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Spectral Analysis of Jacobi matrices and Related Problems

DOCTORAL THESIS

June 4, 2014

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by

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Dissertation submitted to
the Faculty of Nuclear Sciences and Physical Engineering of
Czech Technical University in Prague
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

in the branch of study Mathematical Modelling
of the study program Mathematical Engineering

Prague, June 2014

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List of Symbols

$\overline{A}$ .... the closure of a closable linear operator A
$\mathbb{C}$ .... complex numbers
$\mathbb{C}^{n,n}$ .... space of $n \times n$ complex matrices
$\mathbb{C}^{\infty}$ .... space of complex sequences (indexed by $\mathbb{N}$)
$\mathbb{C}_0^{\lambda}$ .... the complement of the closure of range of a sequence λ into $\mathbb{C}$
$\mathbb{C}[x]$ .... the ring of polynomials in indeterminate x
$\mathbb{C}[[x]]$ ... the ring of formal power series in countably many indeterminates $x = \{x_n\}_{n=1}^{\infty}$
$\mathscr{D}$ .... space of complex sequences with at most finitely many nonvanishing entries
$\deg P$ .... degree of a polynomial P
$\operatorname{der} \lambda$ .... set of finite accumulation points of a sequence λ
$\dim V$ .... dimension of a linear space V
$\operatorname{Dom} A$ .... domain of a linear operator A
δ_x .... Dirac measure supported on $\{x\}$, $\delta_x(\mathcal{M}) = 1$, if $x \in \mathcal{M}$, $\delta_x(\mathcal{M}) = 0$, if $x \notin \mathcal{M}$
$\delta_{x,y}$ .... Kronecker delta, $\delta_{x,y} = 1$, if $x = y$, $\delta_{x,y} = 0$, otherwise
e_n .... the n-th vector of the standard basis of $\ell^2(\mathbb{N})$ (or $\ell^2(\mathbb{Z}_+)$)
E_J .... projection-valued spectral measure of a self-adjoint operator J
Hmp .... Hamburger moment problem
I .... identity operator
$\operatorname{Im} z$ .... imaginary part of a complex number z
$\mathscr{I}_p(\mathcal{H})$ .... p-th class of Schatten-von Neumann operators on a Hilbert space $\mathcal{H}$
$J_\nu(.)$ .... Bessel function of the first kind
$j_{\nu,k}$ .... the k-th positive zero of the Bessel function $J_\nu(.)$
$\operatorname{Ker} A$ .... kernel of a linear operator A
$\ell^p(\mathbb{N})$ .... the Banach space of $x \in \mathbb{C}^\infty$ for which $\sum_{n\in\mathbb{N}} |x_n|^p < \infty$ where $p \geq 1$
$\ell^\infty(\mathbb{N})$ .... the Banach space of bounded complex sequences indexed by $\mathbb{N}$
$L^2(\mathbb{R}, \mathrm{d}\mu)$ .... square-integrable functions on $\mathbb{R}$ with respect to measure μ
LHS .... left-hand side
$\mathbb{N}$ .... positive integers, $\mathbb{N} = \{1, 2, 3, \ldots\}$
OPs .... Orthogonal Polynomials
$\mathscr{P}$ .... set of Pick functions
$\mathbb{R}$ .... real numbers
$\operatorname{Ran} F$ .... range of a map F
$\operatorname{Res}(f, z_0)$ .... residue of a meromorphic function f at z_0
$\operatorname{Re} z$ .... real part of a complex number z
RHS .... right-hand side
$\operatorname{spec}(A)$ .... spectrum of a linear operator A
$\operatorname{spec}_p(A)$ .... point spectrum of a linear operator A

supp μ . . . . . . support of a measure μ
Tr A . . . . . . trace of a trace class operator A
$\overline{z}$ . . . . . . the complex conjugate of complex number z
$\mathbb{Z}$ . . . . . . integer numbers, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$
$\mathbb{Z}_+$ . . . . . . nonegative integers, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$
$(a, b]$ . . . . . . semi-closed interval, $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
$\langle ., . \rangle$ . . . . . . an inner product

Acknowledgement

First of all, I would like to thank my supervisor Prof. Ing. Pavel Šťovíček, DrSc. for being an excellent and inspiring advisor. Cooperation with him was always very fruitful and of great value to me.

