Spectral analysis of non-self-adjoint Jacobi operator associated with Jacobian elliptic functions

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jointly with Petr Siegl

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Contents

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where $b_n \in \mathbb{C}$ and $a_n \in \mathbb{C} \setminus \{0\}$, we associate two operators $J_{\min}$ and $J_{\max}$ acting on $\ell^2(\mathbb{N})$.

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$$J_{\max}^* = C J_{\min} C \quad \text{and} \quad J_{\min}^* = C J_{\max} C$$

  where C is the complex conjugation operator, $(Cx)_n = \overline{x_n}$.

Proper case and spectrum of Jacobi operator

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where the essential spectrum has the simple characterization:
$$\sigma_{ess}(J) = \{z \in \mathbb{C} \mid \mathrm{Ran}(J - z) \text{ is not closed } \}.$$

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The Jacobi matrix associated with Jacobian elliptic functions

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- We will restrict with α to the unit disk $|\alpha| \leq 1$. The spectral properties of $J(\alpha)$ for $|\alpha| > 1$ are very similar to those for $|\alpha| < 1$.

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- JEFs are meromorphic functions in u (with α fixed) as well as meromorphic functions in α (with u fixed). While K is analytic in the cut-plane $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$.

Jacobian elliptic functions - plotting

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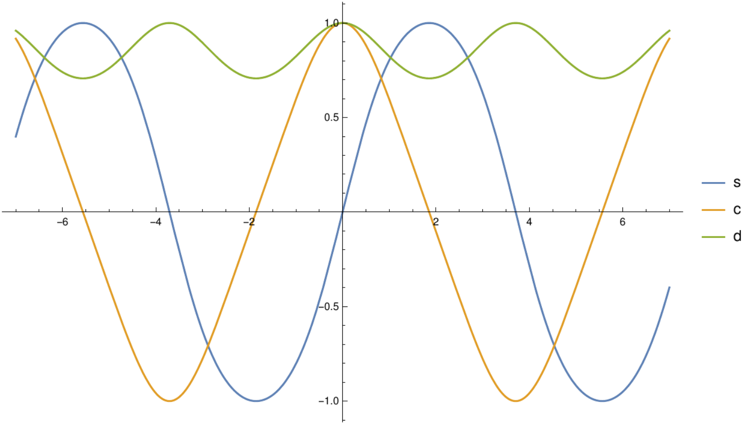

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- We get

$$\mathcal{F}[\mu](z) = \mathrm{cn}(z, \alpha).$$

Consequently, by applying the inverse Fourier transform to the function $\mathrm{cn}(z; \alpha)$, one may recover the spectral measure μ!

Spectral analysis of $J(\alpha)$ for $\alpha \in (-1, 1)$

- For $\alpha \in (-1, 1)$, the evaluation of the inverse Fourier transform yields

$$\mu(t) = \frac{\pi}{\alpha K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1+q^{2n+1}} \left[\delta\left(t - \frac{(2n+1)\pi}{2K} \right) + \delta\left(t + \frac{(2n+1)\pi}{2K} \right) \right]$$

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- In addition, we can also compute the Weyl m-function $m(z;\alpha) := \langle e_1, (J(\alpha)-z)^{-1} e_1 \rangle$, since

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- It results in the formula

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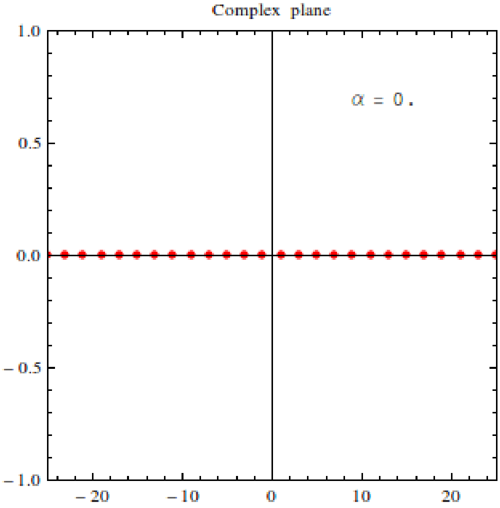
- This implies that the spectrum of $J(\pm 1)$ is purely absolutely continuous and
$$\sigma(J(\pm 1)) = \sigma_{ac}(J(\pm 1)) = \mathbb{R}.$$

Spectrum of $J(\alpha)$ in the self-adjoint case - animation



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- It implies (in the non-self-adjoint case, too!) that

$$\sigma(J(\alpha)) = \frac{\pi}{2K}\left(2\mathbb{Z}+1\right).$$

and all the eigenvalues are simple.

