Spectral analysis of a complex Jacobi matrix associated with Jacobian elliptic functions

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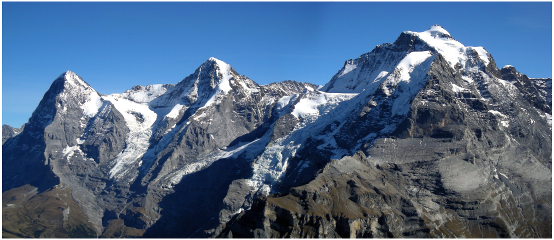

Young Researchers Workshop on Spectral Theory

October 28, 2015

Contents

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- Consequences: $\sigma_r(J(\alpha)) = \emptyset$ and

$$\sigma_{e1}(J(\alpha)) = \sigma_{e2}(J(\alpha)) = \sigma_{e3}(J(\alpha)) = \sigma_{e4}(J(\alpha)).$$

Contents

- For $0 \leq \alpha \leq 1$, the integral (incomplete elliptic of 1st kind)

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- The (copolar) triplet of JEF:

$$\begin{aligned} \mathrm{sn}(u,\alpha) &= \sin \mathrm{am}(u,\alpha), \\ \mathrm{cn}(u,\alpha) &= \cos \mathrm{am}(u,\alpha), \\ \mathrm{dn}(u,\alpha) &= \sqrt{1-\alpha^2 \sin^2 \mathrm{am}(u,\alpha)}. \end{aligned}$$

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- JEFs are meromorphic functions in u (with α fixed) as well as meromorphic functions in α (with u fixed). While K is analytic in the cut-plane $\mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$.

Contents

- We start with the identities

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where we denote $\mu(\cdot) := \langle e_1, E_J(\cdot) e_1 \rangle$.

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- We get

$$\mathcal{F}[\mu](z) = \mathrm{cn}(z,\alpha).$$

Consequently, by applying the inverse Fourier transform to the function $\mathrm{cn}(z;\alpha)$, one may recover the spectral measure μ!

Spectral analysis of $J(\alpha)$ for $\alpha \in (-1, 1)$

- For $\alpha \in (-1, 1)$, the evaluation of the inverse Fourier transform yields

$$\mu(t) = \frac{\pi}{\alpha K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1+q^{2n+1}} \left[\delta\left(t - \frac{(2n+1)\pi}{2K}\right) + \delta\left(t + \frac{(2n+1)\pi}{2K}\right) \right]$$

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- It results in the formula

$$m(z, \alpha) = \frac{2\pi z}{\alpha K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1+q^{2n+1}} \frac{1}{\frac{(2n+1)^2 \pi^2}{4K^2} - z^2}.$$

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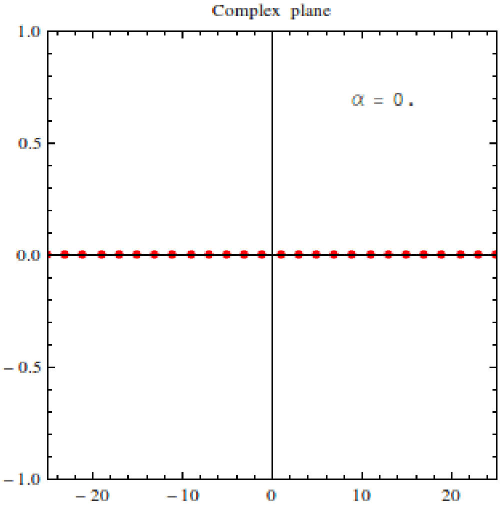
- This implies that the spectrum of $J(\pm 1)$ is purely absolutely continuous and
$$\sigma(J(\pm 1)) = \sigma_{ac}(J(\pm 1)) = \mathbb{R}.$$

Spectrum of $J(\alpha)$ in the self-adjoint case - animation



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  remains true for all $z \in \rho(J(\alpha))$ and $|\alpha| < 1$.
- It implies (in the non-self-adjoint case, too!) that

$$\sigma(J(\alpha)) = \frac{\pi}{2K}\left(2\mathbb{Z}+1\right).$$

  and all the eigenvalues are simple.

