# Constructing measures of orthogonality with applications

František Štampach

Faculty of Nuclear Sciences and Physical Engineering
& Faculty of Information Technology, CTU in Prague

Formal and Analytic Solutions of Differential, Difference and Discrete Equations

August 29, 2013

# Outline

1. **Function $\mathfrak{F}$ and its fundamental properties**
2. **Function $\mathfrak{F}$ and orthogonal polynomials**
3. **Constructing measure of orthogonality**
4. **Application: Generalized Al-Salam-Carlitz I polynomials**

### Definition

Let us define $\mathfrak{F} : \operatorname{Dom}\mathfrak{F} \to \mathbb{C}$ by relation

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty}(-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1}x_{k_1+1}x_{k_2}x_{k_2+1}\dots x_{k_m}x_{k_m+1},$$

where

$$\operatorname{Dom}\mathfrak{F} = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C};\ \sum_{k=1}^{\infty}|x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables let me identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$.

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- However, function $\mathfrak{F}$ is also related with *continued fractions*, *bilateral second order difference equations*, as well as *orthogonal polynomials*.
- In this talk we focus on usage of $\mathfrak{F}$ for description of the *measure of orthogonality* of orthogonal polynomials.

## Some examples

1. Put $x_k = z/(\nu + k)$, then

$$\mathfrak{F}(x) = \Gamma(\nu + 1) z^{-\nu} J_\nu(2z),$$

for $z \in \mathbb{C}$ and $-\nu \notin \mathbb{N}$, where $J_\nu$ is the *Bessel function* of the first kind.

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2. Put $x_k = z^{1/2} q^{(2k-1)/4}$, then

$$\mathfrak{F}(x) = A_q(z) := {}_0\phi_1(; 0; q, -qz),$$

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3. Put

$$x_k = \frac{q^{\frac{1}{2}(\alpha+\gamma+k)-\frac{3}{4}} (q^{\gamma-\alpha+k};q^2)_\infty \, z^{\frac{1}{2}}}{(q^{\gamma-\alpha+k+1};q^2)_\infty \left(1-(1-z)q^{\gamma+k-1}\right)},$$

then

$$\mathfrak{F}(x) = \frac{(q^\gamma;q)_\infty}{((1-z)q^\gamma;q)_\infty} \, {}_1\phi_1(q^\alpha;q^\gamma;q,-q^\gamma z),$$

for $z, \alpha, \gamma \in \mathbb{C}$, $(1-z)q^\gamma \notin q^{-\mathbb{Z}_+}$ and $q \in (0,1)$, where ${}_1\phi_1$ is *q-confluent hypergeometric function* (proof in [F. Š., P. Šťovíček, LAA, 2013]).

## Fundamental property of $\mathfrak{F}$

- For all $x \in \mathrm{Dom}\,\mathfrak{F}$ and $k = 1, 2, \dots$ one has

**Recursive relation**

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \dots, x_k)\,\mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \dots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(T^{k+1} x)$$

where $T$ denotes the left shift operator defined on the space of all sequences:

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**Typical example:** For $x_k = z/(\nu + k - 1)$, the simple recurrence relation for $\mathfrak{F}$ yields the well known formula for Bessel functions:

$$J_{\nu-1}(2z) = \frac{\nu}{z} J_\nu(2z) - J_{\nu+1}(2z).$$

- By the *Favard's theorem*, the couple of polynomial sequences $(\{F_n\}_{n=0}^{\infty}, \{G_n\}_{n=0}^{\infty})$ defined recursively by equation

$$u_{n+1} = (x - \lambda_n)u_n - w_{n-1}^2 u_{n-1}, \quad n = 1, 2, \ldots,$$

where $\lambda_n \in \mathbb{R}$ and $w_n > 0$, and with initial conditions

$$F_0(x) = 1, \qquad F_1(x) = x - \lambda_0,$$
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- As one easily verifies by induction, polynomials $F_n$ and $G_n$ can be expressed in terms of $\mathfrak{F}$,

$$F_n(x) = \prod_{k=0}^{n-1} (x - \lambda_k)\, \mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l - x}\right\}_{l=0}^{n-1}\right), \quad n = 0, 1 \ldots,$$

and

$$G_n(x) = \prod_{k=1}^{n-1} (x - \lambda_k)\, \mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l - x}\right\}_{l=1}^{n-1}\right), \quad n = 0, 1 \ldots,$$

where the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is defined recursively by $\gamma_0 = 1$, $\gamma_{k+1} = w_k/\gamma_k$.

