An inverse spectral problem for NSA

Jacobi and Schrödinger op.

joint with A. Pushnitski

1) SA Jacobi op.: assum.: $a_n > 0$, $b_n \in \mathbb{R}$ both bounded

$$J := \begin{bmatrix} b_0 & a_0 & & & \\ a_0 & b_1 & a_1 & & \\ & a_1 & b_2 & a_2 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \quad \text{on } \ell^2(\mathbb{N}_0)$$

↳ J bounded, $J = J^*$, J has cyclic vector δ_0.

Spectral meas.: $\boxed{\mu := \langle \delta_0, E_J \delta_0 \rangle}$

Spectral map: $\Lambda : J \longmapsto \mu$

Thm.: 1) Injectivity of Λ: The spectral meas. μ determines uniquely

2) Surjectivity of Λ: If μ is a probality meas. on $\mathbb{R}$ with compact & infinite support, then $\exists J$ s.t. $\Lambda(J) = \mu$.

2) NSA Jacobi op.: assum.: relax $b_n \in \mathbb{R}$, i.e. $b_n \in \mathbb{C}$. $(J \neq J^*)$

(Pushnitski – Š., '24, '25)

Hermitisation: $\begin{pmatrix} 0 & J \\ J^* & 0 \end{pmatrix}$ on $\ell^2(\mathbb{N}_0) \oplus \ell^2(\mathbb{N}_0) \simeq \ell^2(\mathbb{N}_0; \mathbb{C}^2)$

unit. equiv. to
$$\mathbb{J} = \begin{bmatrix} B_0 & A_0 & & & \\ A_0 & B_1 & A_1 & & \\ & A_1 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \qquad A_n = \begin{pmatrix} 0 & a_n \\ a_n & 0 \end{pmatrix}, \quad B_n = \begin{pmatrix} 0 & b_n \\ \overline{b_n} & 0 \end{pmatrix}$$

$\mathbb{J} = \mathbb{J}^* \rightsquigarrow$ Spec. measure $\mathbb{J}$: $\Sigma := \mathbb{P}_0 E_{\mathbb{J}} \mathbb{P}_0^*$

2×2 matrix-valued measure

Thm.: There $\exists_1$ even probability meas. ν with compact and infinite support in $\mathbb{R}$ & $\exists_1$ $\psi \in L^\infty(\nu)$ odd and $\|\psi\|_\infty \leq 1$ such that

$$\boxed{d\Sigma = \begin{pmatrix} 1 & \psi \\ \bar{\psi} & 1 \end{pmatrix} d\nu.}$$

Def.: The pair (ν, ψ) is called spectral data of J.

Prop.:

1) $\nu \restriction [0,\infty) = \langle \delta_0, E_{|J|} \delta_0 \rangle$, $(|J| = \sqrt{J^* J})$.
2) Spectrum of $|J|$ has multiplicity equal to
   $\underline{1}$ on $S_1 := \{ s > 0 \mid |\psi(s)| = 1 \}$,
   $\underline{2}$ on $S_2 := \{ s > 0 \mid |\psi(s)| < 1 \}$.
3) $J = J^* \iff \psi$ real-valued.
4) $J = J^* \implies d\mu = (1 + \psi) d\nu$.
5) $J^* J$ has simple spectrum $\iff |\psi| = 1$ ν-a.e.
6) $b_n = 0 \iff \psi \equiv 0$.

$\Lambda(J) := (\nu, \psi)$ ...spectral map

Thm. (Pushnitski – Š, 24):

1) Injectivity: J is uniquely determined by its spectral data
2) Surjectivity: Given i) ν even prob. meas. on $\mathbb{R}$ with comp & infinite support ;
   ii) $\psi \in L^\infty(\nu)$, $\|\psi\|_\infty \leq 1$, ψ odd ;

then $\exists J$ s.t. $\Lambda(J) = (\nu, \psi)$.

3) SA Schrödinger op.: assum.: q real-valued meas. bounded fction on $\mathbb{R}_+$

$$H := -\frac{d}{dx^2} + q \quad \text{in } L^2(\mathbb{R}_+)$$

$$\operatorname{Dom} H = \{ f \in W^{2,2}(\mathbb{R}_+) \mid f(0)\cos\alpha + f'(0)\sin\alpha = 0 \}$$
$$\alpha \in [0, \pi/2]$$

Fund. sys.: φ, θ sol. to $-f'' + qf = \lambda f$ satis.

