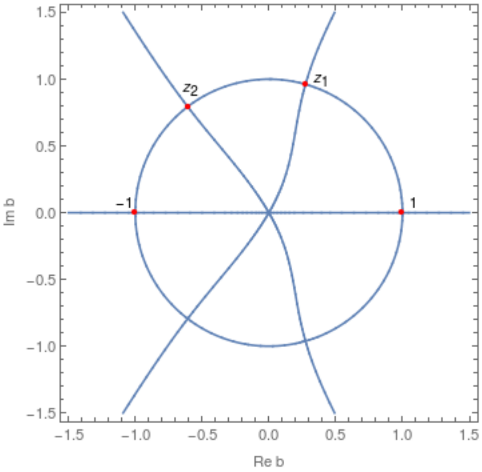

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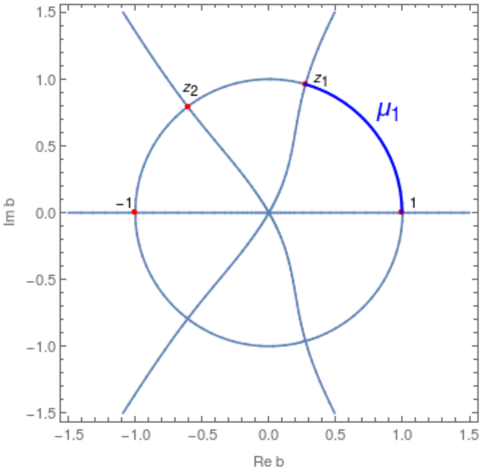

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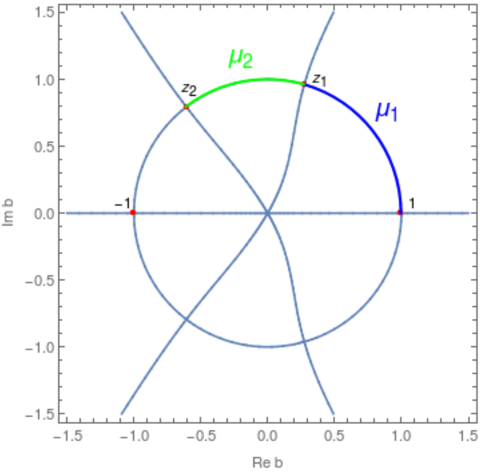

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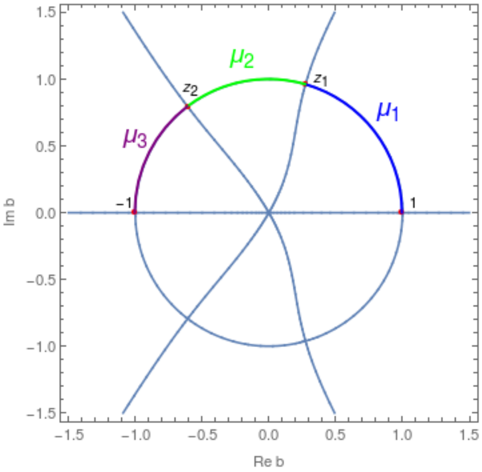

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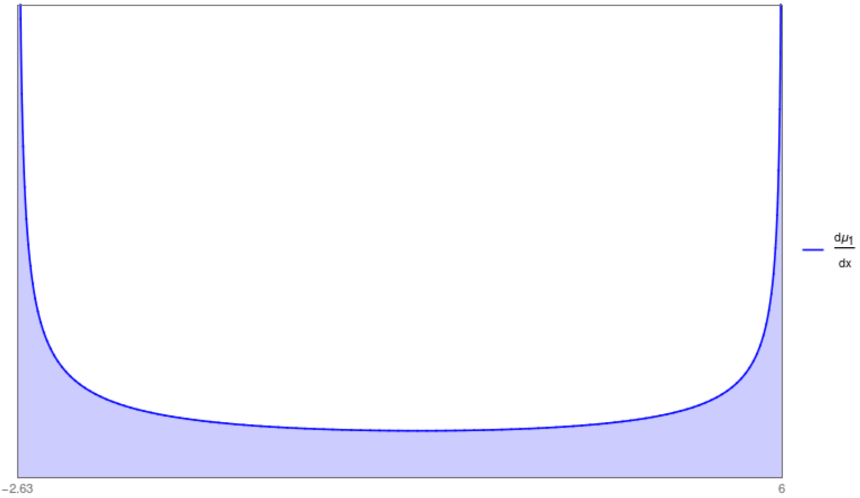