### Proposition

If $\sum_{k\geq 0} \left| \frac{w_k^2}{(x-\lambda_k)(x-\lambda_{k+1})} \right| < \infty$, for some $x \in \mathbb{C}$, then the limit relation

$$\lim_{n\to\infty} \prod_{k=0}^{n-1} (x - \lambda_k)^{-1} F_n(x) = \mathfrak{F}\left( \left\{ \frac{\gamma_k^2}{\lambda_k - x} \right\}_{k=0}^{\infty} \right)$$

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**Typical example:** By setting $\lambda_k = 0$ and $w_k = [4(k+\nu)(k+\nu+1)]^{-1/2}$, polynomials

$$F_n(x) = x^n \mathfrak{F}\left(\left\{\frac{1}{2x(\nu+k)}\right\}_{k=0}^{n-1}\right), \quad n = 0, 1, 2\ldots,$$

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The above limit relation yields the Hurwitz's asymptotic formula for Lommel polynomials

$$\lim_{n\to\infty} \frac{x^n}{2^n \Gamma(\nu+n)} R_{n,\nu}(x) = \left(\frac{x}{2}\right)^{-\nu+1} J_{\nu-1}(x).$$

## Regularization

- The asymptotic behavior of $F_n$, as $n \to \infty$, is expressed in terms of function

$$\Phi(\lambda, w; z) = \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=0}^{\infty}\right)$$

under the assumption that ensures the function to be well defined. This function is meromorphic on $\mathbb{C} \setminus \mathrm{der}(\lambda)$ with poles at $z = \lambda_k$ such that $\lambda_k \notin \mathrm{der}(\lambda)$.

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- Taking into account later application, we restrict sequences $\lambda$ and $w$ such that $\lambda \in \ell^1(\mathbb{Z}_+)$ and $w \in \ell^2(\mathbb{Z}_+)$. Then function
$$\psi_\lambda(z) = \prod_{n=0}^{\infty}(1 - z\lambda_n)$$
is well defined entire function and $\psi_\lambda^{(-1)}(\{0\}) = \{\lambda_n^{-1} : \lambda_n \neq 0,\ n \in \mathbb{Z}_+\}$.

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- Let us define function

$$G(\lambda, w; z) = \begin{cases} \psi_\lambda(z)\Phi(\lambda, w; z^{-1}) & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

Assuming $\lambda \in \ell^1(\mathbb{Z}_+)$ and $w \in \ell^2(\mathbb{Z}_+)$, function $G(\lambda, w; .)$ is entire.

# Markov theorem

- For the limit of the ratio $G_n(z^{-1})/F_n(z^{-1})$, now we have

$$\lim_{n\to\infty}\frac{G_n(z^{-1})}{F_n(z^{-1})} = z\,\frac{G(T\lambda, Tw; z)}{G(\lambda, w; z)},$$

  for all $z \neq 0$ not being zeros of function $G(\lambda, w; .)$.

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**Theorem (Markov)**

Let $\lambda$ be real and $w$ positive sequence and, moreover, both bounded. Then polynomials $\{F_n\}_{n=0}^{\infty}$ are orthogonal with respect to measure $\mu$, for which, it holds

$$\int_{\mathbb{R}} \frac{d\mu(x)}{z-x} = \lim_{n\to\infty} \frac{G_n(z)}{F_n(z)},$$

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- Thus, by the Markov theorem, one finds

$$\int_{\mathbb{R}}\frac{d\mu(x)}{1-xz} = \frac{G(T\lambda, Tw; z)}{G(\lambda, w; z)}.$$

## Proposition

Let $\lambda \in \ell^1(\mathbb{Z}_+)$ be real and $w \in \ell^2(\mathbb{Z}_+)$ be positive sequence. Then all zeros of functions $G(\lambda, w; .)$ and $G(T\lambda, Tw; .)$ are real, simple, and there are infinitely many of them (for each function). Moreover, these two functions have no zero in common.