Proposition

Let $0 < |\alpha| < 1$ and $N \in \mathbb{Z}$, then the vector $v^{(N)}$ given by formulas

$$v_{2k+1}^{(N)} = \mathrm{i}(-1)^k \alpha^k \int_0^{2\pi} e^{-\mathrm{i}(N+1/2)s} \,\mathrm{cn}\left(\frac{Ks}{\pi}, \alpha\right) \mathrm{sn}^{2k}\left(\frac{Ks}{\pi}, \alpha\right) \mathrm{d}s$$

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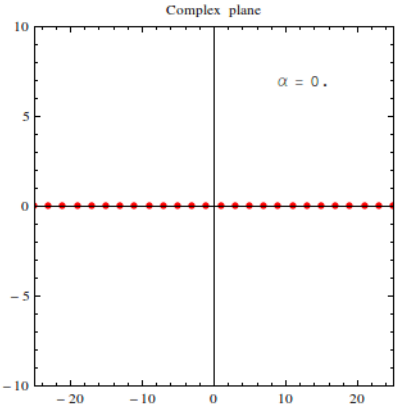
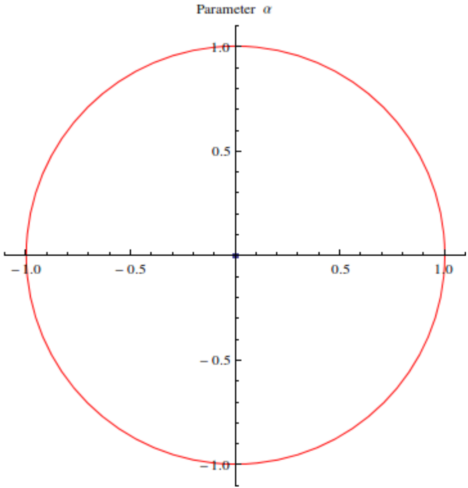
1. $$\|v^{(N)}\| = ? \quad \text{or} \quad \|v^{(N)}\| \sim ? \ \text{ for } N \to \pm\infty.$$
2. Is $\{v^{(N)} \mid N \in \mathbb{Z}\}$ the Riezs basis of $\ell^2(\mathbb{N})$?

Spectrum of $J(\alpha)$ in the non-self-adjoint case - animation





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Spectral analysis of $J(\alpha)$ for $|\alpha| = 1$

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- Then one can verify, indeed, that

$$\lim_{a\to 1-} \frac{\|(J(\alpha)-z)u(a)\|}{\|u(a)\|} = 0, \quad \text{and} \quad \mathrm{w}-\lim_{a\to 1-} u(a) = 0.$$

Spectral analysis of $J(\alpha)$ for $|\alpha| = 1$ - cont.

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$$\int_0^{2K} e^{-\mathrm{i}zt} \begin{Bmatrix} \mathrm{cn}(t,\alpha) \\ \mathrm{dn}(t,\alpha) \end{Bmatrix} \mathrm{sn}^k(t,\alpha)\mathrm{d}t, \qquad \text{as } k \to \infty.$$

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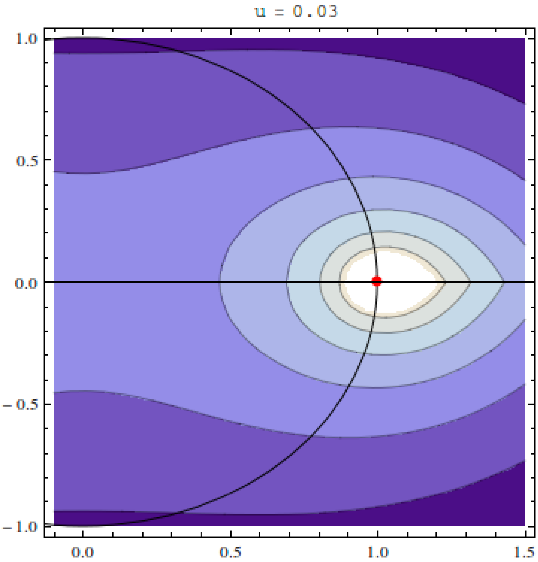
$$u \to |\,\mathrm{sn}(uK(\alpha),\alpha)|$$
in $(0,2)$ for $|\alpha| = 1$, $\alpha \neq \pm 1$.
- It can be shown (not trivial!) that the function has unique global maximum at $u = 1$ for every $|\alpha| = 1$, $\alpha \neq \pm 1$.

Contents



A region in the α-plane where $|\operatorname{sn}(uK(\alpha), \alpha)| < 1$ for $u \in (0, 1)$ fixed.

Thank you!