On the asymptotic eigenvalue distribution of generalized Toeplitz matrices

František Štampach

Czech Technical University, Faculty of Information Technology

Queen's University Belfast Colloquium

September 28, 2018

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$$\mu_n \xrightarrow{w} \mu, \quad \text{i.e.,}\ \forall \varphi \in C_0(\mathbb{C}) : \int_{\mathbb{C}} \varphi \, d\mu_n \to \int_{\mathbb{C}} \varphi \, d\mu.$$

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$$\boxed{2)\ a_{-k} = \overline{a_k}, \quad \forall k \in \mathbb{Z}} \quad \Leftrightarrow \quad a(\mathbb{T}) \subset \mathbb{R}$$

$$\Leftrightarrow \quad T_n(a) = (T_n(a))^*, \quad \forall n \in \mathbb{N}.$$

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$$T_n(a) \xrightarrow{s} T(a) \quad \text{in } \ell^2(\mathbb{N}),$$
where $T(a)$ is the bounded operator given by the respective semi-infinite Toeplitz matrix and $T_n(a)$ is identified with $P_n T(a) P_n$ where P_n is the OG projection onto $\mathrm{span}\{e_1, \ldots, e_n\}$.

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- In total, for s.-a. Toeplitz from the Wiener class it holds that
$$\boxed{\Lambda(T(a)) = \operatorname{spec} T(a) = \left[\min_{z\in\mathbb{T}} a(z), \max_{z\in\mathbb{T}} a(z)\right].}$$

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$$\lim_{n\to\infty} \frac{1}{n} \operatorname{Tr}\left(T_n(a)\right)^k = \frac{1}{2\pi} \int_0^{2\pi} a^k(e^{\mathrm{i}t})\mathrm{d}t, \quad \forall k \in \mathbb{N}.$$

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$$\lim_{n\to\infty} \int_{\mathbb{R}} x^k \mathrm{d}\mu_n(x) = \int_{\mathbb{R}} x^k \mathrm{d}\mu(x), \quad \forall k \in \mathbb{N},$$

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- Roughly speaking:

  "*For n large, the eigenvalues of $T_n(a)$ are distributed as the values of $t \mapsto a(e^{\mathrm{i}t})$.*"

A numerical illustration

Example:

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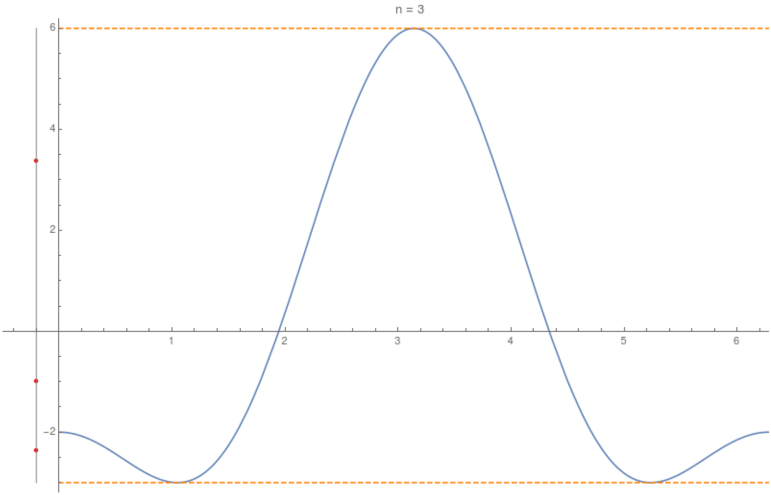

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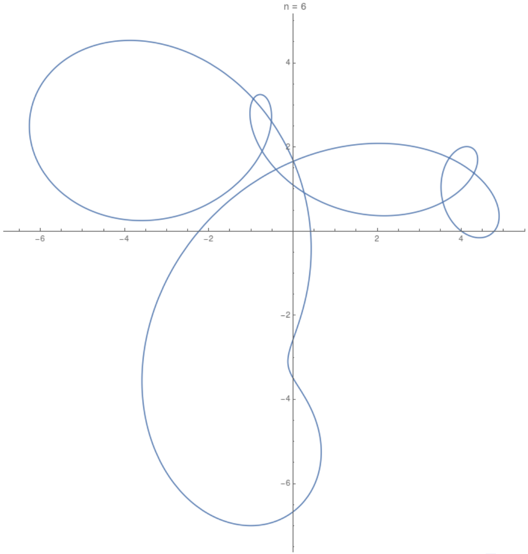
- The relation between $\operatorname{spec} T(b)$ and $\Lambda(T(b)) \ldots$ [`numerical illustration`].