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**Typical example:** If we put $\lambda_k = 0$ and $w_k = [4(\nu + k)(\nu + k + 1)]^{-1/2}$, with $\nu > 0$, then the statement is about zeros of Bessel functions $z^{-\nu+1}J_{\nu-1}(z)$ and $z^{-\nu}J_\nu(z)$.

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$$\operatorname{supp}(\mu) = \{0\} \cup \{z^{-1} : G(\lambda, w, z) = 0\}.$$
- Furthermore, denoting by $\mu_k$, $k \in \mathbb{N}$, zeros of $G(\lambda, w; .)$, we have the Mittag-Leffler expansion
$$\Lambda_0 + \sum_{k=1}^{\infty} \frac{\Lambda_k}{1 - \mu_k^{-1} z} = \frac{G(T\lambda, Tw; z)}{G(\lambda, w; z)}$$
where the convergence of the sum is local uniform in $z \notin \{\mu_k : k \in \mathbb{N}\}$.

## Towards orthogonality

- Numbers $\Lambda_k$ represents jumps of distribution function $F_\mu(x) := \mu((-\infty, x])$ at $x = \mu_k^{-1}$ and $\Lambda_0$ jump at $x = 0$. We can express these jumps as

$$\Lambda_k = \lim_{z \to \mu_k} (1 - \mu_k^{-1} z) \frac{G(T\lambda, Tw; z)}{G(\lambda, w; z)} = -\mu_k^{-1} \frac{G(T\lambda, Tw; \mu_k)}{(\partial_z G)(\lambda, w; \mu_k)}.$$

## Towards orthogonality

- Numbers $\Lambda_k$ represents jumps of distribution function $F_\mu(x) := \mu((-\infty, x])$ at $x = \mu_k^{-1}$ and $\Lambda_0$ jump at $x = 0$. We can express these jumps as

$$\Lambda_k = \lim_{z\to\mu_k}(1-\mu_k^{-1}z)\frac{G(T\lambda, Tw; z)}{G(\lambda, w; z)} = -\mu_k^{-1}\frac{G(T\lambda, Tw; \mu_k)}{(\partial_z G)(\lambda, w; \mu_k)}.$$

- Finally, the orthogonality relation for polynomials $\{F_n\}_{n=0}^{\infty}$ reads

$$\int_{\mathbb{R}} F_m(x)F_n(x)d\mu(x) = \left(\prod_{k=0}^{n-1} w_k^2\right)\delta_{mn}, \quad m, n \in \mathbb{Z}_+.$$

**Theorem**

For $\lambda \in \ell^1(\mathbb{Z}_+)$ be real and $w \in \ell^2(\mathbb{Z}_+)$ positive sequence we introduce function

$$G(\lambda, w; z) = \prod_{n=0}^{\infty}(1 - z\lambda_n)\,\mathfrak{F}\left(\left\{\frac{z\gamma_k^2}{1 - z\lambda_k}\right\}_{k=0}^{\infty}\right),$$

Then the measure of orthogonality $\mu$ of corresponding orthogonal polynomials $\{F_n\}_{n=0}^{\infty}$ is supported by a real sequence with 0, the only cluster point. Moreover, we have

$$\operatorname{supp}(\mu) = \{0\} \cup \{z^{-1} : G(\lambda, w; z) = 0\}.$$

The orthogonality relation reads

$$\int_{\mathbb{R}} F_m(x)F_n(x)d\mu(x) = \left(\prod_{k=0}^{n-1} w_k^2\right)\delta_{mn}, \quad m, n \in \mathbb{Z}_+,$$

and, for $x \in \operatorname{supp}(\mu) \setminus \{0\}$, distribution function $F_\mu(x) := \mu((-\infty, x])$ has jumps

$$F_\mu(x) - F_\mu(x - 0) = -x\frac{G(T\lambda, Tw; x^{-1})}{(\partial_z G)(\lambda, w; x^{-1})}.$$

# Application: Orthogonal polynomial arising from example with ${}_1\phi_1$

- The example with $q$-confluent hypergeometric function introduced at the beginning, slightly reparametrized, yields

$$\mathfrak{F}\left(\left\{\frac{q^{\frac{1}{2}(\delta+k)-\frac{1}{4}}\,(q^{k+1-\delta};q^2)_\infty\,\sqrt{-a}}{(q^{k+2-\delta};q^2)_\infty\,((a+1)q^k-x)}\right\}_{k=0}^{\infty}\right)=\frac{(x^{-1};q)_\infty}{(x^{-1}(a+1);q)_\infty}\,{}_1\phi_1\left(x^{-1}q^{\delta};x^{-1};q,ax^{-1}\right)$$

where $x\notin(a+1)q^{\mathbb{Z}_+}\cup\{0\}$.