Eigenvectors of $J(\alpha)$ for $|\alpha| < 1$

Proposition

Let $0 < |\alpha| < 1$ and $N \in \mathbb{Z}$, then the vector $v^{(N)}$ given by formulas

$$v_{2k+1}^{(N)} = \mathrm{i}(-1)^k \alpha^k \int_0^{2\pi} e^{-\mathrm{i}(N+1/2)s} \operatorname{cn}\left(\frac{Ks}{\pi}, \alpha\right) \operatorname{sn}^{2k}\left(\frac{Ks}{\pi}, \alpha\right) \mathrm{d}s$$

and

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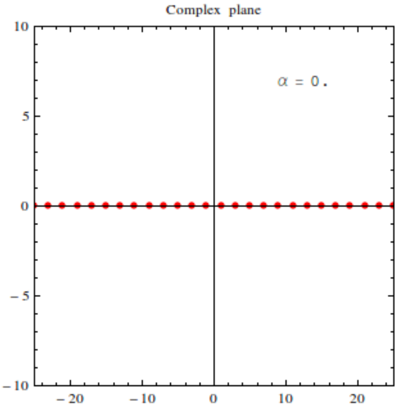
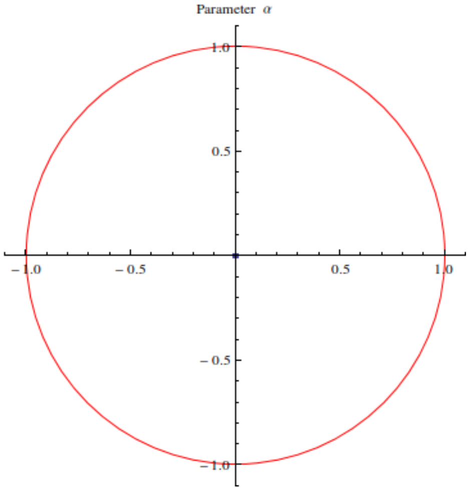
Spectrum of $J(\alpha)$ in the non-self-adjoint case - animation





Spectral analysis of $J(\alpha)$ for $|\alpha| = 1$

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If $|\alpha| = 1$, $\alpha \neq \pm 1$, then

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- Then one can verify, indeed, that
$$\lim_{a \to 1-} \frac{\|(J(\alpha) - z)u(a)\|}{\|u(a)\|} = 0, \quad \text{and} \quad \text{w}-\lim_{a \to 1-} u(a) = 0.$$

Spectral analysis of $J(\alpha)$ for $|\alpha| = 1$ - cont.

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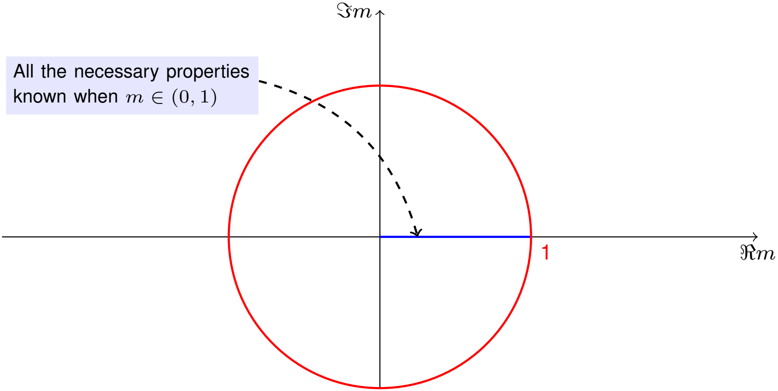
- It can be shown (not trivial!) that the function has unique global maximum at $u = 1$ for every $|\alpha| = 1$, $\alpha \neq \pm 1$.

Contents

On the extremal properties of $|\operatorname{sn}(uK(m) \mid m)|$



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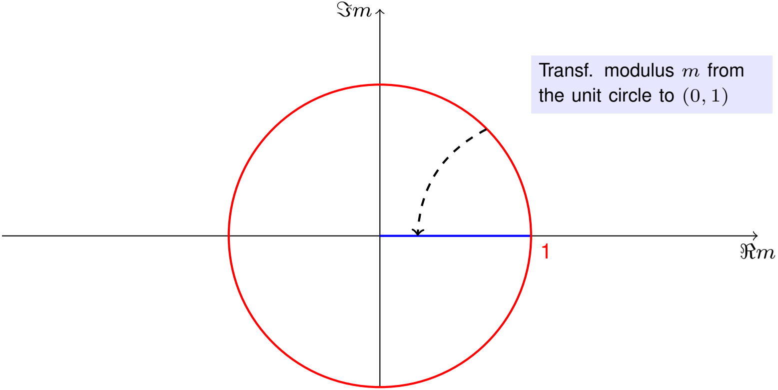

