Continued Fractions Appearing Naturally in Spectral Analysis of Jacobi Operators

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Combinatorics on Words and Mathematical Physics

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Definition

Define $\mathfrak{F} : \ell^1(\mathbb{N}) \to \mathbb{C}$ by

$$\mathfrak{F}(y) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} y_{k_1} y_{k_2} \dots y_{k_m}.$$

- For a finite number of complex variables we identify $\mathfrak{F}(y_1, y_2, \dots, y_n)$ with $\mathfrak{F}(y)$ where $y = (y_1, y_2, \dots, y_n, 0, 0, 0, \dots)$.

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- $\mathfrak{F}$ is well defined on $\ell^1(\mathbb{N})$ due to estimation
$$|\mathfrak{F}(y)| \leq \exp \|y\|_1.$$
- $\mathfrak{F}$ is a continuous functional on $\ell^1(\mathbb{N})$ which is **not** linear. Especially, for any $y \in \ell^1(\mathbb{N})$, it satisfies limit relations
$$\lim_{n\to\infty} \mathfrak{F}(y_1, y_2, \dots, y_n) = \mathfrak{F}(y) \qquad \text{and} \qquad \lim_{n\to\infty} \mathfrak{F}(T^n y) = 1$$
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where T stands for unilateral right-shift operator on the space of complex sequence.
- Function $\mathfrak{F}$ fulfills many nice and simple algebraic and combinatorial identities.
- Function $\mathfrak{F}$ have been developed for investigation of various spectral properties of Jacobi operators and it found many application here (not the scope of this talk).

- For $y \in \ell^1(\mathbb{N})$, $\mathfrak{F}$ satisfies recurrence rule

$$\mathfrak{F}(y) = \mathfrak{F}(Ty) - y_1\,\mathfrak{F}(T^2y).$$

- For $y \in \ell^1(\mathbb{N})$, $\mathfrak{E}$ satisfies recurrence rule

$$\mathfrak{E}(y) = \mathfrak{E}(Ty) - y_1\, \mathfrak{E}(T^2 y).$$

- Consequently, function $\mathfrak{E}$ is related to the Stieltjes continued fraction (S-fraction). For a given $y \in \ell^1(\mathbb{N})$ such that $\mathfrak{E}(y) \neq 0$, it holds

$$\frac{\mathfrak{E}(Ty)}{\mathfrak{E}(y)} = \cfrac{1}{1 - \cfrac{y_1}{1 - \cfrac{y_2}{1 - \cfrac{y_3}{1 - \dots}}}}\,.$$

Recurrence Rule and Continued Fraction

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- The LHS of the last identity can be viewed as a formal power series in countably many indeterminates $y = \{y_k\}_{k=1}^{\infty}$. They forms the ring $\mathbb{C}[[y]]$.
- People realized there is certain connection between an S-fraction and formal power series a long time ago:

  T.J.Stieltjes: *Recherches sur les fractions continues*, 1894-95.

  Particularly, they study cases when $y_k = x e_k$ where e_k is a fixed complex sequence and x is a complex variable:

  L.J.Rogers: *On the representation of certain asymptotic series as convergent continued fractions*, 1907.

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is regarded here as a sequence of convergents A_n/B_n, $n = 0, 1, 2, \dots$, with $A_n, B_n \in \mathbb{C}[a]$ defined by the usual recurrence rules $A_0 = 0$, $A_1 = 1$, $B_0 = B_1 = 1$, and

$$A_n = A_{n-1} - a_{n-1}A_{n-2}, \qquad B_n = B_{n-1} - a_{n-1}B_{n-2}, \qquad n \geq 2.$$

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Thus with every formal S-fraction there is naturally associated a unique formal power series $f(a)$ in the indeterminates a.

Explicit expression for $f(a)$

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A. J. Zajta, W. Pandikow: *Conversion of continued fractions into power series*, **1975**.

P. Flajolet: *Combinatorial aspects of continued fractions*, 1980.

Theorem

In the ring of formal power series in the indeterminates $y_1, \dots, y_n$, one has

$$\log \mathfrak{E}(y_1, \dots, y_n) = -\sum_{N=1}^{\infty} \sum_{\substack{m \in \mathcal{M}(N) \\ d(m) < n}} \alpha(m) \sum_{k=1}^{n-d(m)} \prod_{j=1}^{d(m)} y_{k+j-1}^{m_j}.$$

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- It holds identity $\mathfrak{F}(y_1, \ldots, y_n) = \det(I + Y)$ where Y is an $(n+1) \times (n+1)$ with elements in terms of $y_1, \ldots, y_n$.

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- To find an expression for $\operatorname{Tr} Y^N$ is a hard combinatorial work of the proof.

Logarithm of the power series of S-fraction

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- By using this result one can rediscover the power series expansion $f(a)$ which has been found in 1975.