My thanks also belong to my parents for their faithful support throughout my studies. It is very likely this thesis would not exist without my friends who encouraged me to keep working on theoretical problems of mathematics in difficult times.

At last, I acknowledge gratefully for the financial support from the following grants: Grant No. LC06002 of the Ministry of Education of the Czech Republic and Grants No. 201/09/0811 and No. GA13-11058S of the Czech Science Foundation.

Preface

A semi-infinite symmetric matrix

$$\mathcal{J} = \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

called the Jacobi matrix in honor of Carolus Gustavus Iacobus Iacobi (1804-1851) is the central object of this thesis and everything we present is connected by the Jacobi matrix. First of all, the spectral analysis of linear operators associated with $\mathcal{J}$ is of great interest. A particular class of them are known as discrete Schrödinger operators and their spectral properties play the fundamental role in Quantum Mechanics.

From the mathematical point of view, operators determined by $\mathcal{J}$ constitute a substantial class of operators. For instance, any self-adjoint operator on a separable Hilbert space with simple spectrum is generated (as a minimal closed operator) by some matrix $\mathcal{J}$. This has been proved by M. Stone [40]. Even certain class of closed symmetric operators with deficiency indices $(1, 1)$ are generated by $\mathcal{J}$, see [16]. Therefore Jacobi matrices naturally emerge in models for those phenomena governed by operators of mentioned properties.

Moreover, the spectral theory of operators generated by Jacobi matrices has a significant impact in the theory of Orthogonal Polynomials, the Moment Problem and the theory of Continued Fractions. Several aspects of these connections are discussed in this thesis.

My scientific work as a Ph.D. student has resulted in 7 papers, not all of them being published, however. In present (June 4, 2014), 2 papers have already been published and 2 more have been accepted for publishing. Remaining 3 preprints are still going through the reviewing process in impacted journals. I list these papers in the order as they have been written:

(1) F. Štampach, P. Šťovíček: *On the eigenvalue problem for a particular class of finite Jacobi matrices*, Linear Alg. Appl. **434** (2011) 1336-1353.

(2) F. Štampach, P. Šťovíček: *The characteristic function for Jacobi matrices with applications*, Linear Algebra Appl. **438** (2013) 4130-4155.

(3) F. Štampach, P. Šťovíček: *Special functions and spectrum of Jacobi matrices*, Linear Algebra Appl. (2013) (in press).

(4) F. Štampach, P. Šťovíček: *Factorization of the characteristic function of a Jacobi matrix*, (submitted).

(5) F. Štampach, P. Šťovíček: *Orthogonal polynomials associated with Coulomb wave functions*, J. Math. Anal. Appl. (2014) (in press).

(6) F. Štampach, P. Šťovíček: *The Hahn-Exton q-Bessel function as the characteristic function of a Jacobi matrix*, (submitted).

(7) F. Štampach, P. Šťovíček: *The Nevanlinna parametrization for q-Lommel polynomials in the indeterminate case*, (submitted).

The first paper initiated my work on the characteristic function of a Jacobi matrix. It has been written in continuation of my master studies, however, it also contains results that has not been included in my master thesis.

In the second paper the characteristic function of a Jacobi matrix is introduced and its usage in the spectral analysis of Jacobi operators is described in full details. The theory is applied on several interesting examples ibidem. Even more concrete Jacobi operators with solvable spectra are analyzed in the third paper. Spectral properties of operators under investigation are described in terms of special functions; hypergeometric series and their q-analogues, exclusively.

In the paper (4) a proof of certain logarithm formula and factorization of the characteristic function into the Hadamard's type infinite product is presented. These results have applications concerning spectral zeta functions of Jacobi operators and continued fractions.

The fifth paper deals with a construction of the measure of orthogonality for orthogonal polynomials from a certain class. In addition, a new family of orthogonal polynomials which is a generalization of the well known Lommel polynomials is introduced and investigated.

