An Invitation to the Non-self-adjoint Church

František Štampach

$$A \neq A^*$$

Contents

Classical QM $\rightarrow$ *“Nice world with fancy actors.”*

Classical Quantum Mechanics works with
self-adjoint operators acting on a separable Hilbert space.

Classical QM $\rightarrow$ *"Nice world with fancy actors."*

Classical Quantum Mechanics works with

self-adjoint operators acting on a separable Hilbert space.

Q: Can an operator have empty spectrum?

Classical QM $\rightarrow$ *"Nice world with fancy actors."*

Classical Quantum Mechanics works with

self-adjoint operators acting on a separable Hilbert space.

Q: Can an operator have empty spectrum?

- A bounded $\Longrightarrow \sigma(A) \neq \emptyset$ (Liouville's theorem)

Classical QM $\rightarrow$ *"Nice world with fancy actors."*

Classical Quantum Mechanics works with
self-adjoint operators acting on a separable Hilbert space.

Q: Can an operator have empty spectrum?

- A bounded $\Longrightarrow \sigma(A) \neq \emptyset$ (Liouville's theorem)
- $A = A^*$ (possibly unbounded) $\Longrightarrow \sigma(A) \neq \emptyset$ (the above claim + $r(A) = \|A\|$ for $A = A^*$)

Classical QM $\to$ *"Nice world with fancy actors."*

Classical Quantum Mechanics works with

self-adjoint operators acting on a separable Hilbert space.

Q: Can an operator have empty spectrum?

- A bounded $\Longrightarrow \sigma(A) \neq \emptyset$ (Liouville's theorem)
- $A = A^*$ (possibly unbounded) $\Longrightarrow \sigma(A) \neq \emptyset$ (the above claim + $r(A) = \|A\|$ for $A = A^*$)

Partial A: Well, if it exists, then it has to be an unbounded and non-self-adjoint operator.

Intermezzo - Volterra integral operators

- For $\mathcal{K} \in C([0,1]^2)$, put

$$Kf(x) := \int_0^x \mathcal{K}(x,y) f(y) \mathrm{d}y, \quad x \in [0,1].$$

Intermezzo - Volterra integral operators

- For $\mathcal{K} \in C([0,1]^2)$, put

$$Kf(x) := \int_0^x \mathcal{K}(x,y) f(y) \mathrm{d}y, \quad x \in [0,1].$$

- One has $K \in \mathcal{B}(C([0,1]))$ since $\|K\| \leq \|\mathcal{K}\|_\infty$.

Intermezzo - Volterra integral operators

- For $\mathcal{K} \in C([0,1]^2)$, put

$$Kf(x) := \int_0^x \mathcal{K}(x,y) f(y) \mathrm{d}y, \quad x \in [0,1].$$

- One has $K \in \mathcal{B}(C([0,1]))$ since $\|K\| \leq \|\mathcal{K}\|_\infty$.
- In fact, K is compact (Arzela–Ascoli).

Intermezzo - Volterra integral operators

- For $\mathcal{K} \in C([0,1]^2)$, put

$$Kf(x) := \int_0^x \mathcal{K}(x,y)f(y)\mathrm{d}y, \quad x \in [0,1].$$

- One has $K \in \mathcal{B}(C([0,1]))$ since $\|K\| \leq \|\mathcal{K}\|_\infty$.
- In fact, K is compact (Arzela–Ascoli).

Q: What is $\sigma(K)$?

Intermezzo - Volterra integral operators

- For $\mathcal{K} \in C([0,1]^2)$, put

$$Kf(x) := \int_0^x \mathcal{K}(x,y) f(y) \mathrm{d}y, \quad x \in [0,1].$$

- One has $K \in \mathcal{B}(C([0,1]))$ since $\|K\| \leq \|\mathcal{K}\|_\infty$.
- In fact, K is compact (Arzela–Ascoli).

Q: What is $\sigma(K)$?

- By induction in $n \in \mathbb{N}_0$, one verifies that

$$|K^n f(x)| \leq \frac{x^n \|\mathcal{K}\|_\infty^n}{n!} \|f\|_\infty, \quad \forall x \in [0,1].$$

Intermezzo - Volterra integral operators

- For $\mathcal{K} \in C([0,1]^2)$, put

$$Kf(x) := \int_0^x \mathcal{K}(x,y)f(y)\mathrm{d}y, \quad x \in [0,1].$$

- One has $K \in \mathcal{B}(C([0,1]))$ since $\|K\| \leq \|\mathcal{K}\|_\infty$.
- In fact, K is compact (Arzela–Ascoli).

Q: What is $\sigma(K)$?

- By induction in $n \in \mathbb{N}_0$, one verifies that

$$|K^n f(x)| \leq \frac{x^n \|\mathcal{K}\|_\infty^n}{n!} \|f\|_\infty, \quad \forall x \in [0,1].$$

- Therefore

$$\|K^n\| \leq \frac{\|\mathcal{K}\|_\infty^n}{n!}.$$

Intermezzo - Volterra integral operators

- For $\mathcal{K} \in C([0,1]^2)$, put

$$Kf(x) := \int_0^x \mathcal{K}(x,y)f(y)\mathrm{d}y, \quad x \in [0,1].$$

- One has $K \in \mathcal{B}(C([0,1]))$ since $\|K\| \leq \|\mathcal{K}\|_\infty$.
- In fact, K is compact (Arzela–Ascoli).

Q: What is $\sigma(K)$?

- By induction in $n \in \mathbb{N}_0$, one verifies that

$$|K^n f(x)| \leq \frac{x^n \|\mathcal{K}\|_\infty^n}{n!} \|f\|_\infty, \quad \forall x \in [0,1].$$

- Therefore

$$\|K^n\| \leq \frac{\|\mathcal{K}\|_\infty^n}{n!}.$$

- And thus

$$r(K) = \lim_{n\to\infty} \|K^n\|^{1/n} = 0.$$

Intermezzo - Volterra integral operators

- For $\mathcal{K} \in C([0,1]^2)$, put

$$Kf(x) := \int_0^x \mathcal{K}(x,y)f(y)\mathrm{d}y, \quad x \in [0,1].$$

- One has $K \in \mathcal{B}(C([0,1]))$ since $\|K\| \leq \|\mathcal{K}\|_\infty$.
- In fact, K is compact (Arzela–Ascoli).

Q: What is $\sigma(K)$?

- By induction in $n \in \mathbb{N}_0$, one verifies that

$$|K^n f(x)| \leq \frac{x^n \|\mathcal{K}\|_\infty^n}{n!} \|f\|_\infty, \quad \forall x \in [0,1].$$

- Therefore

$$\|K^n\| \leq \frac{\|\mathcal{K}\|_\infty^n}{n!}.$$

- And thus

$$r(K) = \lim_{n\to\infty} \|K^n\|^{1/n} = 0.$$

- Conclusion:

$$\sigma(K) = \{0\}.$$

Intermezzo - Inverse to the Volterra integral operator

- Fix $\lambda \in \mathbb{C}$ and consider $\mathcal{K}(x,y) := e^{\lambda(x-y)}$. Thus

$$K_\lambda f(x) = \int_0^x e^{\lambda(x-y)} f(y)\mathrm{d}y.$$

Intermezzo - Inverse to the Volterra integral operator

- Fix $\lambda \in \mathbb{C}$ and consider $\mathcal{K}(x,y) := e^{\lambda(x-y)}$. Thus

$$K_\lambda f(x) = \int_0^x e^{\lambda(x-y)} f(y)\mathrm{d}y.$$

- Kernel: Let $K_\lambda f = 0$ for some $f \in C([0,1])$. By differentiation, one gets

$$0 = \frac{\mathrm{d}}{\mathrm{d}x} K_\lambda f(x) = f(x) + \lambda K_\lambda f(x) = f(x), \quad \forall x \in (0,1).$$

Intermezzo - Inverse to the Volterra integral operator

- Fix $\lambda \in \mathbb{C}$ and consider $\mathcal{K}(x,y) := e^{\lambda(x-y)}$. Thus

$$K_\lambda f(x) = \int_0^x e^{\lambda(x-y)} f(y) \mathrm{d}y.$$

- Kernel: Let $K_\lambda f = 0$ for some $f \in C([0,1])$. By differentiation, one gets

$$0 = \frac{\mathrm{d}}{\mathrm{d}x} K_\lambda f(x) = f(x) + \lambda K_\lambda f(x) = f(x), \quad \forall x \in (0,1).$$

- Hence $\operatorname{Ker} K_\lambda = \{0\}$ and K_λ is invertible.