$$\varphi(0,\lambda) = \sin\alpha \qquad \theta(0,\lambda) = \cos\alpha$$
$$\varphi'(0,\lambda) = -\cos\alpha \qquad \theta'(0,\lambda) = \sin\alpha$$

m-func.: $\exists_1\, m = m(\lambda)$ s.t. $\theta(0,\lambda) - m(\lambda)\varphi(0,\lambda) \in L^2(\mathbb{R}_+$
$\lambda \in \mathbb{C}\setminus\mathbb{R}$

Spectr. measure: m is Herglotz-Nevanlinna $\Rightarrow$

$$m(\lambda) = \operatorname{Re} m(i) + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t^2}{1+t^2} \right) d\mu(\lambda)$$

spectral meas. of H

Spectr. map: $\Lambda : \overset{q}{H} \longmapsto \mu$

Thm. (Borg): H (or q) is uniquely determined by μ.
$(q \in L^1_{loc}(\mathbb{R}_+))$

Thm. (Marchenko): As $r \to \infty$, we have

$$\sigma((-\infty, r]) = \begin{cases} \frac{2}{\pi \sin^2\alpha}\sqrt{r} + o(\sqrt{r}), \\ \frac{2}{3\pi} r^{3/2} + o(r^{3/2}). \end{cases}$$

4) NSA Schrödinger op.:

<u>assum.:</u> q <u>complex</u>-valued bounded meas. fction on $\mathbb{R}_+$

<u>Hermitisation:</u> $\begin{pmatrix} 0 & H \\ H^* & 0 \end{pmatrix}$ on $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+) \simeq L^2(\mathbb{R}_+, \mathbb{C}^2)$

$$\mathbb{H} := -\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{=:\mathcal{E}} \frac{d^2}{dx^2} + \underbrace{\begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}}_{=:Q}$$

<u>Fund. sys.:</u> Φ, Θ matrix-val. sol. of $-\mathcal{E}F'' + QF = \lambda F$

$$\Phi(0,\lambda) = \sin\alpha \; I \qquad \Theta(0,\lambda) = \cos\alpha \cdot \mathcal{E}$$

$$\Phi'(0,\lambda) = -\cos\alpha \; I \qquad \Theta'(0,\lambda) = \sin\alpha \cdot \mathcal{E}$$

<u>M-function:</u> $\exists_1 M = M(\lambda)$ s.t. $\Theta(0,\lambda) - \Phi(0,\lambda) M(\lambda) \in L^2(\mathbb{R}_+, \mathbb{C}^2)$

$\lambda \in \mathbb{C} \setminus \mathbb{R}$

<u>Spectral measure:</u> $M(\lambda)^* = M(\bar{\lambda})$, $\dfrac{\operatorname{Im} M(\lambda)}{\operatorname{Im}\lambda} \geq 0$

$$M(\lambda) = \operatorname{Re} M(i) + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\Sigma(t)$$

<u>Spectral measure</u> of $\mathbb{H}$

<u>Thm.:</u> We have $\boxed{d\Sigma = \begin{pmatrix} 1 & \varphi \\ \bar{\varphi} & 1 \end{pmatrix} d\nu}$, where

ν is even meas. on $\mathbb{R}$ & $\varphi \in L^\infty(\nu)$ odd, $\|\varphi\|_\infty \leq 1$.

<u>Def.:</u> The pair (ν, φ) is called <u>spectral data</u> of H.

Thm. (à la Borg): Potential Q is uniquely determined by the spectral measure. Consequently, potential q is uniquely determined by the spectral data (ν, φ)

Thm. (à la Marchenko): As $r \to \infty$,

$$\nu([0,r]) = \begin{cases} \frac{r^{1/2}}{\pi \sin^2 \alpha} + o(r^{-1/2}), & \alpha \neq 0, \\ \frac{r^{3/2}}{3\pi} + o(r^{3/2}), & \alpha = 0, \end{cases}$$

$$\int_0^r \varphi(s)\, d\nu(s) = \begin{cases} - \quad // \quad - \end{cases}$$

($\varphi(s) \longrightarrow 1$ „in a weak average sense")

Further properties:

1) $\Lambda(H) = (\nu, \varphi) \Rightarrow \Lambda(H^*) = (\nu, \bar{\varphi})$

In particular, $H = H^* \Leftrightarrow \varphi$ real.

2) If $H = H^* \Rightarrow d\mu = (1+\varphi)\, d\nu$

3) If λ is a simple singular value of H (i.e. eigv. of $|H|$) then we have „formulas" for

$$\nu(\{\lambda\}) = \text{„…"} \quad \& \quad \varphi(\{\lambda\}) = \text{„…"}$$

in terms of a sol. to $-f'' + qf = \lambda \bar{f}$.