Finally, papers (6) and (7) are closely related. In (6), a number of identities concerning orthogonality and other properties of q-Bessel functions are derived by using spectral analysis of a suitably chosen Jacobi operator. The notion of characteristic function is used here again although the formalism from (2) is no longer applicable. Moreover, the corresponding indeterminate moment problem is solved in (7) by means of derivation of explicit formulas for entire functions from the Nevanlinna parametrization. This completes the work initiated by Koelink in [21] on describing measures of orthogonality of q-Lommel polynomials.

The present thesis is dived into 4 parts. The first part serves as an introduction to papers (1)-(4). We summarize selected results of the highest importance usually provided without proofs that can be found in the papers enclosed. Moreover, there are several explanations and discussion on topics that have not been involved in papers (1)-(4). Either because they can be considered as a common knowledge in the society of mathematicians working on related topics, for example the construction of Jacobi operators from the Jacobi matrix $\mathcal{J}$. Or they do not follow the main idea of the respective paper and form a secondary result, for instance applications concerning continued fractions.

The second part provides some basic informations on orthogonal polynomials and the moment problem with emphasis on the indeterminate case and the Nevanlinna parametrization. Part III consists of reprints of (1)-(3) and preprint of (4) which serves as the complementary material to the first introductory part with full details. Similarly the last part is composed of reprint to (5) and preprints to papers (6) and (7) which supplement and provide additional results on problems indicated in the second part.

At the end, it is to be said a lot of results are simplified or not even mentioned in the general survey parts I and II. The truly interested reader is encouraged to read papers (1)-(7).

Part I

Characteristic Function of a Jacobi Matrix

I.1 Jacobi operator

Differences between terms *Jacobi matrix* and *Jacobi operator* are sometimes neglected in literature. While the former means usually a formal semi-infinite tridiagonal matrix, the meaning of the latter needs not be always fully clear. Let us denote by $\mathcal{J}$ a Jacobi matrix of the form

$$\mathcal{J} = \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{1}$$

where $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ and $w = \{w_n\}_{n=1}^{\infty}$ are given sequences.

In this section, we describe the standard construction of densely defined linear operators acting on $\ell^2(\mathbb{N})$ associated with a matrix $\mathcal{J}$. We assume $\lambda \subset \mathbb{R}$ and $w \subset \mathbb{R} \setminus \{0\}$, although everything can be done in the general case of complex λ and w as well, see [3].

Let us denote by $\mathbb{C}^\infty$ the linear space of all complex sequences indexed by $\mathbb{N}$ and by $\mathscr{D}$ its subspace of those sequences having at most finitely many nonvanishing entries. In other words, $\mathscr{D}$ coincides with the linear hull of the standard basis $\{e_n \mid n \in \mathbb{N}\}$ of $\ell^2(\mathbb{N})$. Notice matrix $\mathcal{J}$ acts linearly on $\mathbb{C}^\infty$ and $\mathscr{D}$ is a $\mathcal{J}$-invariant subspace. Action $\mathcal{J}x$, for $x \in \mathbb{C}^\infty$, is to be understood as the formal matrix product while treating x as a column vector.

Let us denote by $\dot{J}$ the restriction of $\mathcal{J}$ to $\mathscr{D}$. In general, operator $\dot{J}$ is not closed on $\ell^2(\mathbb{N})$, however, it is always closable on $\ell^2(\mathbb{N})$ since it is symmetric. Let us introduce $J_{\text{min}} := \overline{\dot{J}}$, the closure of $\dot{J}$. The subscript min indicates J_{min} is the smallest closed restriction of $\mathcal{J}$ having $\{e_n \mid n \in \mathbb{N}\}$ in its domain.

Another natural candidate of an operator associated with Jacobi matrix $\mathcal{J}$ is so called maximal domain operator J_{max}, see [18, Chp. 3, §2.]. That is J_{max} is the restriction of $\mathcal{J}$ on the set

$$\text{Dom}\, J_{\text{max}} := \{x \in \ell^2(\mathbb{N}) \mid \mathcal{J}x \in \ell^2(\mathbb{N})\}.$$

Clearly $\dot{J} \subset J_{\text{max}}$. Further, operators $\dot{J}$, J_{min}, and J_{max} and their adjoints are related as follows.