A numerical illustration

Example: $$b(z) = 3\mathrm{i}z^{-1} - (1+2\mathrm{i})z^2 + (1+\mathrm{i})z^3 - (1-\mathrm{i})z^5$$



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- But one has

$$\Lambda(T(b)) = \bigcap_{\rho>0} \operatorname{spec}(T(b_\rho)),$$

  where

$$b_\rho(z) := b(\rho z).$$

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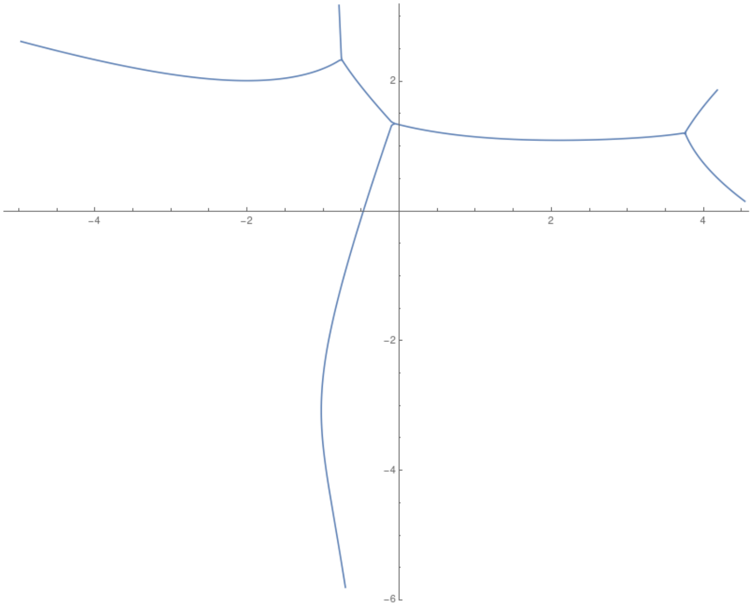
Theorem (Schmidt, Spitzer, Ullman, 1960-67):

$\Lambda(T(b))$ is a connected set that equals the union of a finite number of pairwise disjoint open analytic arcs and a finite number of the so-called exceptional points (roughly speaking: branching points and endpoints).

The example once more



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Proposition:

There is a smooth function $g : \mathbb{C} \setminus \Lambda(T(b))$ such that

$$\boxed{\lim_{n\to\infty} \left|\det(T_n(b) - \lambda)\right|^{1/n} = g(\lambda)}$$

locally uniformly in $\mathbb{C} \setminus \Lambda(T(b))$.

AED for the banded Toeplitz matrices

- The formula for AED for s.-a. Toeplitz matrices followed from the first Szegő limit formula:

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{Tr}\left(T_n(b)\right)^k = \frac{1}{2\pi}\int_0^{2\pi} b^k(e^{\mathrm{i}t})\mathrm{d}t, \quad \forall k \in \mathbb{N}.$$

- M. Kac (1954) proved that the above formula holds also when the self-adjointness is relaxed.
- However, it cannot be extended from polynomials to continuous functions because $\mathbb{C}[z]$ is not dense in $C_0(\mathbb{C})$.
- The derivation of AED in the non-self-adjoint case is based on a more detailed asymptotic analysis of the determinant of Toeplitz matrices (Szegő, Widom):

Proposition:

There is a smooth function $g : \mathbb{C} \setminus \Lambda(T(b))$ such that

$$\lim_{n\to\infty} \left|\det(T_n(b) - \lambda)\right|^{1/n} = g(\lambda)$$

locally uniformly in $\mathbb{C} \setminus \Lambda(T(b))$. In addition, one has

$$g(\lambda) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left|b(\rho e^{\mathrm{i}t}) - \lambda\right|\mathrm{d}t\right),$$

where $|z_r(\lambda)| < \rho < |z_{r+1}(\lambda)|$. (Recall $\lambda \in \Lambda(T(b)) \Leftrightarrow |z_r(\lambda)| = |z_{r+1}(\lambda)|$.)