Intermezzo - Inverse to the Volterra integral operator

- Fix $\lambda \in \mathbb{C}$ and consider $\mathcal{K}(x,y) := e^{\lambda(x-y)}$. Thus

$$K_\lambda f(x) = \int_0^x e^{\lambda(x-y)} f(y) \mathrm{d}y.$$

- Kernel: Let $K_\lambda f = 0$ for some $f \in C([0,1])$. By differentiation, one gets

$$0 = \frac{\mathrm{d}}{\mathrm{d}x} K_\lambda f(x) = f(x) + \lambda K_\lambda f(x) = f(x), \quad \forall x \in (0,1).$$

- Hence $\mathrm{Ker}\, K_\lambda = \{0\}$ and K_λ is invertible.
- Range: It is easy to see that: $\mathrm{Ran}\, K_\lambda \subset \{g \in C^1([0,1]) \mid g(0) = 0\}$.

Intermezzo - Inverse to the Volterra integral operator

- Fix $\lambda \in \mathbb{C}$ and consider $\mathcal{K}(x,y) := e^{\lambda(x-y)}$. Thus
$$K_\lambda f(x) = \int_0^x e^{\lambda(x-y)} f(y)\mathrm{d}y.$$
- Kernel: Let $K_\lambda f = 0$ for some $f \in C([0,1])$. By differentiation, one gets
$$0 = \frac{\mathrm{d}}{\mathrm{d}x} K_\lambda f(x) = f(x) + \lambda K_\lambda f(x) = f(x), \quad \forall x \in (0,1).$$
- Hence $\operatorname{Ker} K_\lambda = \{0\}$ and K_λ is invertible.
- Range: It is easy to see that: $\operatorname{Ran} K_\lambda \underbrace{=}_{\text{in fact}} \{g \in C^1([0,1]) \mid g(0) = 0\}$.

Intermezzo - Inverse to the Volterra integral operator

- Fix $\lambda \in \mathbb{C}$ and consider $\mathcal{K}(x,y) := e^{\lambda(x-y)}$. Thus
$$K_\lambda f(x) = \int_0^x e^{\lambda(x-y)} f(y) \mathrm{d}y.$$
- Kernel: Let $K_\lambda f = 0$ for some $f \in C([0,1])$. By differentiation, one gets
$$0 = \frac{\mathrm{d}}{\mathrm{d}x} K_\lambda f(x) = f(x) + \lambda K_\lambda f(x) = f(x), \quad \forall x \in (0,1).$$
- Hence $\operatorname{Ker} K_\lambda = \{0\}$ and K_λ is invertible.
- Range: It is easy to see that: $\operatorname{Ran} K_\lambda \underbrace{=}_{\text{in fact}} \{g \in C^1([0,1]) \mid g(0) = 0\}$.
- The opposite inclusion: ? $\exists f \in C([0,1])$ such that $K_\lambda f = g$?

Intermezzo - Inverse to the Volterra integral operator

- Fix $\lambda \in \mathbb{C}$ and consider $\mathcal{K}(x,y) := e^{\lambda(x-y)}$. Thus

$$K_\lambda f(x) = \int_0^x e^{\lambda(x-y)} f(y) \mathrm{d}y.$$

- Kernel: Let $K_\lambda f = 0$ for some $f \in C([0,1])$. By differentiation, one gets

$$0 = \frac{\mathrm{d}}{\mathrm{d}x} K_\lambda f(x) = f(x) + \lambda K_\lambda f(x) = f(x), \quad \forall x \in (0,1).$$

- Hence $\mathrm{Ker}\, K_\lambda = \{0\}$ and K_λ is invertible.
- Range: It is easy to see that: $\mathrm{Ran}\, K_\lambda \underbrace{=}_{\text{in fact}} \{g \in C^1([0,1]) \mid g(0) = 0\}.$
- The opposite inclusion: ? $\exists f \in C([0,1])$ such that $K_\lambda f = g$?

$$\int_0^x e^{-\lambda y} f(y) \mathrm{d}y = e^{-\lambda x} g(x) \qquad \rightsquigarrow$$

Intermezzo - Inverse to the Volterra integral operator

- Fix $\lambda \in \mathbb{C}$ and consider $\mathcal{K}(x,y) := e^{\lambda(x-y)}$. Thus

$$K_\lambda f(x) = \int_0^x e^{\lambda(x-y)} f(y) \mathrm{d}y.$$

- Kernel: Let $K_\lambda f = 0$ for some $f \in C([0,1])$. By differentiation, one gets

$$0 = \frac{\mathrm{d}}{\mathrm{d}x} K_\lambda f(x) = f(x) + \lambda K_\lambda f(x) = f(x), \quad \forall x \in (0,1).$$

- Hence $\operatorname{Ker} K_\lambda = \{0\}$ and K_λ is invertible.
- Range: It is easy to see that: $\operatorname{Ran} K_\lambda \underbrace{=}_{\text{in fact}} \{g \in C^1([0,1]) \mid g(0) = 0\}.$
- The opposite inclusion: ? $\exists f \in C([0,1])$ such that $K_\lambda f = g$?

$$\int_0^x e^{-\lambda y} f(y) \mathrm{d}y = e^{-\lambda x} g(x) \qquad \rightsquigarrow \qquad e^{-\lambda x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{-\lambda x} g(x) \right),$$

Intermezzo - Inverse to the Volterra integral operator

- Fix $\lambda \in \mathbb{C}$ and consider $\mathcal{K}(x,y) := e^{\lambda(x-y)}$. Thus

$$K_\lambda f(x) = \int_0^x e^{\lambda(x-y)} f(y)\mathrm{d}y.$$

- Kernel: Let $K_\lambda f = 0$ for some $f \in C([0,1])$. By differentiation, one gets

$$0 = \frac{\mathrm{d}}{\mathrm{d}x} K_\lambda f(x) = f(x) + \lambda K_\lambda f(x) = f(x), \quad \forall x \in (0,1).$$

- Hence $\mathrm{Ker}\, K_\lambda = \{0\}$ and K_λ is invertible.
- Range: It is easy to see that: $\mathrm{Ran}\, K_\lambda \underbrace{=}_{\text{in fact}} \{g \in C^1([0,1]) \mid g(0) = 0\}$.
- The opposite inclusion: ? $\exists f \in C([0,1])$ such that $K_\lambda f = g$?

$$\int_0^x e^{-\lambda y} f(y)\mathrm{d}y = e^{-\lambda x} g(x) \qquad \rightsquigarrow \qquad e^{-\lambda x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\lambda x} g(x)\right),$$

$$f(x) = e^{\lambda x} \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\lambda x} g(x)\right)$$

Intermezzo - Inverse to the Volterra integral operator

- Fix $\lambda \in \mathbb{C}$ and consider $\mathcal{K}(x,y) := e^{\lambda(x-y)}$. Thus

$$K_\lambda f(x) = \int_0^x e^{\lambda(x-y)} f(y) \mathrm{d}y.$$

- Kernel: Let $K_\lambda f = 0$ for some $f \in C([0,1])$. By differentiation, one gets

$$0 = \frac{\mathrm{d}}{\mathrm{d}x} K_\lambda f(x) = f(x) + \lambda K_\lambda f(x) = f(x), \quad \forall x \in (0,1).$$

- Hence $\operatorname{Ker} K_\lambda = \{0\}$ and K_λ is invertible.
- Range: It is easy to see that: $\operatorname{Ran} K_\lambda \underbrace{=}_{\text{in fact}} \{g \in C^1([0,1]) \mid g(0) = 0\}$.
- The opposite inclusion: ? $\exists f \in C([0,1])$ such that $K_\lambda f = g$?

$$\int_0^x e^{-\lambda y} f(y) \mathrm{d}y = e^{-\lambda x} g(x) \qquad \rightsquigarrow \qquad e^{-\lambda x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\lambda x} g(x)\right),$$

$$f(x) = e^{\lambda x} \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\lambda x} g(x)\right) = g'(x) - \lambda g(x)$$

Intermezzo - Inverse to the Volterra integral operator

- Fix $\lambda \in \mathbb{C}$ and consider $\mathcal{K}(x,y) := e^{\lambda(x-y)}$. Thus

$$K_\lambda f(x) = \int_0^x e^{\lambda(x-y)} f(y) \mathrm{d}y.$$

- Kernel: Let $K_\lambda f = 0$ for some $f \in C([0,1])$. By differentiation, one gets

$$0 = \frac{\mathrm{d}}{\mathrm{d}x} K_\lambda f(x) = f(x) + \lambda K_\lambda f(x) = f(x), \quad \forall x \in (0,1).$$

- Hence $\operatorname{Ker} K_\lambda = \{0\}$ and K_λ is invertible.
- Range: It is easy to see that: $\operatorname{Ran} K_\lambda \underbrace{=}_{\text{in fact}} \{g \in C^1([0,1]) \mid g(0) = 0\}$.
- The opposite inclusion: ? $\exists f \in C([0,1])$ such that $K_\lambda f = g$?