Transf. modulus m from the unit circle to $(0, 1)$

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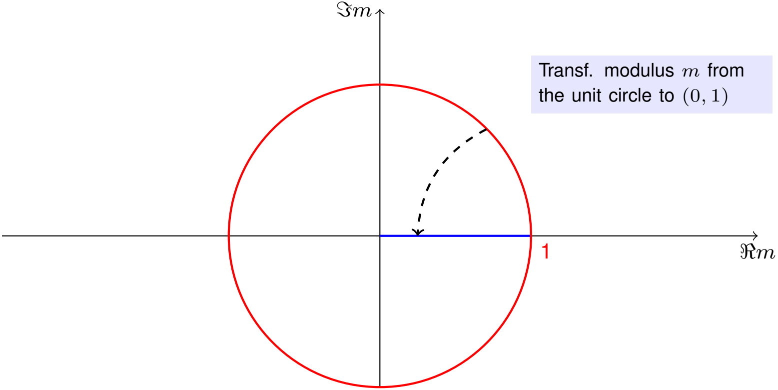
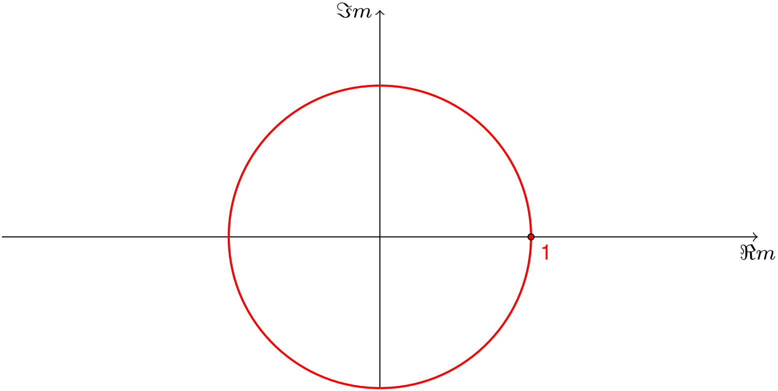


Transf. modulus m from the unit circle to $(0,1)$

$$\operatorname{sn}^2(uK(m) \mid m) = \frac{1}{\sqrt{m}} \frac{c_1^2 + s^2 s_1^2 \cos^2\theta - cc_1 + \mathrm{i} ss_1 dd_1}{c_1^2 + s^2 s_1^2 \cos^2\theta + cc_1 - \mathrm{i} ss_1 dd_1},$$

where $\Im m > 0, m = e^{4\mathrm{i}\theta}$ and $s = \operatorname{sn}(uK(\cos^2\theta) \mid \cos^2\theta)$, $s_1 = \operatorname{sn}(uK(\sin^2\theta) \mid \sin^2\theta)$, etc.

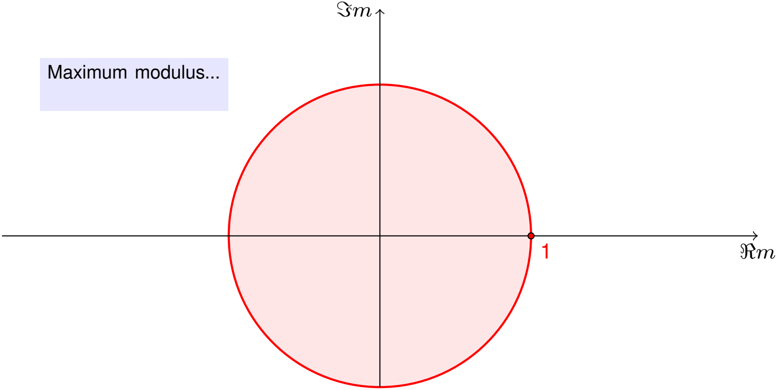
On the extremal properties of $|\operatorname{sn}(uK(m) \mid m)|$



$|\operatorname{sn}(uK(m) \mid m)| \leq 1$ for all $m \in \partial\mathbb{D}$ with the equality only for $m = 1$.

On the extremal properties of $|\operatorname{sn}(uK(m) \mid m)|$

Maximum modulus...