Hirschman's formula for AED

Theorem (Hirschman Jr., 1967):

For a banded Toeplitz matrices, the AED μ exists and is absolutely continuous. On each analytic arc of $\Lambda(T(b))$, its density reads

$$\frac{\mathrm{d}\mu}{\mathrm{d}s}(\lambda) = \frac{1}{2\pi}\frac{1}{g(\lambda)}\left|\frac{\partial g}{\partial \vec{n}}(\lambda+) - \frac{\partial g}{\partial \vec{n}}(\lambda-)\right|,$$

where $\mathrm{d}s$ stands for the arc-length measure on the respective arc and $\partial g/\partial\vec{n}(\lambda\pm)$ are one-sided limits of the directional derivative $\partial g/\partial\vec{n}$ w.r.t. a unit normal vector to the arc at λ (depends on a chosen orientation but the formula for the density does not).

Final comments on Toeplitz matrices

- All the essential results for the *limiting set* $\Lambda(T(b))$ (Schmidt & Spitzer) and the *limiting measure* μ (Hirschman) concern banded Toeplitz matrices only.

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- But there are still a lot of matrices in the Wienner class but not with a rational symbol and basically nothing is known for them...

Open problem:

Can one deduce a description of

1. the set $\Lambda(T(a))$ / generalize Schmidt and Spitzer's result
2. the AED μ / generalize Hirschman's result

for some non-rational symbols?

Contents

1 Self-adjoint Toeplitz matrices

2 Non-self-adjoint banded Toeplitz matrices

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4 Non-Self-adjoint KMS matrices

KMS matrix

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i.e.,

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  2. A. Kuijlaars, W. Van Assche, 1999: orthogonal polynomials with variable coefficients.

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$\boxed{\text{2) } a \text{ is real-valued}}$ $\quad\Leftrightarrow\quad T_n(a) = (T_n(a))^*\,, \quad \forall n \in \mathbb{N}.$

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($\sim$ the Wiener class)

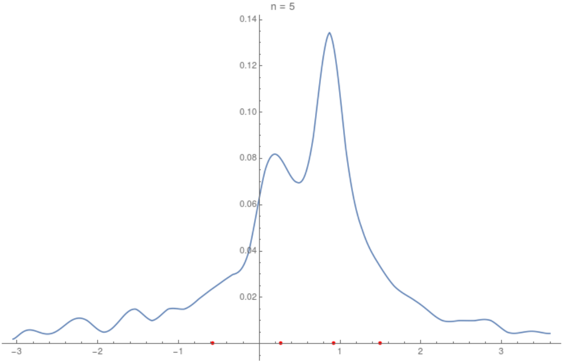
2) a is real-valued $\quad \Leftrightarrow \quad T_n(a) = (T_n(a))^*, \quad \forall n \in \mathbb{N}.$

Theorem (Kac, Murdock, Szegő, 1953)

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{Tr}(\varphi(T_n(a))) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \varphi(a(x,t)) \mathrm{d}x\mathrm{d}t, \quad \forall \varphi \in C(\mathbb{R}).$$

A numerical illustration

Example: $$a(x,t) = 2x^3 e^{-2\mathrm{i}t} + xe^{-\mathrm{i}t} + (1 - x^2) + xe^{\mathrm{i}t} + 2x^3 e^{-2\mathrm{i}t}$$



The AED μ:

$$\mu((\alpha,\beta)) = \frac{1}{2\pi} \left| \{(x,t) \in [0,1] \times [0,2\pi] \mid \alpha < a(x,t) < \beta\} \right|$$

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- Let us look at the numerics first...

Example

$$a(x,t) = \mathrm{i}xe^{-\mathrm{i}t} + 3(1-x^2) + \mathrm{i}xe^{\mathrm{i}t}$$

$$T_n(a) = \begin{pmatrix} 3-\frac{3}{n^2} & \frac{\mathrm{i}}{n} & & & \\ \frac{\mathrm{i}}{n} & 3-\frac{12}{n^2} & \frac{2\mathrm{i}}{n} & & \\ & \frac{2\mathrm{i}}{n} & 3-\frac{27}{n^2} & \frac{3\mathrm{i}}{n} & \\ & \ddots & \ddots & \ddots & \ddots \\ & & & \frac{n-1}{n}\mathrm{i} & 3 \end{pmatrix},$$

