Continued Fractions Appearing Naturally in Spectral Analysis of Jacobi Operators

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May 2013

1 Function & and its properties

Stieltjes continued fractions and formal power series

3 The Rogers-Ramanujan continued fraction

A note on the role of continued fractions in spectral analysis of Jacobi operators

Define $\mathfrak{E}: \ell^1(\mathbb{N}) \to \mathbb{C}$ by

$$\mathfrak{E}(y) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} y_{k_1} y_{k_2} \cdots y_{k_m}$$

• For a finite number of complex variables we identify $\mathfrak{E}(y_1, y_2, \ldots, y_n)$ with $\mathfrak{E}(y)$ where $y = (y_1, y_2, \ldots, y_n, 0, 0, 0, \ldots)$.

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 € is a continuous functional on ℓ¹(ℕ) which is not linear. Especially, for any y ∈ ℓ¹(ℕ), it satisfies limit relations

$$\lim_{n\to\infty}\mathfrak{E}(y_1,y_2,\ldots,y_n)=\mathfrak{E}(y) \quad \text{and} \quad \lim_{n\to\infty}\mathfrak{E}(T^ny)=1$$

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- Function & fulfills many nice and simple algebraic and combinatorial identities.
- Function & have been developed for investigation of various spectral properties of Jacobi operators and it found many application here (not the scope of this talk).

• For $y \in \ell^1(\mathbb{N})$, \mathfrak{E} satisfies recurrence rule

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Recurrence Rule and Continued Fraction

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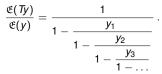
• Consequently, function \mathfrak{E} is related to the Stieltjes continued fraction (S-fraction). For a given $y \in \ell^1(\mathbb{N})$ such that $\mathfrak{E}(y) \neq 0$, it holds

$$\frac{\mathfrak{E}(Ty)}{\mathfrak{E}(y)} = \frac{1}{1 - \frac{y_1}{1 - \frac{y_2}{1 - \frac{y_3}{1 - \frac{y_3}{1 - \dots}}}}}$$

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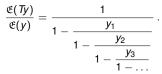
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 The LHS of the last identity can be viewed as a formal power series in countably many indeterminates y = {y_k}_{k=1}[∞]. They forms the ring C[[y]]. • For $y \in \ell^1(\mathbb{N})$, \mathfrak{E} satisfies recurrence rule

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- People realized there is certain connection between an S-fraction and formal power series a long time ago:

T.J.Stieltjes: Recherches sur les fractions continues, 1894-95.

Particularly, they study cases when $y_k = xe_k$ where e_k is a fixed complex sequence and x is a complex variable:

L.J.Rogers: On the representation of certain asymptotic series as convergent continued fractions, 1907. • But how exactly one associates the S-fraction with a formal power series?

- But how exactly one associates the S-fraction with a formal power series?
- Let $a = \{a_k\}_{k=1}^{\infty} \subset \mathbb{C}$. The formal Stieltjes continued fraction

$$\frac{1}{1} - \frac{a_1}{1} - \frac{a_2}{1} - \frac{a_3}{1} - \cdots$$

$$A_n = A_{n-1} - a_{n-1}A_{n-2}, \qquad B_n = B_{n-1} - a_{n-1}B_{n-2}, \qquad n \ge 2.$$

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Thus with every formal S-fraction there is naturally associated a unique formal power series f(a) in the indeterminates a.

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Theorem

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A. J. Zajta, W. Pandikow: Conversion of continued fractions into power series, 1975.

P. Flajolet: Combinatorial aspects of continued fractions, 1980.

In the ring of formal power series in the indeterminates y_1, \ldots, y_n , one has

$$\log \mathfrak{E}(y_1,\ldots,y_n) = -\sum_{N=1}^{\infty} \sum_{\substack{m \in \mathcal{M}(N) \\ d(m) < n}} \alpha(m) \sum_{k=1}^{n-d(m)} \prod_{j=1}^{d(m)} y_{k+j-1}^{m_j}$$

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• To find an expression for $Tr Y^N$ is a hard combinatorial work of the proof.

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- By using this result one can rediscover the power series expansion *f*(*a*) which has been found in 1975.

• Perhaps the simplest example is obtained if we set $a_j = z$, for all $j \in \mathbb{N}$, in the formal S-fraction. The formula for logarithm then yields

$$\log\left(\frac{1}{1} - \frac{z}{1} - \frac{z}{1} - \frac{z}{1} - \cdots\right) = \sum_{N=1}^{\infty} \frac{1}{2N} \binom{2N}{N} z^N = \log\frac{2}{1 + \sqrt{1 - 4z}}$$

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• Since $c(z) = 2/(1 + \sqrt{1-4z})$ is known to be the generating function for the Catalan numbers, one derives this way an identity relating $\beta(m)$ with Catalan numbers,

$$\sum_{m\in\mathcal{M}(N)}\beta(m)=C_N:=\frac{1}{N+1}\binom{2N}{N}.$$

The Rogers-Ramanujan continued fraction

• The generalized Rogers-Ramanujan continued fraction

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represents a more involved example.