$$\int_0^x e^{-\lambda y} f(y) \mathrm{d}y = e^{-\lambda x} g(x) \qquad \rightsquigarrow \qquad e^{-\lambda x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{-\lambda x} g(x) \right),$$

$$f(x) = e^{\lambda x} \frac{\mathrm{d}}{\mathrm{d}x} \left(e^{-\lambda x} g(x) \right) = g'(x) - \lambda g(x) \in C([0,1]).$$

An operator with empty spectrum

- Define

$$Tg := g', \qquad \operatorname{Dom} T := \operatorname{Ran} K_\lambda = \{g \in C^1([0,1]) \mid g(0) = 0\}.$$

An operator with empty spectrum

- Define

$$Tg := g', \qquad \operatorname{Dom} T := \operatorname{Ran} K_\lambda = \{g \in C^1([0,1]) \mid g(0) = 0\}.$$

- We've got, for arbitrary $\lambda \in \mathbb{C}$,

$$(K_\lambda)^{-1} g = g' - \lambda g = (T - \lambda) g$$

An operator with empty spectrum

- Define
$$Tg := g', \qquad \operatorname{Dom} T := \operatorname{Ran} K_\lambda = \{g \in C^1([0,1]) \mid g(0) = 0\}.$$
- We've got, for arbitrary $\lambda \in \mathbb{C}$,
$$(K_\lambda)^{-1} g = g' - \lambda g = (T - \lambda) g \qquad \rightsquigarrow \qquad K_\lambda = (T - \lambda)^{-1}.$$

An operator with empty spectrum

- Define
$$Tg := g', \qquad \operatorname{Dom} T := \operatorname{Ran} K_\lambda = \{g \in C^1([0,1]) \mid g(0) = 0\}.$$
- We've got, for arbitrary $\lambda \in \mathbb{C}$,
$$(K_\lambda)^{-1} g = g' - \lambda g = (T - \lambda) g \qquad \rightsquigarrow \qquad K_\lambda = (T - \lambda)^{-1}.$$
- Conclusion: $\sigma(T) = \emptyset$.

An operator with empty spectrum

- Define
$$Tg := g', \qquad \operatorname{Dom} T := \operatorname{Ran} K_\lambda = \{g \in C^1([0,1]) \mid g(0) = 0\}.$$
- We've got, for arbitrary $\lambda \in \mathbb{C}$,
$$(K_\lambda)^{-1} g = g' - \lambda g = (T - \lambda) g \qquad \rightsquigarrow \qquad K_\lambda = (T - \lambda)^{-1}.$$
- Conclusion: $\sigma(T) = \emptyset$.

Remark: The operator T acts on the Banach space $C([0,1])$. A similar example works on the Hilbert space $L^2(0,1)$:

$$Tg := g', \qquad \operatorname{Dom} T := \{g \in AC([0,1]) \mid g(0) = 0\},$$

where

$$AC([0,1]) = \{g \text{ a.c. on } [0,1] \mid g' \in L^2(0,1)\}.$$

Then $\sigma(T) = \emptyset$.

Contents

Parts of the spectrum - Introduction

- Spectrum:

$$\lambda \in \rho(T) \equiv \mathbb{C} \setminus \sigma(T) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad T - \lambda \text{ is invertible with bounded inverse.}$$

Parts of the spectrum - Introduction

- Spectrum:

$$\lambda \in \rho(T) \equiv \mathbb{C} \setminus \sigma(T) \quad \overset{\text{def}}{\Longleftrightarrow} \quad T - \lambda \text{ is invertible with bounded inverse.}$$

- Parts of the spectrum:

(i) $T - \lambda$ not injective $\quad \rightsquigarrow \quad \lambda \in \sigma_p(T)$

(ii) $T - \lambda$ injective but not surjective $\begin{cases} \mathrm{Ran}(T-\lambda) \text{ dense} & \rightsquigarrow \quad \lambda \in \sigma_c(T) \\ \mathrm{Ran}(T-\lambda) \text{ not dense} & \rightsquigarrow \quad \lambda \in \sigma_r(T) \end{cases}$

Parts of the spectrum - Introduction

- Spectrum:

$$\lambda \in \rho(T) \equiv \mathbb{C} \setminus \sigma(T) \quad \overset{\text{def}}{\Longleftrightarrow} \quad T - \lambda \text{ is invertible with bounded inverse.}$$

- Parts of the spectrum:

(i) $T - \lambda$ not injective $\quad \rightsquigarrow \quad \lambda \in \sigma_p(T)$

(ii) $T - \lambda$ injective but not surjective $\begin{cases} \mathrm{Ran}(T-\lambda) \text{ dense} & \rightsquigarrow \quad \lambda \in \sigma_c(T) \\ \mathrm{Ran}(T-\lambda) \text{ not dense} & \rightsquigarrow \quad \lambda \in \sigma_r(T) \end{cases}$

- Physical (QM) interpretation:

(a) $\sigma_p(T)$ - bound states
(b) $\sigma_c(T)$ - scattering states
(c) $\sigma_r(T)$ - no interpretation (???)

Parts of the spectrum - Introduction

- Spectrum:

$$\lambda \in \rho(T) \equiv \mathbb{C} \setminus \sigma(T) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad T - \lambda \text{ is invertible with bounded inverse.}$$

- Parts of the spectrum:

(i) $T - \lambda$ not injective $\quad \rightsquigarrow \quad \lambda \in \sigma_p(T)$

(ii) $T - \lambda$ injective but not surjective $\begin{cases} \mathrm{Ran}(T-\lambda) \text{ dense} & \rightsquigarrow \quad \lambda \in \sigma_c(T) \\ \mathrm{Ran}(T-\lambda) \text{ not dense} & \rightsquigarrow \quad \lambda \in \sigma_r(T) \end{cases}$

- Physical (QM) interpretation:

(a) $\sigma_p(T)$ - bound states
(b) $\sigma_c(T)$ - scattering states
(c) $\sigma_r(T)$ - no interpretation (???)

- (ad a) Intuition: “Bound states are not many - cannot form a continuous set” (orthogonal eigenvectors, separability).

Parts of the spectrum - Introduction

- Spectrum:

$$\lambda \in \rho(T) \equiv \mathbb{C} \setminus \sigma(T) \quad \overset{\text{def}}{\iff} \quad T - \lambda \text{ is invertible with bounded inverse.}$$

- Parts of the spectrum:

(i) $T - \lambda$ not injective $\quad \rightsquigarrow \quad \lambda \in \sigma_p(T)$

(ii) $T - \lambda$ injective but not surjective $\begin{cases} \operatorname{Ran}(T-\lambda) \text{ dense} & \rightsquigarrow \quad \lambda \in \sigma_c(T) \\ \operatorname{Ran}(T-\lambda) \text{ not dense} & \rightsquigarrow \quad \lambda \in \sigma_r(T) \end{cases}$

- Physical (QM) interpretation:

(a) $\sigma_p(T)$ - bound states
(b) $\sigma_c(T)$ - scattering states
(c) $\sigma_r(T)$ - no interpretation (???)

- (ad a) Intuition: “Bound states are not many - cannot form a continuous set” (orthogonal eigenvectors, separability).
- (ad c) Residual spectrum not present in QM: $T = T^* \implies \sigma_r(T) = \emptyset$.

Parts of the spectrum - Introduction

- Spectrum:

$$\lambda \in \rho(T) \equiv \mathbb{C} \setminus \sigma(T) \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad T - \lambda \text{ is invertible with bounded inverse.}$$

- Parts of the spectrum:

(i) $T - \lambda$ not injective $\quad \rightsquigarrow \quad \lambda \in \sigma_p(T)$

(ii) $T - \lambda$ injective but not surjective $\begin{cases} \operatorname{Ran}(T-\lambda) \text{ dense} & \rightsquigarrow \quad \lambda \in \sigma_c(T) \\ \operatorname{Ran}(T-\lambda) \text{ not dense} & \rightsquigarrow \quad \lambda \in \sigma_r(T) \end{cases}$

- Physical (QM) interpretation:

(a) $\sigma_p(T)$ - bound states
(b) $\sigma_c(T)$ - scattering states
(c) $\sigma_r(T)$ - no interpretation (???)

- (ad a) Intuition: “Bound states are not many - cannot form a continuous set” (orthogonal eigenvectors, separability).
- (ad c) Residual spectrum not present in QM: $T = T^* \quad \Longrightarrow \quad \sigma_r(T) = \emptyset$.

The above comments fail to hold if $T \neq T^*$.