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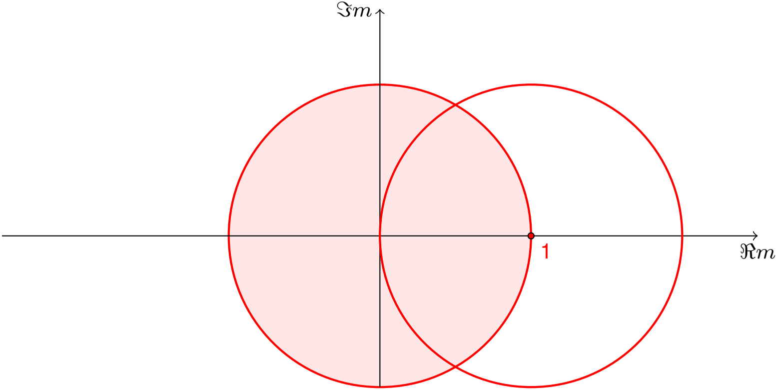
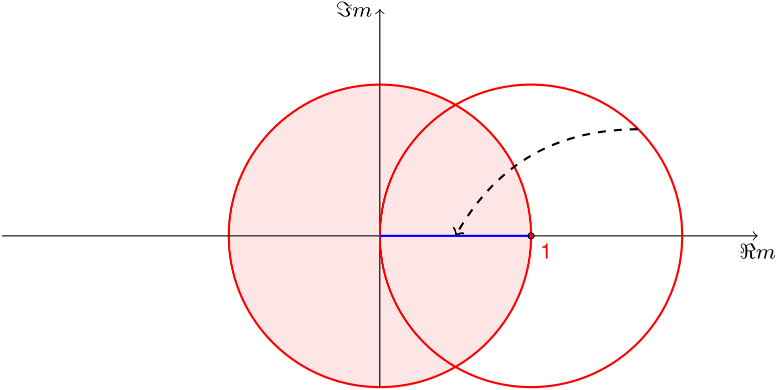


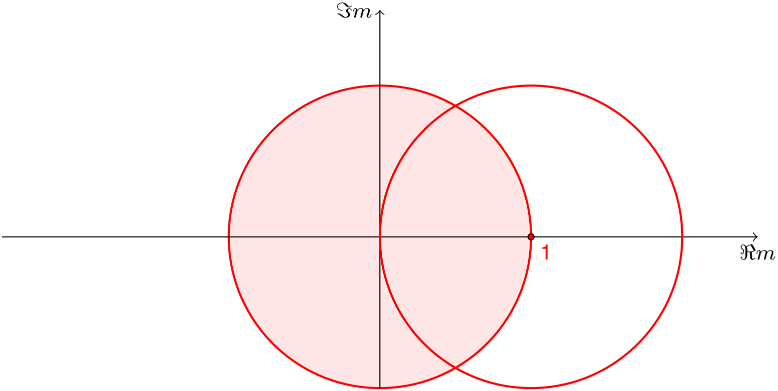
Another circle $\mathbb{D}_1 = \{z \mid |z-1| = 1\}$

On the extremal properties of $|\operatorname{sn}(uK(m) \mid m)|$



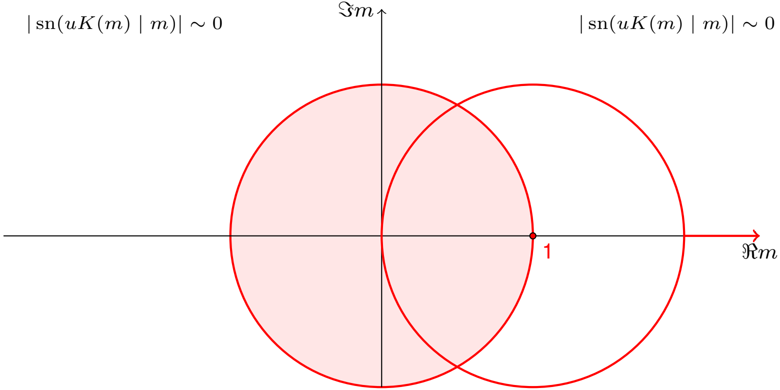
Another circle $\mathbb{D}_1 = \{z \mid |z-1| = 1\}$ and another transformation formula (not displayed) ...

On the extremal properties of $|\operatorname{sn}(uK(m) \mid m)|$



$|\operatorname{sn}(uK(m) \mid m)| < 1$ for all $m \in \partial\mathbb{D}_1$

On the extremal properties of $|\operatorname{sn}(uK(m) \mid m)|$



In additon, $\lim_{\epsilon\to 0+} |\operatorname{sn}(uK(m \pm \mathrm{i}\epsilon) \mid m + \pm \mathrm{i}\epsilon)| < 1$ for all $m \geq 2$ and

the function $m \mapsto \operatorname{sn}(uK(m) \mid m)$ is bounded.