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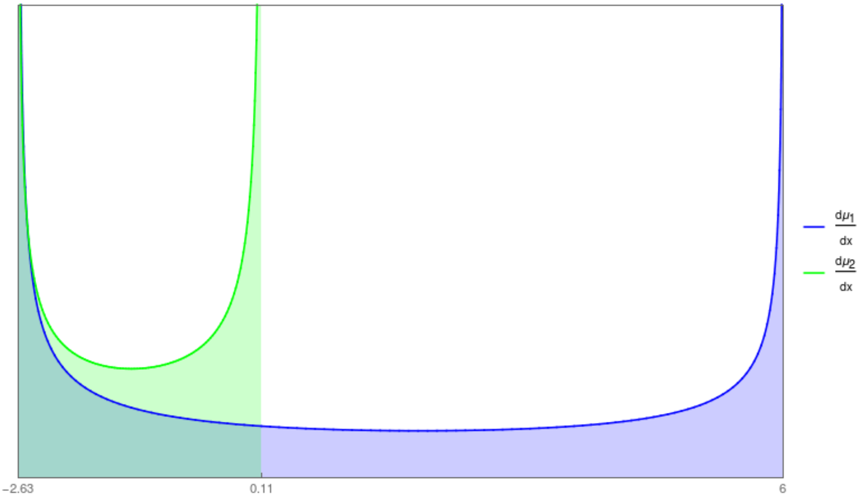

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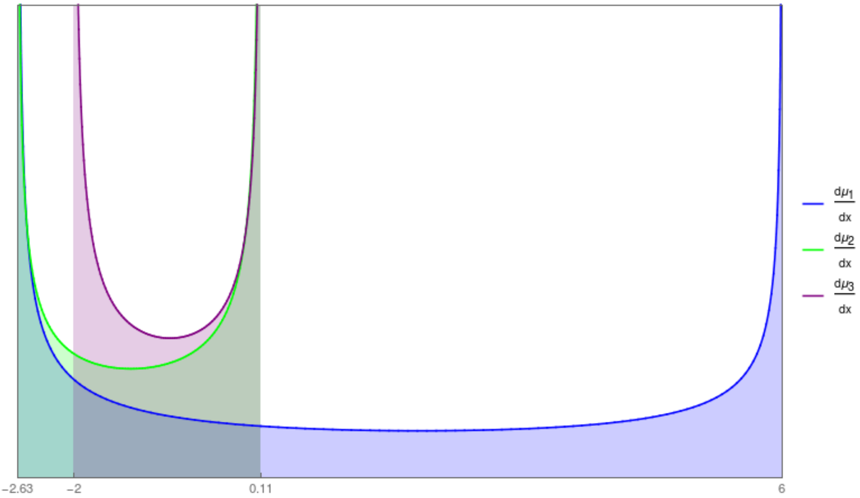

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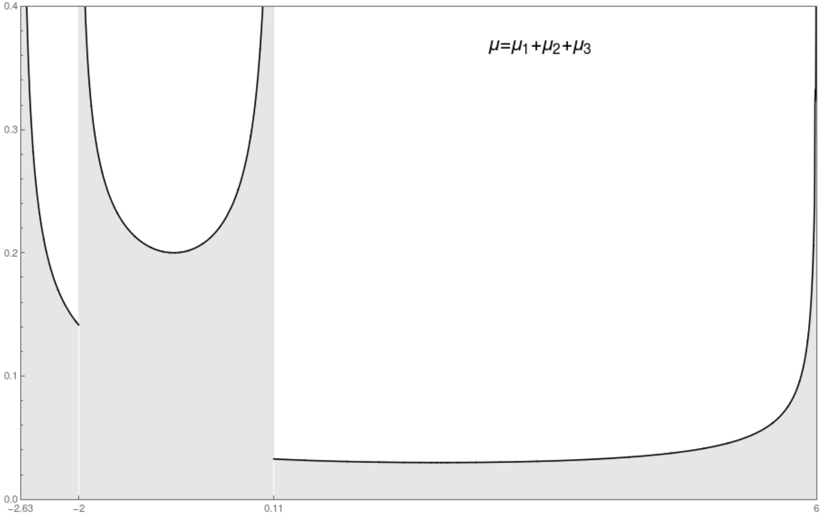