Simple but important observation

- Easy exercise:

$$\operatorname{Ker} T^* = (\operatorname{Ran} T)^\perp.$$

Simple but important observation

- Easy exercise:
$$\operatorname{Ker} T^* = (\operatorname{Ran} T)^\perp.$$
- Corollaries of $\operatorname{Ker}(T^* - \overline{\lambda}) = (\operatorname{Ran}(T - \lambda))^\perp$:
$$\begin{aligned} (i) \quad & \lambda \in \sigma_r(T) \quad \implies \quad \overline{\lambda} \in \sigma_p(T^*) \\ (ii) \quad & \lambda \in \sigma_p(T) \quad \implies \quad \overline{\lambda} \in \sigma_p(T^*) \cup \sigma_r(T^*) \end{aligned}$$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$

- $\|R\| = \|L\| = 1$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define
$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$
- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}$.

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$

- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}.$
- $L^* = R.$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define
$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$
- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}.$
- $L^* = R.$
-
$$Lx = \lambda x$$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$

- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}.$
- $L^* = R.$
-

$$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n$$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$

- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}$.
- $L^* = R$.
- $$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1$$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define
$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$
- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}.$
- $L^* = R.$
- $$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1 \in \ell^2(\mathbb{N}) \quad \Leftrightarrow \quad |\lambda| < 1$$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define
$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$
- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}$.
- $L^* = R$.
- $$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1 \in \ell^2(\mathbb{N}) \quad \Leftrightarrow \quad |\lambda| < 1$$
- Conclusion: $\sigma_p(L) = \mathbb{D}$ and $\sigma(L) = \overline{\mathbb{D}}$.

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define
$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$
- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}$.
- $L^* = R$.
-
$$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1 \in \ell^2(\mathbb{N}) \quad \Leftrightarrow \quad |\lambda| < 1$$
- Conclusion: $\sigma_p(L) = \mathbb{D}$ and $\sigma(L) = \overline{\mathbb{D}}$.
-
$$Rx = \lambda x$$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define
$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \quad \text{and} \quad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$
- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}$.
- $L^* = R$.
- $$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1 \in \ell^2(\mathbb{N}) \quad \Leftrightarrow \quad |\lambda| < 1$$
- Conclusion: $\sigma_p(L) = \mathbb{D}$ and $\sigma(L) = \overline{\mathbb{D}}$.
- $$Rx = \lambda x \quad \Leftrightarrow \quad x_1 = 0 \ \wedge \ x_{n-1} = \lambda x_n$$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$

- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}$.
- $L^* = R$.
-

$$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1 \in \ell^2(\mathbb{N}) \quad \Leftrightarrow \quad |\lambda| < 1$$

- Conclusion: $\sigma_p(L) = \mathbb{D}$ and $\sigma(L) = \overline{\mathbb{D}}$.
-

$$Rx = \lambda x \quad \Leftrightarrow \quad x_1 = 0 \wedge x_{n-1} = \lambda x_n \quad \Leftrightarrow \quad x = 0$$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \quad \text{and} \quad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$

- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}$.
- $L^* = R$.
-

$$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1 \in \ell^2(\mathbb{N}) \quad \Leftrightarrow \quad |\lambda| < 1$$

- Conclusion: $\sigma_p(L) = \mathbb{D}$ and $\sigma(L) = \overline{\mathbb{D}}$.
-

$$Rx = \lambda x \quad \Leftrightarrow \quad x_1 = 0 \wedge x_{n-1} = \lambda x_n \quad \Leftrightarrow \quad x = 0$$

- Conclusion: $\sigma_p(R) = \emptyset$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$

- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}$.
- $L^* = R$.
-

$$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1 \in \ell^2(\mathbb{N}) \quad \Leftrightarrow \quad |\lambda| < 1$$

- Conclusion: $\sigma_p(L) = \mathbb{D}$ and $\sigma(L) = \overline{\mathbb{D}}$.
-

$$Rx = \lambda x \quad \Leftrightarrow \quad x_1 = 0 \wedge x_{n-1} = \lambda x_n \quad \Leftrightarrow \quad x = 0$$

- Conclusion: $\sigma_p(R) = \emptyset \quad \Rightarrow \quad \sigma_r(L) = \emptyset$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define
$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$
- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}$.
- $L^* = R$.
- $$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1 \in \ell^2(\mathbb{N}) \quad \Leftrightarrow \quad |\lambda| < 1$$
- Conclusion: $\sigma_p(L) = \mathbb{D}$ and $\sigma(L) = \overline{\mathbb{D}}$.
- $$Rx = \lambda x \quad \Leftrightarrow \quad x_1 = 0 \wedge x_{n-1} = \lambda x_n \quad \Leftrightarrow \quad x = 0$$
- Conclusion: $\sigma_p(R) = \emptyset \quad \Rightarrow \quad \sigma_r(L) = \emptyset \quad \Rightarrow \quad \sigma_c(L) = \mathbb{T}$.

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \quad \text{and} \quad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$

- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}$.
- $L^* = R$.
-

$$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1 \in \ell^2(\mathbb{N}) \quad \Leftrightarrow \quad |\lambda| < 1$$

- Conclusion: $\sigma_p(L) = \mathbb{D}$ and $\sigma(L) = \overline{\mathbb{D}}$.
-

$$Rx = \lambda x \quad \Leftrightarrow \quad x_1 = 0 \wedge x_{n-1} = \lambda x_n \quad \Leftrightarrow \quad x = 0$$

- Conclusion: $\sigma_p(R) = \emptyset \quad \Rightarrow \quad \sigma_r(L) = \emptyset \quad \Rightarrow \quad \sigma_c(L) = \mathbb{T}$.
-

$$\sigma_p(L) = \mathbb{D} \quad \Rightarrow \quad \mathbb{D} \subset \sigma_p(R) \cup \sigma_r(R)$$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define
$$R(x_1,x_2,\dots) := (0,x_1,x_2,\dots) \qquad \text{and} \qquad L(x_1,x_2,\dots) := (x_2,x_3,\dots).$$
- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}$.
- $L^* = R$.
- $$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1 \in \ell^2(\mathbb{N}) \quad \Leftrightarrow \quad |\lambda| < 1$$
- Conclusion: $\sigma_p(L) = \mathbb{D}$ and $\sigma(L) = \overline{\mathbb{D}}$.
- $$Rx = \lambda x \quad \Leftrightarrow \quad x_1 = 0 \,\wedge\, x_{n-1} = \lambda x_n \quad \Leftrightarrow \quad x = 0$$
- Conclusion: $\sigma_p(R) = \emptyset \quad \Rightarrow \quad \sigma_r(L) = \emptyset \quad \Rightarrow \quad \sigma_c(L) = \mathbb{T}$.
- $$\sigma_p(L) = \mathbb{D} \quad \Rightarrow \quad \mathbb{D} \subset \underbrace{\sigma_p(R)}_{=\emptyset} \cup \sigma_r(R) = \sigma_r(R).$$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define
$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$
- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}.$
- $L^* = R.$
- $$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1 \in \ell^2(\mathbb{N}) \quad \Leftrightarrow \quad |\lambda| < 1$$
- Conclusion: $\sigma_p(L) = \mathbb{D}$ and $\sigma(L) = \overline{\mathbb{D}}$.
- $$Rx = \lambda x \quad \Leftrightarrow \quad x_1 = 0 \wedge x_{n-1} = \lambda x_n \quad \Leftrightarrow \quad x = 0$$
- Conclusion: $\sigma_p(R) = \emptyset \quad \Rightarrow \quad \sigma_r(L) = \emptyset \quad \Rightarrow \quad \sigma_c(L) = \mathbb{T}$.
- $$\sigma_p(L) = \mathbb{D} \quad \Rightarrow \quad \mathbb{D} \subset \underbrace{\sigma_p(R)}_{=\emptyset} \cup \sigma_r(R) = \sigma_r(R).$$
- Conclusion: $\sigma_c(R) = \mathbb{T}$ (since $\mathbb{T} \ni \lambda \in \sigma_r(R) \quad \Rightarrow \quad \overline{\lambda} \in \sigma_p(L) = \mathbb{D}$).