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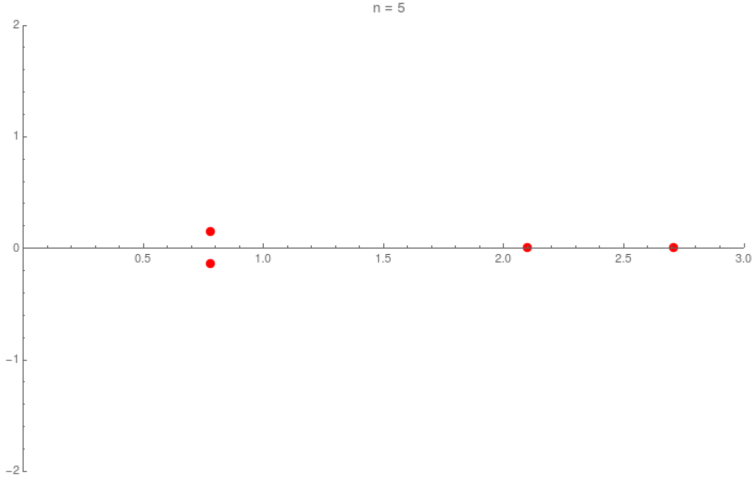

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The special KMS matrices (Sampling Jacobi matrices):

$$\boxed{a(x,t) = \alpha(x)e^{-\mathrm{i}t} + \beta(x) + \alpha(x)e^{\mathrm{i}t},}$$

where $\alpha^2, \beta \in \mathbb{C}[x]$ of low degree.

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- In this setting, there is a close connection between $\det(z - T_n(a))$ and the hypergeometric orthogonal polynomials. We make use of some special properties of these polynomials.

The strategy for the derivation of the limiting measure

Definition:

The *Cauchy transform* of a Borel measure μ is a function defined by

$$C_\mu(z) := \int_{\mathbb{C}} \frac{\mathrm{d}\mu(x)}{z - x}, \quad z \in \mathbb{C} \setminus \operatorname{supp} \mu.$$

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Theorem

Let μ_n is a sequence of probability measures supported uniformly in a compact set $K \subset \mathbb{C}$. Assume that

$$\lim_{n\to\infty} C_{\mu_n}(z) = C(z), \quad \text{a.e. } z \in \mathbb{C}.$$

Then C is the Cauchy transform of a probability measure μ which is a weak limit of μ_n for $n \to \infty$. Moreover, one has

$$\mu = \frac{1}{\pi} \partial_{\overline{z}} C \quad \text{in the generalized sense.}$$

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- The main difficultly of the strategy: $\boxed{p_n(z) \sim ?}$ for $n \to \infty$.
- There are many powerful methods for the asymptotic analysis but it usually requires a more detailed knowledge about p_n (generating functions, integral representations, recurrences,...).

An appetizer - one simple example

$$\alpha(x) = \sqrt{ax}, \quad (a > 0),$$

$$\beta(x) = \mathrm{i}x,$$

$$T_n = \begin{pmatrix} \beta\left(\frac{1}{n}\right) & \alpha\left(\frac{1}{n}\right) & & \\ \alpha\left(\frac{1}{n}\right) & \beta\left(\frac{2}{n}\right) & \alpha\left(\frac{2}{n}\right) & \\ & \ddots & \ddots & \ddots \\ & & \alpha\left(\frac{n-1}{n}\right) & \beta\left(1\right) \end{pmatrix},$$

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- The characteristic polynomial of T_n can be identified with the Charlier polynomials:

$$p_n(z) = C_n^{(-an)}\left(-an - \mathrm{i}zn - 1\right),$$

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- The analysis is very technical (steepest descent, Stokes phenomenon).

An appetizer - final results

- Define the curve: $\boxed{\gamma(x) := x + \mathrm{i}y(x),}\quad x \in (-2\sqrt{a}, 2\sqrt{a}),$

  where y is the solution of

$$y'(x) = -\frac{\Im \log\left((1+\xi_+)/(1+\xi_-)\right)}{\Re \log\left((1+\xi_+)/(1+\xi_-)\right)}, \quad y(2\sqrt{a}) = 1,$$

  $\xi_\pm = \xi_\pm(z, a)$ are the two solutions of $a\xi^2 - (1 + \mathrm{i}z)\xi - 1 = 0$, and $z = x + \mathrm{i}y(x)$.