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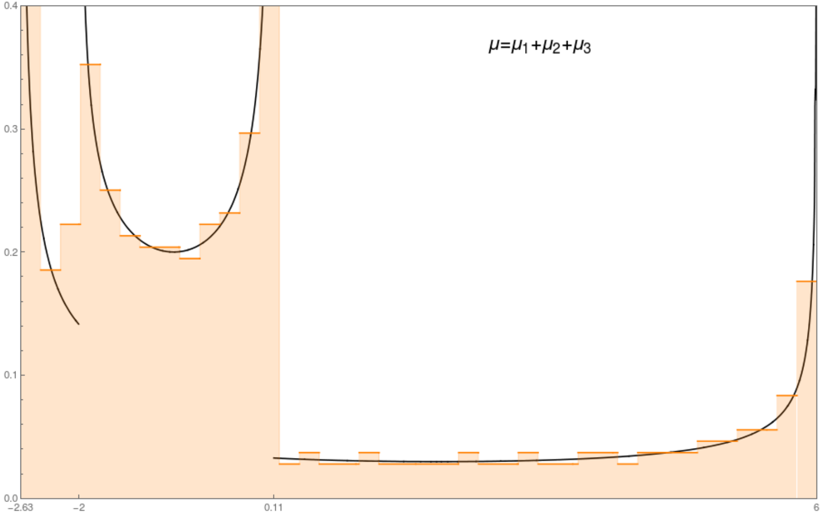

# Contents


1. Toeplitz matrices with real spectrum
2. The asymptotic eigenvalue distribution
3. Connections to the Hamburger Moment Problem and Orthogonal Polynomials

# The limiting measure as a solution of the HMP

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(If a counter-example exists, $\mathbb{C} \setminus \Lambda(b)$ has to be disconnected.)

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1. the Jordan curve intersects $\mathbb{R}$ at exactly two points whose $b$-images are the endpoints of the interval $\Lambda(b) = [\alpha, \beta]$;
2. the OGPs $\{p_n\}$ belong to the Blumenthal–Nevai class $M((\beta - \alpha)/2, (\alpha + \beta)/2)$, i.e.,

$$\lim_{n\to\infty} a_n = \frac{\beta - \alpha}{4} \quad \text{and} \quad \lim_{n\to\infty} b_n = \frac{\alpha + \beta}{2}.$$

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$$\Lambda(b) = \operatorname{supp}\mu = \left[0, \frac{(r+s)^{r+s}}{r^r s^s}\right] \supset \operatorname{spec} T_n(b) \quad \forall n \in \mathbb{N}.$$

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- Explicit formulas for the Jacobi parameters $a_n$ and $b_n$ are not known in general but we have

$$2\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{(r+s)^{r+s}}{2r^r s^s}.$$

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- Jacobi parameters:

$$a_1^2 = 6a^2, \quad a_k^2 = \frac{9(6k-5)(6k-1)(3k-1)(3k+1)}{4(4k-3)(4k-1)^2(4k+1)}a^2, \quad \text{for } k > 1.$$

and

$$b_1 = 3a, \quad b_k = \frac{3(36k^2-54k+13)}{2(4k-5)(4k-1)}a, \quad \text{for } k > 1.$$

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$$2^n p_n(x) = r_n^{(\alpha,\beta)}(x;c) - \frac{4}{27} r_{n-1}^{(\alpha,\beta)}(x;c+1) - \frac{256}{729} r_{n-2}^{(\alpha,\beta)}(x;c+2), \quad n \in \mathbb{N},$$

where $\alpha = 1/2$, $\beta = -2/3$, and $c = -1/6$.

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$$r_n^{(\alpha,\beta)}(x;c) := \frac{2^n(c+\alpha+\beta+1)_n(c+1)_n}{(2c+\alpha+\beta+1)_{2n}} P_n^{(\alpha,\beta)}(2x-1;c), \quad n \in \mathbb{N}_0,$$

it holds

$$2^n p_n(x) = r_n^{(\alpha,\beta)}(x;c) - \frac{4}{27} r_{n-1}^{(\alpha,\beta)}(x;c+1) - \frac{256}{729} r_{n-2}^{(\alpha,\beta)}(x;c+2), \quad n \in \mathbb{N},$$

where $\alpha = 1/2$, $\beta = -2/3$, and $c = -1/6$.
- This relation and the known properties of the associated Jacobi polynomials allow to derive other formulas for $p_n$ such as: an explicit representation, a generating function, ...

Thank you!