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define
$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$
- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}$.
- $L^* = R$.
- $$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1 \in \ell^2(\mathbb{N}) \quad \Leftrightarrow \quad |\lambda| < 1$$
- Conclusion: $\sigma_p(L) = \mathbb{D}$ and $\sigma(L) = \overline{\mathbb{D}}$.
- $$Rx = \lambda x \quad \Leftrightarrow \quad x_1 = 0 \,\wedge\, x_{n-1} = \lambda x_n \quad \Leftrightarrow \quad x = 0$$
- Conclusion: $\sigma_p(R) = \emptyset \quad \Rightarrow \quad \sigma_r(L) = \emptyset \quad \Rightarrow \quad \sigma_c(L) = \mathbb{T}$.
- $$\sigma_p(L) = \mathbb{D} \quad \Rightarrow \quad \mathbb{D} \subset \underbrace{\sigma_p(R)}_{=\emptyset} \cup \sigma_r(R) = \sigma_r(R).$$
- Conclusion: $\sigma_c(R) = \mathbb{T}$ (since $\mathbb{T} \ni \lambda \in \sigma_r(R) \quad \Rightarrow \quad \overline{\lambda} \in \sigma_p(L) = \mathbb{D}$).
- In total:

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \quad \text{and} \quad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$

- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}$.
- $L^* = R$.
- $$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1 \in \ell^2(\mathbb{N}) \quad \Leftrightarrow \quad |\lambda| < 1$$
- Conclusion: $\sigma_p(L) = \mathbb{D}$ and $\sigma(L) = \overline{\mathbb{D}}$.
- $$Rx = \lambda x \quad \Leftrightarrow \quad x_1 = 0 \wedge x_{n-1} = \lambda x_n \quad \Leftrightarrow \quad x = 0$$
- Conclusion: $\sigma_p(R) = \emptyset \quad \Rightarrow \quad \sigma_r(L) = \emptyset \quad \Rightarrow \quad \sigma_c(L) = \mathbb{T}$.
- $$\sigma_p(L) = \mathbb{D} \quad \Rightarrow \quad \mathbb{D} \subset \underbrace{\sigma_p(R)}_{=\emptyset} \cup \sigma_r(R) = \sigma_r(R).$$
- Conclusion: $\sigma_c(R) = \mathbb{T}$ (since $\mathbb{T} \ni \lambda \in \sigma_r(R) \quad \Rightarrow \quad \overline{\lambda} \in \sigma_p(L) = \mathbb{D}$).
- In total:

$$\sigma_p(L) = \mathbb{D}, \quad \sigma_c(L) = \mathbb{T}, \quad \sigma_r(L) = \emptyset$$

Two good candidates for counter-examples

- On $\ell^2(\mathbb{N})$, define
$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \qquad \text{and} \qquad L(x_1, x_2, \dots) := (x_2, x_3, \dots).$$
- $\|R\| = \|L\| = 1 \quad \Rightarrow \quad \sigma(R), \sigma(L) \subset \overline{\mathbb{D}}$.
- $L^* = R$.
- $$Lx = \lambda x \quad \Leftrightarrow \quad x_{n+1} = \lambda x_n \quad \Leftrightarrow \quad x_{n+1} = \lambda^n x_1 \in \ell^2(\mathbb{N}) \quad \Leftrightarrow \quad |\lambda| < 1$$
- Conclusion: $\sigma_p(L) = \mathbb{D}$ and $\sigma(L) = \overline{\mathbb{D}}$.
- $$Rx = \lambda x \quad \Leftrightarrow \quad x_1 = 0 \;\wedge\; x_{n-1} = \lambda x_n \quad \Leftrightarrow \quad x = 0$$
- Conclusion: $\sigma_p(R) = \emptyset \quad \Rightarrow \quad \sigma_r(L) = \emptyset \quad \Rightarrow \quad \sigma_c(L) = \mathbb{T}$.
- $$\sigma_p(L) = \mathbb{D} \quad \Rightarrow \quad \mathbb{D} \subset \underbrace{\sigma_p(R)}_{=\emptyset} \cup \sigma_r(R) = \sigma_r(R).$$
- Conclusion: $\sigma_c(R) = \mathbb{T}$ (since $\mathbb{T} \ni \lambda \in \sigma_r(R) \quad \Rightarrow \quad \overline{\lambda} \in \sigma_p(L) = \mathbb{D}$).
- In total:
$$\sigma_p(L) = \mathbb{D}, \quad \sigma_c(L) = \mathbb{T}, \quad \sigma_r(L) = \emptyset$$
$$\sigma_p(R) = \emptyset, \quad \sigma_c(R) = \mathbb{T}, \quad \sigma_r(R) = \mathbb{D}$$

Contents

Where NSA operators can appear

- **Operator and Spectral Theory:** general properties of NSA operators possessing additional symmetry, real spectrum, similarity to SA operators, basiness of eigenvectors (completeness, Schauder, Riezs), perturbation theory, spectral approximation, pseudospectral analysis, generalized eigenvalue problems - matrix pencils, ...
- **Mathematical Physics:** optics, damped systems, quantum resonances, hydro- and magnetohydrodynamics, superconductivity, graphene, NSA QM - $\mathcal{PT}$-symmetry (?), ...
- **Approximation theory and OGPs:** asymptotic analysis, zero distribution of OGPs with non-standard parameters, complex orthogonality, Riemann–Hilbert problem, real-rootedness of various polynomial families, Multiple OGPs, Hermite–Padé approximants, number theory, ...
- **Complex analysis:** location of zeros, reality of zeros of entire functions - special functions, Laguerre–Pólya class, ...
- **Numerical mathematics:** reliability of approximations, pseudospectrum, stability, ...
- And much more...

Where NSA operators can appear

- **Operator and Spectral Theory:** general properties of NSA operators possessing additional symmetry, real spectrum, similarity to SA operators, basiness of eigenvectors (completeness, Schauder, Riezs), perturbation theory, spectral approximation, pseudospectral analysis, generalized eigenvalue problems - matrix pencils, ...
- **Mathematical Physics:** optics, damped systems, quantum resonances, hydro- and magnetohydrodynamics, superconductivity, graphene, NSA QM - $\mathcal{PT}$-symmetry (?), ...
- **Approximation theory and OGPs:** asymptotic analysis, zero distribution of OGPs with non-standard parameters, complex orthogonality, Riemann–Hilbert problem, real-rootedness of various polynomial families, Multiple OGPs, Hermite–Padé approximants, number theory, ...
- **Complex analysis:** location of zeros, reality of zeros of entire functions - special functions, Laguerre–Pólya class, ...
- **Numerical mathematics:** reliability of approximations, pseudospectrum, stability, ...
- And much more...

Lets look briefly on two topics...

NSA operators with purely real spectrum

- We saw examples of NSA operators with properties being “far” from those of SA operators.

NSA operators with purely real spectrum

- We saw examples of NSA operators with properties being “far” from those of SA operators.
- NSA operators which in a sense resembles properties of SA operators (with no obvious reason) are of special interest.

NSA operators with purely real spectrum

- We saw examples of NSA operators with properties being “far” from those of SA operators.
- NSA operators which in a sense resembles properties of SA operators (with no obvious reason) are of special interest.
- Usually, the investigation focuses on operators from well-understood particular families like:

NSA operators with purely real spectrum

- We saw examples of NSA operators with properties being “far” from those of SA operators.
- NSA operators which in a sense resembles properties of SA operators (with no obvious reason) are of special interest.
- Usually, the investigation focuses on operators from well-understood particular families like: Schrödinger, Sturm–Liouville, Dirac, Jacobi, Hankel, Toeplitz, locally Toeplitz, KMS, various integral operators, BD pencils,...

NSA operators with purely real spectrum

- We saw examples of NSA operators with properties being “far” from those of SA operators.
- NSA operators which in a sense resembles properties of SA operators (with no obvious reason) are of special interest.
- Usually, the investigation focuses on operators from well-understood particular families like: Schrödinger, Sturm–Liouville, Dirac, Jacobi, Hankel, Toeplitz, locally Toeplitz, KMS, various integral operators, BD pencils,...

Q: Under what conditions can have a NSA operator/matrix (from one of the above class) purely real spectrum?

NSA operators with purely real spectrum

- We saw examples of NSA operators with properties being “far” from those of SA operators.
- NSA operators which in a sense resembles properties of SA operators (with no obvious reason) are of special interest.
- Usually, the investigation focuses on operators from well-understood particular families like:
  Schrödinger, Sturm–Liouville, Dirac, Jacobi, Hankel, Toeplitz, locally Toeplitz, KMS, various integral operators, BD pencils,...

Q: Under what conditions can have a NSA operator/matrix (from one of the above class) purely real spectrum?

- Ignoring some results (perturbation-like, Krein or Pontryagin spaces,...), we can say that there are not many (general) sufficient conditions guaranteeing reality of the spectrum.

NSA operators with purely real spectrum

- We saw examples of NSA operators with properties being "far" from those of SA operators.
- NSA operators which in a sense resembles properties of SA operators (with no obvious reason) are of special interest.
- Usually, the investigation focuses on operators from well-understood particular families like:
  Schrödinger, Sturm–Liouville, Dirac, Jacobi, Hankel, Toeplitz, locally Toeplitz, KMS, various integral operators, BD pencils,...

Q: Under what conditions can have a NSA operator/matrix (from one of the above class) purely real spectrum?