Lemma 1: It holds

$$(\dot{J})^* = J_{\text{min}}^* = J_{\text{max}}, \quad J_{\text{max}}^* = J_{\text{min}}.$$

Proof. The first equality is clear since $J_{\text{min}} = \overline{\dot{J}}$. In order to prove the second equality we recall the definition of the adjoint,

$$\text{Dom}\, J_{\text{min}}^* = \{y \in \ell^2(\mathbb{N}) \mid (\exists y^* \in \ell^2)(\forall x \in \text{Dom}\, J_{\text{min}})(\langle y, J_{\text{min}}x\rangle = \langle y^*, x\rangle)\}.$$

According to the definition of J_{min} and the continuity of the scalar product we can restrict with x to be from $\mathscr{D}$, or even further to require $\langle y, J_{\text{min}}e_j\rangle = \langle y^*, e_j\rangle$ holds for all $j \in \mathbb{N}$, by linearity. Thus,

$$\text{Dom}\, J_{\text{min}}^* = \{y \in \ell^2(\mathbb{N}) \mid (\exists y^* \in \ell^2)(\forall j \in \mathbb{N})(\langle y, J_{\text{min}}e_j\rangle = \langle y^*, e_j\rangle)\}.$$

Since $\langle y, J_{\text{min}}e_j\rangle$ coincides with the j-th element of $\mathcal{J}y$ we conclude $y \in \ell^2(\mathbb{N})$ belongs to the domain of J_{min}^* if and only if $\mathcal{J}y \in \ell^2(\mathbb{N})$. Hence $\text{Dom}\, J_{\text{min}}^* = \text{Dom}\, J_{\text{max}}$ and the equality $J_{\text{min}}^* = J_{\text{max}}$ follows. Equality $J_{\text{max}}^* = J_{\text{min}}$ is obtained by taking adjoints in the last equation. ☐

As a consequence of the last lemma, we see $J_{\max}$ is closed. Further we have $J_{\min} \subset J_{\max}$ and, in addition, any closed linear operator T which is a restriction of $\mathcal{J}$ on a domain containing the standard basis of $\ell^2(\mathbb{N})$ satisfies $J_{\min} \subset T \subset J_{\max}$.

If $J_{\min} = J_{\max}$ there is a unique densely defined self-adjoint operator whose matrix in the standard basis of $\ell^2(\mathbb{N})$ coincides with (1), the Jacobi operator. In this case, we drop the subscripts min and max and write just J referring the Jacobi operator in question. With some abuse of notation, we can even use the same letter J for the Jacobi matrix (1).

If $J_{\min} \neq J_{\max}$ there is infinitely many densely defined operators associated with $\mathcal{J}$. In fact, the deficiency indices of the symmetric operator $J_{\min}$ are either $(0,0)$ or $(1,1)$. Indeed, by Lemma 1, a non-trivial solution $x = x(\xi)$ of the equation

$$J_{\min}^* x = \xi x,$$

where $\xi \in \mathbb{C}$, $\operatorname{Im} \xi \neq 0$, is a solution of the system of difference equations

$$\lambda_1 x_1 + w_1 x_2 = \xi x_1, \quad w_{n-1} x_n + \lambda_n x_n + w_n x_{n+1} = \xi x_n, \quad n \geq 2,$$

and it is determined uniquely up to a multiplicative constant. Indeed, by fixing the value x_1, other components of the vector x can be computed recursively since $w_n \neq 0$, $\forall n \in \mathbb{N}$. Consequently,

$$0 \leq \dim \operatorname{Ker}(J_{\min}^* - \xi) \leq 1.$$

Taking into account $x(\bar{\xi}) = \overline{x(\xi)}$, we conclude deficiency indices are either $(1,1)$ or $(0,0)$ depending on whether x belongs or does not belong to $\ell^2(\mathbb{N})$, respectively.

In the case $J_{\min} \neq J_{\max}$, deficiency indices of $J_{\min}$ are $(1,1)$. By application of the von-Neumann theory of self-adjoint extensions to $J_{\min}$, we obtain a one-parameter family of self-adjoint operators J_κ, $\kappa \in \mathbb{R} \cup \{\infty\}$, for which $J_{\min} \subset J_\kappa \subset J_{\max}$. The explicit description of the set $\operatorname{Dom} J_\kappa$ in terms of boundary conditions can be found, for instance, in [42, Chp. 2, Sec. 6].