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Theorem

The power series expansion R(z; q) in the variable z of the generalized Rogers-Ramanujan continued fraction fulfills

$$R(z;q) = 1 + \sum_{N=1}^{\infty} \left(\sum_{m \in \mathcal{M}(N)} \beta(m) q^{\epsilon_1(m)} \right) (-z)^N$$

and

$$\log R(z;q) = \sum_{N=1}^{\infty} \left(\sum_{m \in \mathcal{M}(N)} \alpha(m) q^{\epsilon_1(m)} \right) (-z)^N$$

where

$$\epsilon_1(m) = \sum_{j=1}^{d(m)} (j-1)m_j.$$

• For 0 < q < 1 and $z \in \mathbb{C}$, it can be shown

$$\mathfrak{E}\left(\{-zq^{k-1}\}_{k=1}^{\infty}\right) = {}_0\phi_1(;0;q,z).$$

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 Its convergents are expressible in terms of the q-Fibonacci numbers of the first and second kind:

$$F_0(q) = 0, \ F_1(q) = 1, \quad F_n(q) = F_{n-1}(q) + q^{n-2}F_{n-2}(q) \quad \text{for } n \ge 2,$$

and

$$\hat{F}_0(q) = 0, \ \hat{F}_1(q) = 1, \quad \hat{F}_n(q) = \hat{F}_{n-1}(q) + q^{n-1}\hat{F}_{n-2}(q) \quad \text{for } n \ge 2.$$

See L. Carlitz: Fibonacci notes 3: q-Fibonacci numbers, 1974.

František Štampach (FNSPE, CTU)

Continued Fraction in Spectral Analysis

$$F_{\infty}(q) = \lim_{n \to \infty} F_n(q) = 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 5q^9 + \dots,$$

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• By using the relation between \mathfrak{E} and $_0\phi_1$ again, one finds well known identities

$$F_{\infty}(q) = {}_{0}\phi_{1}(; 0; q, q), \qquad \hat{F}_{\infty}(q) = {}_{0}\phi_{1}(; 0; q, q^{2}).$$

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 The celebrated Rogers-Ramanujan identities extend this identity to a much stronger result by showing

$$_{0}\phi_{1}(; 0; q, q) = \prod_{\mathbb{N} \ni n \equiv 1, 4 \mod 5} (1 - q^{n})^{-1}$$

and

$$_{0}\phi_{1}(;0;q,q^{2}) = \prod_{\mathbb{N} \ni n \equiv 2,3 \mod 5} (1-q^{n})^{-1}$$

The Rogers-Ramanujan continued fraction

• Power series formulas for R(q) and $\log R(q)$ yields

$$R(q) = 1 + \sum_{\ell=1}^{\infty} \sum_{m \in \mathbb{N}^{\ell}} (-1)^{|m|} \beta(m) q^{m_1 + 2m_2 + \ldots + \ell m_{\ell}}$$

and

$$\log R(q) = \sum_{\ell=1}^{\infty} \sum_{m \in \mathbb{N}^{\ell}} (-1)^{|m|} \alpha(m) q^{m_1 + 2m_2 + \ldots + \ell m_{\ell}}.$$

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• The summands can be expressed in terms of *q*-Fibonacci numbers:

$$\sum_{m\in\mathbb{N}^{\ell}}(-1)^{|m|}\,\beta(m)\,q^{m_1+2m_2+\ldots+\ell m_{\ell}}=\frac{(-1)^{\ell}q^{(\ell+1)\ell/2}}{F_{\ell+1}(q)F_{\ell+2}(q)},$$

and

$$\sum_{m\in\mathbb{N}^{\ell}} (-1)^{|m|} \alpha(m) \, q^{m_1+2m_2+\ldots+\ell m_{\ell}} = \log\left(\frac{\hat{F}_{\ell+1}(q)F_{\ell+1}(q)}{\hat{F}_{\ell}(q)F_{\ell+2}(q)}\right),$$

for $\ell \in \mathbb{N}$.

A note on the role of generalized continued fractions in spectral analysis

• Under some assumptions on sequences $\{a_n\}$ and $\{b_n\}$, the generalized continued fraction

$$\frac{1}{z-b_1} - \frac{a_1^2}{z-b_2} - \frac{a_2^2}{z-b_3} - \frac{a_3^2}{z-b_4} - \cdots$$

converges locally uniformly on $\mathbb{C} \setminus \mathbb{R}$.

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• This function, known as *the Weyl m-function* m(z) in the theory of Jacobi operators, plays a fundamental role in the spectral theory of those operators.

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Consequently, results concerning continued fractions are of much interest even in the Mathematical Physics community!

Thank you, and enjoy Beskydy!