- Ignoring some results (perturbation-like, Krein or Pontryagin spaces,...), we can say that there are not many (general) sufficient conditions guaranteeing reality of the spectrum.
- Instead, there exist several very involved concrete examples:

i) $-\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + \mathrm{i}x^3$ on $L^2(\mathbb{R})$, ii) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sin(x)\dfrac{\mathrm{d}}{\mathrm{d}x}\right) + \dfrac{\mathrm{d}}{\mathrm{d}x}$ on $L^2(-\pi,\pi)$,

iii) some Jacobi matrices.

Toeplitz matrices with real eigenvalues

- Toeplitz matrix:

$$T_n(b) = \left(a_{j-k}\right)_{j,k=0}^{n-1} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \dots & a_{-n+2} \\ a_2 & a_1 & a_0 & \dots & a_{-n+3} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_0 \end{pmatrix},$$

where $a_n \in \mathbb{C}$.

Toeplitz matrices with real eigenvalues

- Toeplitz matrix:

$$T_n(b) = \left(a_{j-k}\right)_{j,k=0}^{n-1} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \dots & a_{-n+2} \\ a_2 & a_1 & a_0 & \dots & a_{-n+3} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_0 \end{pmatrix},$$

where $a_n \in \mathbb{C}$.

- Symbol of $T(b)$:

$$b(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

Toeplitz matrices with real eigenvalues

- Toeplitz matrix:

$$T_n(b) = \left(a_{j-k}\right)_{j,k=0}^{n-1} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \dots & a_{-n+2} \\ a_2 & a_1 & a_0 & \dots & a_{-n+3} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_0 \end{pmatrix},$$

  where $a_n \in \mathbb{C}$.
- Symbol of $T(b)$:

$$b(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

Theorem:

Assume (for simplicity!) that b is a Laurent polynomial. Then

$$\sigma(T_n(b)) \subset \mathbb{R}, \quad \forall n \in \mathbb{N} \quad \iff \quad b^{-1}(\mathbb{R}) \text{ contains a Jordan curve.}$$

Toeplitz matrices - explicit examples

- Tridiagonal Toeplitz matrix:

$$b(z) = z^{-1} + az, \qquad (a \in \mathbb{C} \setminus \{0\}).$$

Then

$$\sigma(T_n(b)) \subset \mathbb{R}, \quad \forall n \in \mathbb{N} \quad \Longleftrightarrow \quad a > 0.$$

Toeplitz matrices - explicit examples

- Tridiagonal Toeplitz matrix:

$$b(z) = z^{-1} + az, \qquad (a \in \mathbb{C} \setminus \{0\}).$$

Then

$$\sigma(T_n(b)) \subset \mathbb{R}, \quad \forall n \in \mathbb{N} \quad \Longleftrightarrow \quad a > 0.$$

- Four-diagonal Toeplitz matrix:

$$b(z) = z^{-1} + az + bz^2, \qquad (a \in \mathbb{C},\ b \in \mathbb{C} \setminus \{0\}).$$

Then

$$\sigma(T_n(b)) \subset \mathbb{R}, \quad \forall n \in \mathbb{N} \quad \Longleftrightarrow \quad a^3 \geq 27b^2 > 0.$$

Toeplitz matrices - explicit examples

- Tridiagonal Toeplitz matrix:

$$b(z) = z^{-1} + az, \qquad (a \in \mathbb{C} \setminus \{0\}).$$

Then

$$\sigma(T_n(b)) \subset \mathbb{R}, \quad \forall n \in \mathbb{N} \quad \Longleftrightarrow \quad a > 0.$$

- Four-diagonal Toeplitz matrix:

$$b(z) = z^{-1} + az + bz^2, \qquad (a \in \mathbb{C},\ b \in \mathbb{C} \setminus \{0\}).$$

Then

$$\sigma(T_n(b)) \subset \mathbb{R}, \quad \forall n \in \mathbb{N} \quad \Longleftrightarrow \quad a^3 \geq 27b^2 > 0.$$

- A banded Toeplitz matrix:

$$b(z) = z^{-r}\,(1 + az)^{r+s}, \qquad (r, s \in \mathbb{N},\ a \in \mathbb{R} \setminus \{0\}).$$

Then $\sigma(T_n(b)) \subset \mathbb{R},\ \forall n \in \mathbb{N}$.

Toeplitz matrices - explicit examples

- Tridiagonal Toeplitz matrix:

$$b(z) = z^{-1} + az, \qquad (a \in \mathbb{C} \setminus \{0\}).$$

Then

$$\sigma(T_n(b)) \subset \mathbb{R}, \quad \forall n \in \mathbb{N} \quad \Longleftrightarrow \quad a > 0.$$

- Four-diagonal Toeplitz matrix:

$$b(z) = z^{-1} + az + bz^2, \qquad (a \in \mathbb{C},\ b \in \mathbb{C} \setminus \{0\}).$$

Then

$$\sigma(T_n(b)) \subset \mathbb{R}, \quad \forall n \in \mathbb{N} \quad \Longleftrightarrow \quad a^3 \geq 27b^2 > 0.$$

- A banded Toeplitz matrix:

$$b(z) = z^{-r}\left(1 + az\right)^{r+s}, \qquad (r, s \in \mathbb{N},\ a \in \mathbb{R} \setminus \{0\}).$$

Then $\sigma(T_n(b)) \subset \mathbb{R},\ \forall n \in \mathbb{N}$.

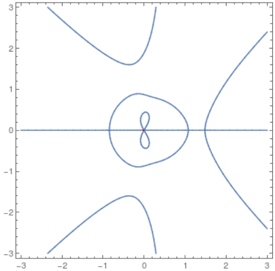
- A non-banded Toeplitz matrix:

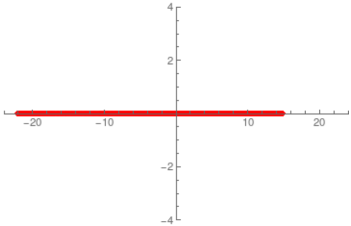
$$b(z) = e^{az} + e^{b/z}, \qquad (ab > 0).$$

Then $\sigma(T_n(b)) \subset \mathbb{R},\ \forall n \in \mathbb{N}$.

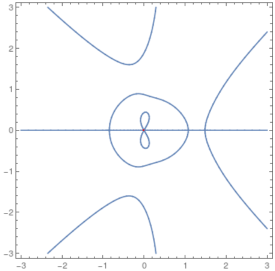
Toeplitz matrices - numerical examples

$$b(z) = z^{-3} - z^{-2} + 7z^{-1} + 9z - 2z^2 + 2z^3 - z^4$$

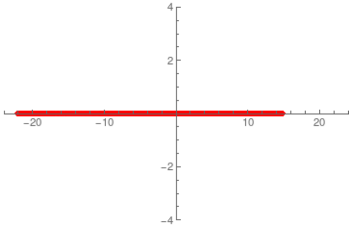


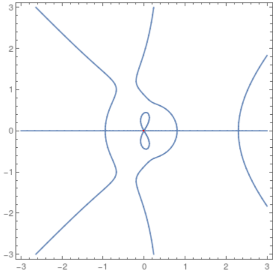


Toeplitz matrices - numerical examples

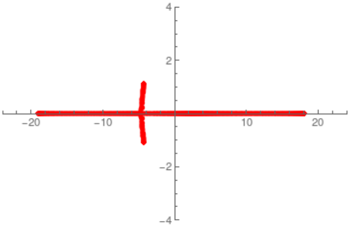


$$b(z) = z^{-3} - z^{-2} + 7z^{-1} + 9z - 2z^2 + 2z^3 - z^4$$





$$b(z) = z^{-3} - z^{-2} + 7z^{-1} + 9z + 2z^2 + 2z^3 - z^4$$



Spectral approximation

- Let $\mathcal{H}$ be separable Hilbert space, $\{e_n\}_{n=1}^{\infty}$ ONB of $\mathcal{H}$, and $A \in \mathcal{B}(\mathcal{H})$.

Spectral approximation

- Let $\mathcal{H}$ be separable Hilbert space, $\{e_n\}_{n=1}^{\infty}$ ONB of $\mathcal{H}$, and $A \in \mathcal{B}(\mathcal{H})$.
- Let P_n be OG projection onto $\mathcal{H}_n := \operatorname{span}\{e_1, \dots, e_n\}$. Put $A_n := P_n A \upharpoonright \mathcal{H}_n$.

Spectral approximation

- Let $\mathcal{H}$ be separable Hilbert space, $\{e_n\}_{n=1}^{\infty}$ ONB of $\mathcal{H}$, and $A \in \mathcal{B}(\mathcal{H})$.
- Let P_n be OG projection onto $\mathcal{H}_n := \mathrm{span}\{e_1, \dots, e_n\}$. Put $A_n := P_n A \restriction \mathcal{H}_n$.