To distinguish two cases when $J_{\min}$ has deficiency indices $(0,0)$ or $(1,1)$, various terminology is used in literature. For example, Teschl [42] defines $\mathcal{J}$ to be in the *limit point*, or *limit circle* case if J_{min} is, or is not self-adjoint, referring to the Weyl theory of ordinary differential operators. While Akhiezer [2] uses the terms $\mathcal{J}$ being of *type C* in the letter case, and of *type D* in the former case. Furthermore, Beckerman [3] says $\mathcal{J}$ is *proper* if $J_{\min} = J_{\max}$, although he deals with a more general situation with a complex Jacobi matrix.

I.2 Function $\mathfrak{F}$

The main aim of Part I is to define a function in terms of the diagonal sequence λ and the off-diagonal sequence w from the Jacobi matrix (1) such that its zeros are related with the spectrum of the Jacobi operator $J_{\max}$. For this purpose we introduce a function called $\mathfrak{F}$ which has been defined and studied in [32], for the first time.

Definition 2: Define $\mathfrak{F} : D \to \mathbb{C}$,

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1}, \tag{2}$$

where

$$D = \left\{ x \in \mathbb{C}^{\infty} \;\middle|\; \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables we identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$. By convention, we also put $\mathfrak{F}(\emptyset) = 1$ where $\emptyset$ is the empty sequence.

Note the function $\mathfrak{F}$ is indeed well defined on the domain D since one has the estimate

$$|\mathfrak{F}(x)| \leq \exp\left(\sum_{k=1}^{\infty} |x_k x_{k+1}| \right).$$

Note also that the domain D is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$.

Besides the application using $\mathfrak{F}$ in the spectral analysis of Jacobi operators, function $\mathfrak{F}$ is closely related with solutions of unilateral or bilateral difference equations of the second order, theory of continued fractions, or orthogonal polynomials. Of course, $\mathfrak{F}$ is also a nice mathematical object of independent interest. At the start, we summarize several algebraic and combinatorial properties of $\mathfrak{F}$ of which proofs can be found in [32, 33].

First, $\mathfrak{F}$ satisfies the relation

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \dots, x_k)\, \mathfrak{F}(T^k x) - \mathfrak{F}(x_1, \dots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(T^{k+1} x), \quad k \in \mathbb{N}, \tag{3}$$

where $x \in D$ and T stands for the shift operator from the left defined on $\mathbb{C}^{\infty}$ by the relation $(Tx)_n = x_{n+1}$. In particular, for $k = 1$ one gets the rule

$$\mathfrak{F}(x) = \mathfrak{F}(Tx) - x_1 x_2 \mathfrak{F}(T^2 x). \tag{4}$$

In addition, one has the symmetry property

$$\mathfrak{F}(x_1, x_2, \dots, x_{k-1}, x_k) = \mathfrak{F}(x_k, x_{k-1}, \dots, x_2, x_1).$$

A particular case of (3) also yields

$$\mathfrak{F}(x_1, x_2, \dots, x_{k+1}) = \mathfrak{F}(x_1, x_2, \dots, x_k) - x_k x_{k+1}\, \mathfrak{F}(x_1, x_2, \dots, x_{k-1}). \tag{5}$$

Next, $\mathfrak{F}$ is a continuous functional on $\ell^2(\mathbb{N})$ and, for $x \in D$, one has

$$\lim_{n \to \infty} \mathfrak{F}(T^n x) = 1 \quad \text{and} \quad \lim_{n \to \infty} \mathfrak{F}(x_1, x_2, \dots, x_n) = \mathfrak{F}(x). \tag{6}$$

Alternatively, $\mathfrak{F}(x_1, \dots, x_n)$ can be expressed as the determinant of $n \times n$ matrix $X^{(n)}$, with entries defined by

$$X_{ij}^{(n)} = \begin{cases} 1, & \text{if } i = j, \\ x_i, & \text{if } |i - j| = 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $i, j \in \{1, \dots, n\}$. Thus, $\mathfrak{F}(x_1, \dots, x_n) = \det X^{(n)}$ and, taking into account the second limit relation in (6), for $x \in D$, one gets

$$\mathfrak{F}(x) = \lim_{n \to \infty} \det X^{(n)}.$$

With a concrete choice of $x \in \mathbb{C}^{\infty}$, the definition relation (2) can be sometimes further simplified resulting in a power series form which can be written in terms of special functions. Usually they are hypergeometric series or their q-analogues. The following illustrative example is concerned with the Bessel functions of the first kind. The corresponding derivation has been worked out in [32, Sec. 2].