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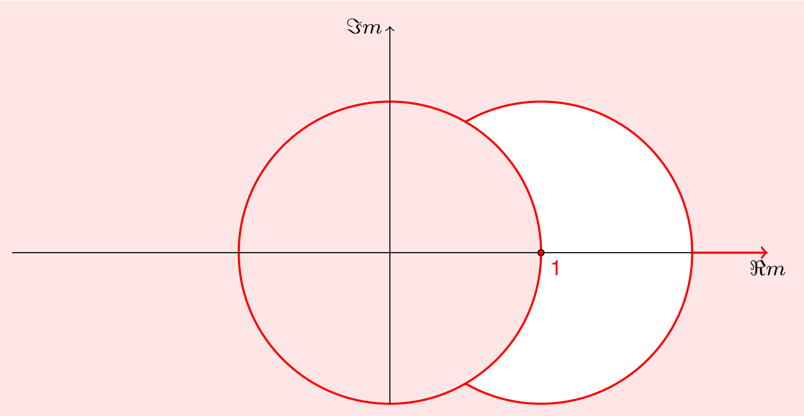
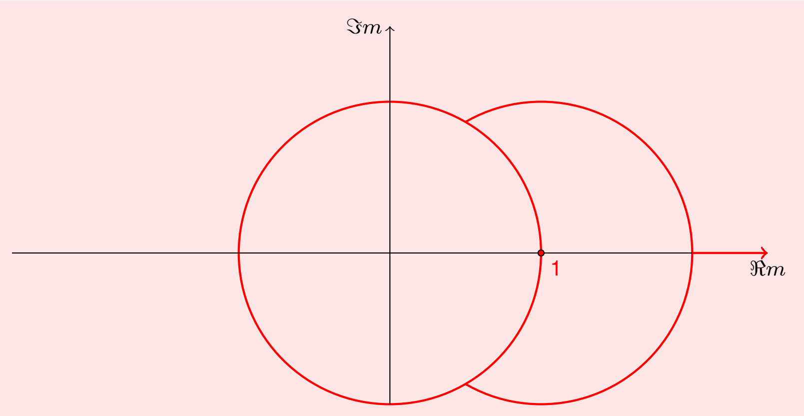


$|\operatorname{sn}(uK(m) \mid m)| \leq 1$ for all $m \notin \mathbb{D}_1 \setminus \mathbb{D}$ with the equality only for $m = 1$.

On the extremal properties of $|\operatorname{sn}(uK(m) \mid m)|$



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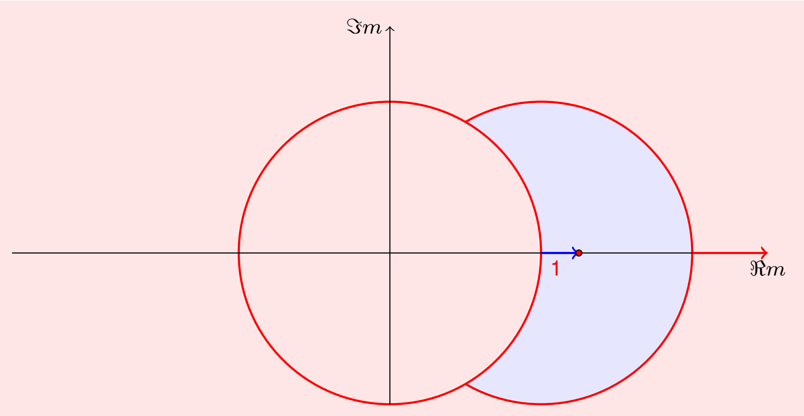

If $\dfrac{1}{2} < u < 1$ the global maximum of $m \mapsto |\operatorname{sn}(uK(m) \mid m)|$

is located in $(1, 2)$ with the value $> 1.$

On the extremal properties of $|\operatorname{sn}(uK(m) \mid m)|$ - main theorem

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3. For $u \in (1/2, 1)$, the function $m \mapsto |\operatorname{sn}(K(m)u \mid m)|$ has a global maximum located in the interval $(1, 2)$ with the value exceeding 1, i.e.,
$$\max_{m \in \mathbb{C}} |\operatorname{sn}(K(m)u \mid m)| = |\operatorname{sn}(K(m^*)u \mid m^*)| > 1 \ \text{ for some } m^* \in (1,2).$$

References:

1. P. Siegl, F. Š.: *On extremal properties of Jacobian elliptic functions with complex modulus*, Math. Anal. Appl. (2016), arXiv:1512.06089.
2. P. Siegl, F. Š.: *Spectral analysis of non-self-adjoint Jacobi operator associated with Jacobian elliptic functions*, arXiv:1603.01052.

Thank you!