Q: Does $\sigma(A_n)$ approximate $\sigma(A)$ for $n \to \infty$?

Spectral approximation

- Let $\mathcal{H}$ be separable Hilbert space, $\{e_n\}_{n=1}^{\infty}$ ONB of $\mathcal{H}$, and $A \in \mathcal{B}(\mathcal{H})$.
- Let P_n be OG projection onto $\mathcal{H}_n := \operatorname{span}\{e_1, \dots, e_n\}$. Put $A_n := P_n A \upharpoonright \mathcal{H}_n$.

Q: Does $\sigma(A_n)$ approximate $\sigma(A)$ for $n \to \infty$?

- Limit points of evls:

$$\Lambda(A) := \{\lambda \in \mathbb{C} \mid \liminf_{n\to\infty} \operatorname{dist}(\lambda, \sigma(A_n)) = 0\}.$$

Spectral approximation

- Let $\mathcal{H}$ be separable Hilbert space, $\{e_n\}_{n=1}^{\infty}$ ONB of $\mathcal{H}$, and $A \in \mathcal{B}(\mathcal{H})$.
- Let P_n be OG projection onto $\mathcal{H}_n := \mathrm{span}\{e_1, \dots, e_n\}$. Put $A_n := P_n A \upharpoonright \mathcal{H}_n$.

Q: Does $\sigma(A_n)$ approximate $\sigma(A)$ for $n \to \infty$?

- Limit points of evls:

$$\Lambda(A) := \{\lambda \in \mathbb{C} \mid \liminf_{n\to\infty} \mathrm{dist}(\lambda, \sigma(A_n)) = 0\}.$$

Q: Is any relation between $\sigma(A)$ and $\Lambda(A)$?

Spectral approximation

- Let $\mathcal{H}$ be separable Hilbert space, $\{e_n\}_{n=1}^{\infty}$ ONB of $\mathcal{H}$, and $A \in \mathcal{B}(\mathcal{H})$.
- Let P_n be OG projection onto $\mathcal{H}_n := \operatorname{span}\{e_1, \ldots, e_n\}$. Put $A_n := P_n A \restriction \mathcal{H}_n$.

Q: Does $\sigma(A_n)$ approximate $\sigma(A)$ for $n \to \infty$?

- Limit points of evls:

$$\Lambda(A) := \{\lambda \in \mathbb{C} \mid \liminf_{n\to\infty} \operatorname{dist}(\lambda, \sigma(A_n)) = 0\}.$$

Q: Is any relation between $\sigma(A)$ and $\Lambda(A)$?

A: Not in general. But...

Claim:

$$A = A^* \quad \Rightarrow \quad \sigma(A) \subset \Lambda(A).$$

Claim:

$$A = A^* \quad \Rightarrow \quad \sigma(A) \subset \Lambda(A).$$

Proof:

Claim:

$$A = A^* \quad \Rightarrow \quad \sigma(A) \subset \Lambda(A).$$

Proof:

- Let $\lambda \notin \Lambda(A)$. Then $\exists \epsilon > 0$ such that $B_\epsilon(\lambda) \cap \sigma(A_n) = \emptyset, \ \forall n > n_0$.

Claim:

$$A = A^* \quad \Rightarrow \quad \sigma(A) \subset \Lambda(A).$$

Proof:

- Let $\lambda \notin \Lambda(A)$. Then $\exists \epsilon > 0$ such that $B_\epsilon(\lambda) \cap \sigma(A_n) = \emptyset, \ \forall n > n_0$.
- Important ingredient: For any $T = T^*$, one has

$$\|(T-\lambda)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}, \quad \lambda \notin \sigma(T).$$

Claim:

$$A = A^* \quad \Rightarrow \quad \sigma(A) \subset \Lambda(A).$$

Proof:

- Let $\lambda \notin \Lambda(A)$. Then $\exists \epsilon > 0$ such that $B_\epsilon(\lambda) \cap \sigma(A_n) = \emptyset, \ \forall n > n_0$.
- Important ingredient: For any $T = T^*$, one has

$$\|(T-\lambda)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}, \quad \lambda \notin \sigma(T).$$

- So in this case: $\|(A_n - \lambda)^{-1}\| \leq 1/\epsilon, \ \forall n > n_0$.

Claim:

$$A = A^* \quad \Rightarrow \quad \sigma(A) \subset \Lambda(A).$$

Proof:

- Let $\lambda \notin \Lambda(A)$. Then $\exists \epsilon > 0$ such that $B_\epsilon(\lambda) \cap \sigma(A_n) = \emptyset, \ \forall n > n_0$.
- Important ingredient: For any $T = T^*$, one has

$$\|(T - \lambda)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}, \quad \lambda \notin \sigma(T).$$

- So in this case: $\|(A_n - \lambda)^{-1}\| \leq 1/\epsilon, \ \forall n > n_0$.
- Put

$$B_n := (A_n - \lambda)^{-1} \oplus I \in B_{1/\epsilon} \subset \mathcal{B}(\mathcal{H}) \text{ (weakly precompact).}$$

Claim:

$$A = A^* \quad \Rightarrow \quad \sigma(A) \subset \Lambda(A).$$

Proof:

- Let $\lambda \notin \Lambda(A)$. Then $\exists \epsilon > 0$ such that $B_\epsilon(\lambda) \cap \sigma(A_n) = \emptyset, \ \forall n > n_0$.
- Important ingredient: For any $T = T^*$, one has

$$\|(T - \lambda)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}, \quad \lambda \notin \sigma(T).$$

- So in this case: $\|(A_n - \lambda)^{-1}\| \leq 1/\epsilon, \ \forall n > n_0$.
- Put

$$B_n := (A_n - \lambda)^{-1} \oplus I \in B_{1/\epsilon} \subset \mathcal{B}(\mathcal{H}) \text{ (weakly precompact).}$$

- $\exists B \in \mathcal{B}(\mathcal{H})$ such that $B_n \xrightarrow{w} B$ (cheating slightly).

Claim:

$$A = A^* \quad \Rightarrow \quad \sigma(A) \subset \Lambda(A).$$

Proof:

- Let $\lambda \notin \Lambda(A)$. Then $\exists \epsilon > 0$ such that $B_\epsilon(\lambda) \cap \sigma(A_n) = \emptyset, \ \forall n > n_0$.
- Important ingredient: For any $T = T^*$, one has

$$\|(T - \lambda)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}, \quad \lambda \notin \sigma(T).$$

- So in this case: $\|(A_n - \lambda)^{-1}\| \leq 1/\epsilon, \ \forall n > n_0$.
- Put

$$B_n := (A_n - \lambda)^{-1} \oplus I \in B_{1/\epsilon} \subset \mathcal{B}(\mathcal{H}) \text{ (weakly precompact).}$$

- $\exists B \in \mathcal{B}(\mathcal{H})$ such that $B_n \xrightarrow{w} B$ (cheating slightly).
- It is easy to verify that $P_n \xrightarrow{s} I$. Consequently, $AP_n \xrightarrow{s} A$.

Claim:

$$A = A^* \quad \Rightarrow \quad \sigma(A) \subset \Lambda(A).$$

Proof:

- Let $\lambda \notin \Lambda(A)$. Then $\exists \epsilon > 0$ such that $B_\epsilon(\lambda) \cap \sigma(A_n) = \emptyset, \ \forall n > n_0$.
- Important ingredient: For any $T = T^*$, one has

$$\|(T-\lambda)^{-1}\| = \frac{1}{\mathrm{dist}(\lambda, \sigma(T))}, \quad \lambda \notin \sigma(T).$$

- So in this case: $\|(A_n - \lambda)^{-1}\| \leq 1/\epsilon, \ \forall n > n_0$.
- Put

$$B_n := (A_n - \lambda)^{-1} \oplus I \in B_{1/\epsilon} \subset \mathcal{B}(\mathcal{H}) \text{ (weakly precompact).}$$

- $\exists B \in \mathcal{B}(\mathcal{H})$ such that $B_n \xrightarrow{w} B$ (cheating slightly).
- It is easy to verify that $P_n \xrightarrow{s} I$. Consequently, $AP_n \xrightarrow{s} A$.
- Taking the weak limits in

$$(A - \lambda)P_n B_n = P_n,$$

Claim:

$$A = A^* \quad \Rightarrow \quad \sigma(A) \subset \Lambda(A).$$

Proof:

- Let $\lambda \notin \Lambda(A)$. Then $\exists \epsilon > 0$ such that $B_\epsilon(\lambda) \cap \sigma(A_n) = \emptyset, \ \forall n > n_0$.
- Important ingredient: For any $T = T^*$, one has

$$\|(T - \lambda)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}, \quad \lambda \notin \sigma(T).$$

- So in this case: $\|(A_n - \lambda)^{-1}\| \leq 1/\epsilon, \ \forall n > n_0$.
- Put

$$B_n := (A_n - \lambda)^{-1} \oplus I \in B_{1/\epsilon} \subset \mathcal{B}(\mathcal{H}) \text{ (weakly precompact).}$$

- $\exists B \in \mathcal{B}(\mathcal{H})$ such that $B_n \xrightarrow{w} B$ (cheating slightly).
- It is easy to verify that $P_n \xrightarrow{s} I$. Consequently, $AP_n \xrightarrow{s} A$.
- Taking the weak limits in

$$(A - \lambda)P_n B_n = P_n,$$

one gets $(A - \lambda)B = I$ and similarly also $B(A - \lambda) = I$.