Example 3: For $y, \nu \in \mathbb{C}$, $\nu \notin -\mathbb{N}$, one has

$$\mathfrak{F}\left(\left\{\frac{y}{\nu+k}\right\}_{k=1}^{\infty}\right) = \Gamma(\nu+1) y^{-\nu} J_{\nu}(2y). \tag{7}$$

At the end of this section, let us shortly discuss the link between $\mathfrak{F}$ and solutions of unilateral difference equation of the form

$$\alpha_{n-1}u_{n-1} + \beta_n u_n + \alpha_n u_{n+1} = 0, \quad n = 2, 3, 4, \dots, \tag{8}$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$ are given sequences satisfying the convergence condition

$$\sum_{n=1}^{\infty} \left| \frac{\alpha_n^2}{\beta_n \beta_{n+1}} \right| < \infty. \tag{9}$$

Let $x = x(\alpha, \beta) \in \mathbb{C}^{\infty}$ be an arbitrary sequence fulfilling the recurrence rule

$$x_n x_{n+1} = \frac{\alpha_n^2}{\beta_n \beta_{n+1}}, \quad n \in \mathbb{N},$$

which is determined unambiguously up to a multiplicative constant.

Condition (9) implies $x \in D$ and we define $\varphi, \psi \in \mathbb{C}^{\infty}$ by the relations

$$\varphi_n = (-1)^n \frac{(\alpha!)_{n-1}}{(\beta!)_n} \mathfrak{F}(T^n x), \quad n \in \mathbb{N},$$

and

$$\psi_n = (-1)^{n+1} \frac{(\beta!)_{n-1}}{(\alpha!)_{n-1}} \mathfrak{F}(x_1, \dots, x_{n-1}), \quad n \in \mathbb{N},$$

where we put

$$(y!)_n = \prod_{k=1}^{n} y_k, \quad n \in \mathbb{Z}_+,$$

for any $y \in \mathbb{C}^{\infty}$. It is a straightforward application of (4) and (5) to verify both sequences φ and ψ solve difference equation (8). For two solutions $u, v \in \mathbb{C}^{\infty}$ of (8) the Wronskian is introduced as

$$\mathcal{W}(u, v) = \alpha_n \left(u_n v_{n+1} - u_{n+1} v_n\right),$$

see [42, Chp. 1]. Number $\mathcal{W}(u, v)$ is a constant independent of the index n. Moreover, two solutions are linearly dependent if and only if their Wronskian vanishes. By using formula (3) one gets

$$\mathcal{W}(\varphi, \psi) = \mathfrak{F}(x).$$

Thus, sequences φ and ψ form a couple of two linearly independent solutions of the difference equation (8) if and only if $\mathfrak{F}(x) \neq 0$.

Similarly can be treated the bilateral difference equation of the form (8) where $n \in \mathbb{Z}$. Providing the corresponding convergence condition as (9) one can express a couple of solutions of the bilateral difference equation in terms of $\mathfrak{F}$. However, in this case, the definition of $\mathfrak{F}$ has to be slightly generalized, see [33, Subsec. 2.3] for a detailed discussion.

I.3 Characteristic function of a Jacobi matrix

With the aid of $\mathfrak{F}$, one can describe spectral properties of a Jacobi operator $J_{\max}$ providing certain convergence condition in terms of the sequences λ and w is satisfied, see (12) below. First of all, one can introduce a function whose zeros describe the spectrum of the Jacobi operator $J_{\max}$. In addition, eigenvectors or even the Green function of $J_{\max}$ can be described in terms of $\mathfrak{F}$ as well. The explicit formulas have been derived in [33] where the reader can find all the omitted proofs from this introductory section.

For the sake of simplicity and taking into consideration concrete applications, we assume $\lambda \subset \mathbb{R}$ and $w \subset \mathbb{R} \setminus \{0\}$, although the case of complex sequences λ and w can be treated as well. Even matrix (1) need not be necessarily symmetric.