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  where $x\notin(a+1)q^{\mathbb{Z}_+}\cup\{0\}$.

- This identity correspond with the polynomial sequence $U_n(a,\delta;q,x)$, $n\in\mathbb{Z}_+$, which is generated by recursion

$$v_{n+1}=\left(x-(a+1)q^n\right)v_n+aq^{n+\delta-1}(1-q^{n-\delta})v_{n-1},\quad n\in\mathbb{N},$$

  with initial setting $U_0(a,\delta;q,x)=1$ and $U_1(a,\delta;q,x)=x-a-1$.

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- Thus, in this case, sequences $\{\lambda_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$ are as follows:

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- For $\delta=0$, polynomials $U_n(a,0;q,x)$ are known as Al-Salam-Carlitz I and are listed in the $q$-Askey scheme. They can be expressed as

$$U_n(a,0;q,x)=(-a)^n q^{\binom{n}{2}}\,{}_2\phi_1\left(q^{-n},x^{-1};0;q,a^{-1}qx\right).$$

- In this case, one deduces

$$G(T^k\lambda, T^k w; x) = {}_1\tilde{\phi}_1(xq^\delta; q^k x; q, aq^k x), \qquad \text{for } k = 0, 1, 2, \ldots,$$

  where ${}_1\tilde{\phi}_1$ denoted regularized $q$-confluent hypergeometric function defined by

$${}_1\tilde{\phi}_1(a; b; q, z) := (b; q)_\infty {}_1\phi_1(a; b; q, z).$$

Consequently, assuming $a < 0$ and $\delta < 1$, we can use the general theorem to find:

## The orthogonality of generalized Al-Salam-Carlitz I polynomials

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Consequently, assuming $a < 0$ and $\delta < 1$, we can use the general theorem to find:

**i)** Polynomials $\{U_n(a, \delta; q, x) : n \in \mathbb{Z}_+\}$ satisfy the orthogonality relation

$$\int_{\mathbb{R}} U_m(a, \delta; q, x) U_n(a, \delta; q, x) d\mu(x) = (-a)^n q^{n\delta + n(n-1)/2} (q^{1-\delta}; q)_n \delta_{mn}$$

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**ii)** Measure $\mu$ is supported by the set

$$\operatorname{supp}(\mu) = \{x^{-1} \in \mathbb{C} : {}_1\tilde{\phi}_1\left(xq^\delta; x; q, ax\right) = 0\} \cup \{0\},$$

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- If $\delta = 0$ these results yields orthogonality for Al-Salam-Carlitz I polynomials, which can be described fully explicitly.

# Other properties - Generating function

Suppose $x \neq 0$ then the generating function for $U_n(a, \delta; q, x)$ reads:

**i)** if $\delta < 0$,

$$\sum_{n=0}^{\infty} \frac{U_n(a, \delta; q, x)}{(q^{-\delta}; q)_{n+1}} t^n = \sum_{k=0}^{\infty} \frac{(aq^{\delta} t; q)_k (q^{\delta} t; q)_k}{(xt; q)_{k+1}} q^{-k\delta}, \quad |xt| < 1,$$

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**iii)** if $\delta > 0$,

-unknown-

# Other properties - Structure relations

- For $n \in \mathbb{Z}_+$, it holds

$$\mathcal{D}_q U_n(a,\delta;q,x) = \frac{1-q^{n-\delta}}{1-q} U_{n-1}(a,\delta;q,x) - q^n \frac{1-q^{-\delta}}{1-q} U_{n-1}(a,\delta-1;q,x).$$

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- On the other hand, it seems there is no simple formula which would generalize the *backward shift* for Al-Salam-Carlitz I polynomials, which reads

$$(a-x)(1-x)U_n(a,0;q,q^{-1}x) - aU_n(a,0;q,x) = xq^{-n}U_{n+1}(a,0;q,x).$$

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- Consequently, we do not know if there is a second order $q$-difference equation for polynomials $U_n(a,\delta;q,x)$.

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Thank you!