Claim:

$$A = A^* \quad \Rightarrow \quad \sigma(A) \subset \Lambda(A).$$

Proof:

- Let $\lambda \notin \Lambda(A)$. Then $\exists \epsilon > 0$ such that $B_\epsilon(\lambda) \cap \sigma(A_n) = \emptyset, \ \forall n > n_0$.
- Important ingredient: For any $T = T^*$, one has

$$\|(T-\lambda)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}, \quad \lambda \notin \sigma(T).$$

- So in this case: $\|(A_n - \lambda)^{-1}\| \leq 1/\epsilon, \ \forall n > n_0$.
- Put

$$B_n := (A_n - \lambda)^{-1} \oplus I \in B_{1/\epsilon} \subset \mathcal{B}(\mathcal{H}) \text{ (weakly precompact).}$$

- $\exists B \in \mathcal{B}(\mathcal{H})$ such that $B_n \xrightarrow{w} B$ (cheating slightly).
- It is easy to verify that $P_n \xrightarrow{s} I$. Consequently, $AP_n \xrightarrow{s} A$.
- Taking the weak limits in

$$(A-\lambda)P_n B_n = P_n,$$

  one gets $(A-\lambda)B = I$ and similarly also $B(A-\lambda) = I$.
- Hence $B = (A-\lambda)^{-1}$ and $\lambda \notin \sigma(A)$.

Few comments:

- In general, $\sigma(A) \neq \Lambda(A)$ for $A = A^*$.

Few comments:

- In general, $\sigma(A) \neq \Lambda(A)$ for $A = A^*$.
- Additional conditions are necessary to have the equality $\sigma(A) = \Lambda(A)$ (and several are known for operators from particular classes).

Few comments:

- In general, $\sigma(A) \neq \Lambda(A)$ for $A = A^*$.
- Additional conditions are necessary to have the equality $\sigma(A) = \Lambda(A)$ (and several are known for operators from particular classes).
- For $A \neq A^*$, the inclusion $\sigma(A) \subset \Lambda(A)$ does not hold (take the shift operators L, R).

Few comments:

- In general, $\sigma(A) \neq \Lambda(A)$ for $A = A^*$.
- Additional conditions are necessary to have the equality $\sigma(A) = \Lambda(A)$ (and several are known for operators from particular classes).
- For $A \neq A^*$, the inclusion $\sigma(A) \subset \Lambda(A)$ does not hold (take the shift operators L, R).

Q: Are there other conditions guaranteeing $\sigma(A) \subset \Lambda(A)$ or $\Lambda(A) \subset \sigma(A)$?

Few comments:

- In general, $\sigma(A) \neq \Lambda(A)$ for $A = A^*$.
- Additional conditions are necessary to have the equality $\sigma(A) = \Lambda(A)$ (and several are known for operators from particular classes).
- For $A \neq A^*$, the inclusion $\sigma(A) \subset \Lambda(A)$ does not hold (take the shift operators L, R).

Q: Are there other conditions guaranteeing $\sigma(A) \subset \Lambda(A)$ or $\Lambda(A) \subset \sigma(A)$?

- If A is compact, then $\sigma(A) = \Lambda(A)$.

Few comments:

- In general, $\sigma(A) \neq \Lambda(A)$ for $A = A^*$.
- Additional conditions are necessary to have the equality $\sigma(A) = \Lambda(A)$ (and several are known for operators from particular classes).
- For $A \neq A^*$, the inclusion $\sigma(A) \subset \Lambda(A)$ does not hold (take the shift operators L, R).

Q: Are there other conditions guaranteeing $\sigma(A) \subset \Lambda(A)$ or $\Lambda(A) \subset \sigma(A)$?

- If A is compact, then $\sigma(A) = \Lambda(A)$.

Q: If A is an unbounded sectorial operator with discrete spectrum, does one have $\Lambda(A) = \sigma(A)$?

Few comments:

- In general, $\sigma(A) \neq \Lambda(A)$ for $A = A^*$.
- Additional conditions are necessary to have the equality $\sigma(A) = \Lambda(A)$ (and several are known for operators from particular classes).
- For $A \neq A^*$, the inclusion $\sigma(A) \subset \Lambda(A)$ does not hold (take the shift operators L, R).

Q: Are there other conditions guaranteeing $\sigma(A) \subset \Lambda(A)$ or $\Lambda(A) \subset \sigma(A)$?

- If A is compact, then $\sigma(A) = \Lambda(A)$.

Q: If A is an unbounded sectorial operator with discrete spectrum, does one have $\Lambda(A) = \sigma(A)$?

- If $T(b)$ is possibly NSA banded Toeplitz operator on $\ell^2(\mathbb{N})$ (or Wiener–Hopf operator), then $\Lambda(A) \subset \sigma(A)$.

Few comments:

- In general, $\sigma(A) \neq \Lambda(A)$ for $A = A^*$.
- Additional conditions are necessary to have the equality $\sigma(A) = \Lambda(A)$ (and several are known for operators from particular classes).
- For $A \neq A^*$, the inclusion $\sigma(A) \subset \Lambda(A)$ does not hold (take the shift operators L, R).

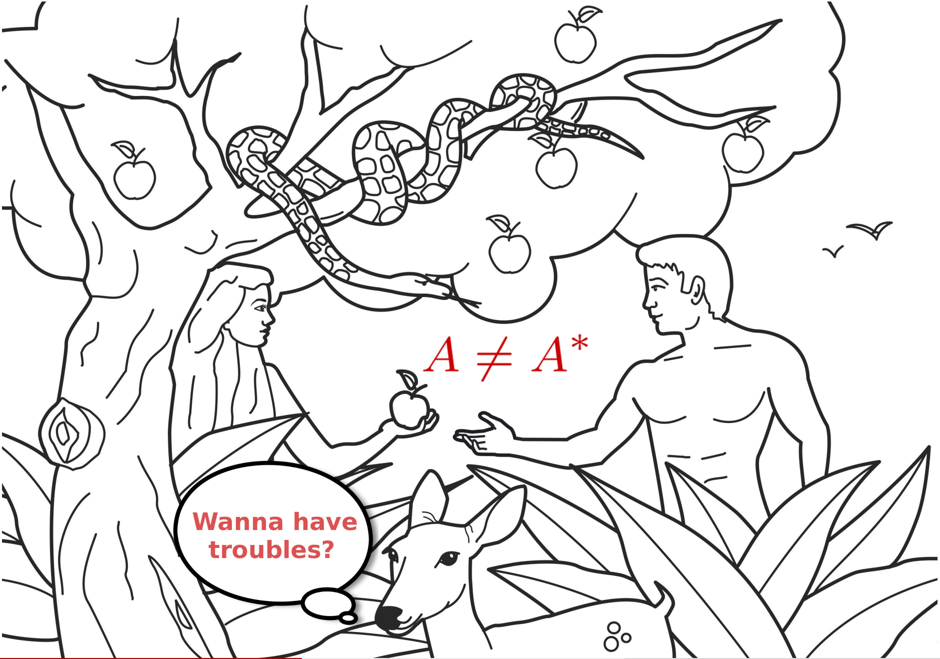
Q: Are there other conditions guaranteeing $\sigma(A) \subset \Lambda(A)$ or $\Lambda(A) \subset \sigma(A)$?

- If A is compact, then $\sigma(A) = \Lambda(A)$.

Q: If A is an unbounded sectorial operator with discrete spectrum, does one have $\Lambda(A) = \sigma(A)$?

- If $T(b)$ is possibly NSA banded Toeplitz operator on $\ell^2(\mathbb{N})$ (or Wiener–Hopf operator), then $\Lambda(A) \subset \sigma(A)$.

Q: Are there other classes of NSA operators for which $\Lambda(A) \subset \sigma(A)$?


$A \neq A^*$
Wanna have troubles?