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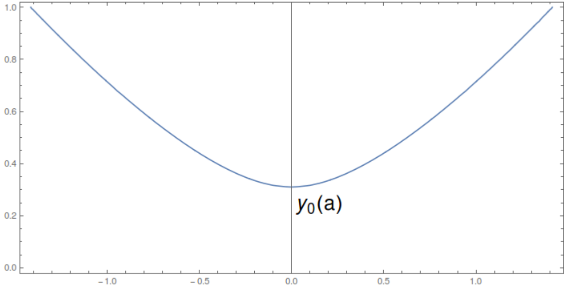

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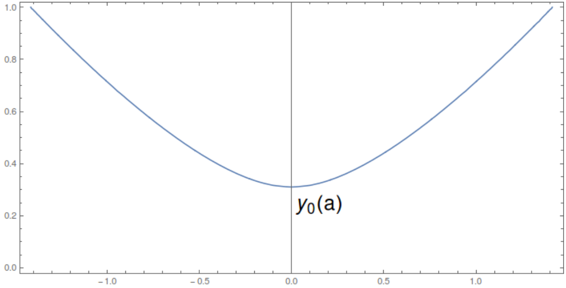
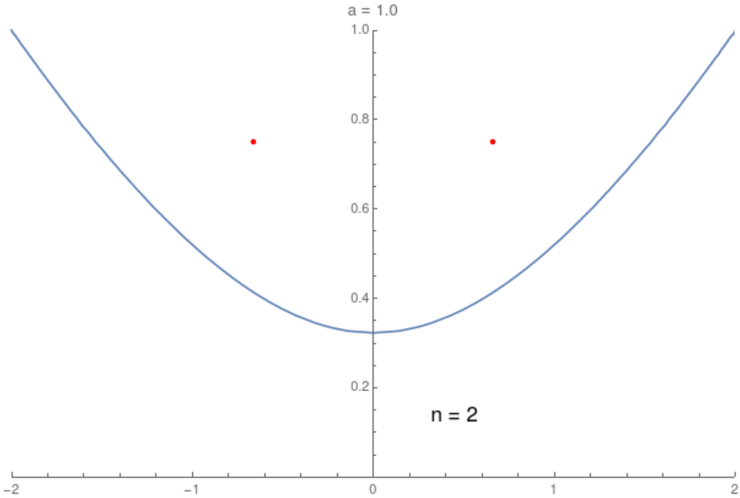


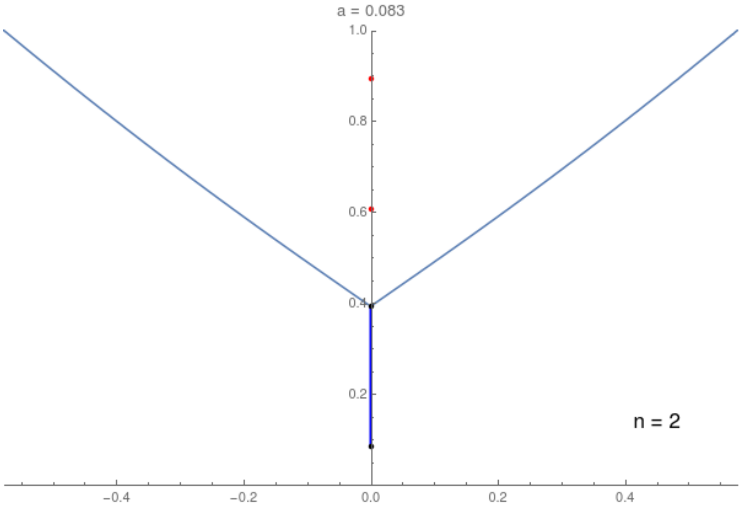
- $y_0(a)$ is the imaginary coordinate of the intersection of the curve and the imaginary line.

Regime 1: $a > y_0(a)$



$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(x) = \frac{1}{2\pi}\frac{|\log\left((1+\xi_+)/(1+\xi_-)\right)|^2}{\Re\log\left((1+\xi_+)/(1+\xi_-)\right)}, \qquad |x| < 2\sqrt{a}.$$

Regime 2: $a < y_0(a)$



$$\frac{d\mu}{dx}(x) = \frac{1}{2\pi} \frac{|\log((1+\xi_+)/(1+\xi_-))|^2}{\Re \log((1+\xi_+)/(1+\xi_-))}, \quad \text{and} \quad \frac{d\mu}{dy}(y) = 1.$$

More to see at: `http://users.fit.cvut.cz/~stampfra`

Thank you!