Simple Example

- Perhaps the simplest example is obtained if we set $a_j = z$, for all $j \in \mathbb{N}$, in the formal S-fraction. The formula for logarithm then yields

$$\log\left(\frac{1}{1} - \frac{z}{1} - \frac{z}{1} - \frac{z}{1} - \dots\right) = \sum_{N=1}^{\infty} \frac{1}{2N}\binom{2N}{N} z^N = \log \frac{2}{1+\sqrt{1-4z}}\,.$$

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- Since $c(z) = 2/(1+\sqrt{1-4z})$ is known to be the generating function for the Catalan numbers, one derives this way an identity relating $\beta(m)$ with Catalan numbers,

$$\sum_{m\in\mathcal{M}(N)} \beta(m) = C_N := \frac{1}{N+1}\binom{2N}{N}.$$

The Rogers-Ramanujan continued fraction

- The generalized Rogers-Ramanujan continued fraction

$$\frac{1}{1}\,\underset{+}{}\,\frac{z}{1}\,\underset{+}{}\,\frac{qz}{1}\,\underset{+}{}\,\frac{q^2z}{1}\,\underset{+}{}\,\frac{q^3z}{1}\,\underset{+}{}\,\cdots$$

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Theorem

The power series expansion $R(z;q)$ in the variable z of the generalized Rogers-Ramanujan continued fraction fulfills

$$R(z;q) = 1 + \sum_{N=1}^{\infty}\left(\sum_{m\in\mathcal{M}(N)} \beta(m)\, q^{\epsilon_1(m)}\right)(-z)^N$$

and

$$\log R(z;q) = \sum_{N=1}^{\infty}\left(\sum_{m\in\mathcal{M}(N)} \alpha(m)\, q^{\epsilon_1(m)}\right)(-z)^N$$

where

$$\epsilon_1(m) = \sum_{j=1}^{d(m)} (j-1)m_j.$$

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- For $0 < q < 1$ and $z \in \mathbb{C}$, it can be shown

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- By putting $z = q$ one arrives at the Rogers-Ramanujan continued fraction

$$\frac{1}{1}\,{+}\,\frac{q}{1}\,{+}\,\frac{q^2}{1}\,{+}\,\frac{q^3}{1}\,{+}\,\cdots.$$

- For $0 < q < 1$ and $z \in \mathbb{C}$, it can be shown

$$\mathfrak{F}\left(\{-zq^{k-1}\}_{k=1}^{\infty}\right) = {}_0\phi_1(;0;q,z).$$

- Consequently, for $R(z;q)$ it holds

$$R(z;q) = {}_0\phi_1(;0;q,qz)/\,{}_0\phi_1(;0;q,z).$$

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- Its convergents are expressible in terms of the q-Fibonacci numbers of the first and second kind:

$$F_0(q) = 0,\ F_1(q) = 1, \quad F_n(q) = F_{n-1}(q) + q^{n-2}F_{n-2}(q) \quad \text{for } n \geq 2,$$

  and

$$\hat{F}_0(q) = 0,\ \hat{F}_1(q) = 1, \quad \hat{F}_n(q) = \hat{F}_{n-1}(q) + q^{n-1}\hat{F}_{n-2}(q) \quad \text{for } n \geq 2.$$

See L. Carlitz: *Fibonacci notes 3: q-Fibonacci numbers*, 1974.

The Rogers-Ramanujan continued fraction

- For $0 < q < 1$, there exists the limits

$$F_\infty(q) = \lim_{n\to\infty} F_n(q) = 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 5q^9 + \ldots,$$
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- The celebrated Rogers-Ramanujan identities extend this identity to a much stronger result by showing

$${}_0\phi_1(\,;0;q,q) = \prod_{\mathbb{N}\ni n\equiv 1,4 \bmod 5} (1-q^n)^{-1}$$

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$${}_0\phi_1(\,;0;q,q^2) = \prod_{\mathbb{N}\ni n\equiv 2,3 \bmod 5} (1-q^n)^{-1}.$$

- Power series formulas for $R(q)$ and $\log R(q)$ yields

$$R(q) = 1 + \sum_{\ell=1}^{\infty} \sum_{m \in \mathbb{N}^{\ell}} (-1)^{|m|} \beta(m)\, q^{m_1+2m_2+\ldots+\ell m_\ell}$$

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- The summands can be expressed in terms of q-Fibonacci numbers:

$$\sum_{m \in \mathbb{N}^\ell} (-1)^{|m|}\, \beta(m)\, q^{m_1+2m_2+\ldots+\ell m_\ell} = \frac{(-1)^\ell q^{(\ell+1)\ell/2}}{F_{\ell+1}(q) F_{\ell+2}(q)},$$

and

$$\sum_{m \in \mathbb{N}^\ell} (-1)^{|m|} \alpha(m)\, q^{m_1+2m_2+\ldots+\ell m_\ell} = \log\left(\frac{\hat{F}_{\ell+1}(q) F_{\ell+1}(q)}{\hat{F}_\ell(q) F_{\ell+2}(q)}\right),$$

for $\ell \in \mathbb{N}$.

- Under some assumptions on sequences $\{a_n\}$ and $\{b_n\}$, the generalized continued fraction

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Consequently, results concerning continued fractions are of much interest even in the Mathematical Physics community!

Thank you, and enjoy Beskydy!