The main idea can be motivated by the following connection between $\mathfrak{F}$ and the characteristic polynomial of a finite Jacobi matrix. Let $J_n \in \mathbb{C}^{n,n}$ denotes the $n \times n$ principal submatrix of matrix (1), i.e.,

$$J_n = \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & \ddots & \ddots & \ddots & \\ & & w_{n-2} & \lambda_{n-1} & w_{n-1} \\ & & & w_{n-1} & \lambda_n \end{pmatrix},$$

then for all $z \in \mathbb{C}$ it holds

$$\det(J_n - zI_n) = \left(\prod_{k=1}^{n}(\lambda_k - z)\right) \mathfrak{F}\left(\frac{\gamma_1^2}{\lambda_1 - z}, \frac{\gamma_2^2}{\lambda_2 - z}, \dots, \frac{\gamma_n^2}{\lambda_n - z}\right) \tag{10}$$

where $\{\gamma_k\}_{k=1}^{n}$ is any sequence satisfying the recurrence $\gamma_k\gamma_{k+1} = w_k$, for $k \geq 1$.

Looking at the formula (10) one can try to send $n \to \infty$. Of course both sides can diverge, however, if the sequence in the argument of $\mathfrak{F}$ belongs to the domain D, one can extract this term hoping the resulting function reflects some spectral properties of an operator associated with matrix (1). By this way one arrives at the function

$$F_{\mathcal{J}}(z) := \mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - z}\right\}_{n=1}^{\infty}\right) \tag{11}$$

which we refer to as the *characteristic function* associated with a Jacobi matrix $\mathcal{J}$. Function $F_{\mathcal{J}}$ is well defined whenever the sequence in the argument of $\mathfrak{F}$ on the RHS of (11) belongs to D. We proved this to be guaranteed for all $z \in \mathbb{C}_0^{\lambda} := \mathbb{C} \setminus \overline{\operatorname{Ran}\lambda}$ under the assumption that there exists at least one $z_0 \in \mathbb{C}_0^{\lambda}$ such that

$$\sum_{n=1}^{\infty}\left|\frac{w_n^2}{(\lambda_n - z_0)(\lambda_{n+1} - z_0)}\right| < \infty. \tag{12}$$

Condition (12) determines a class of Jacobi matrices for which the characteristic function is defined. Moreover, (12) together with reality of λ and w implies $J_{\max}$ to be self-adjoint, see [33, Theorem 15], and hence we omit the subscript and write simply J for the Jacobi operator in question.

Let $\operatorname{der}(\lambda)$ stands for the set of all finite accumulation points of the sequence λ. By a closer inspection one finds that, under the assumptions of (12) the function $F_{\mathcal{J}}(z)$

is meromorphic on $\mathbb{C}\setminus\operatorname{der}(\lambda)$ with poles at the points $z=\lambda_n$ for some $n\in\mathbb{N}$ (not belonging to $\operatorname{der}(\lambda)$, however). For any such z, the order of the pole is less than or equal to $r(z)$ where

$$r(z):=\sum_{k=1}^{\infty}\delta_{z,\lambda_k}$$

is the number of members of the sequence λ coinciding with z (hence $r(z)=0$ for $z\in\mathbb{C}_0^{\lambda}$).

If condition (12) is fulfilled, the part of $\operatorname{spec}(J)$ not intersecting $\operatorname{der}(\lambda)$ coincides with the set

$$\mathfrak{Z}(\mathcal{J}):=\left\{z\in\mathbb{C}\setminus\operatorname{der}(\lambda)\;\Big|\;\lim_{u\to z}(u-z)^{r(z)}F_{\mathcal{J}}(u)=0\right\}.$$

Of course, $\mathfrak{Z}(\mathcal{J})\cap\mathbb{C}_0^{\lambda}$ is nothing but the set of zeros of $F_{\mathcal{J}}(z)$. More precisely, we have proved

$$\operatorname{spec}(J)\setminus\operatorname{der}(\lambda)=\operatorname{spec}_p(J)\setminus\operatorname{der}(\lambda)=\mathfrak{Z}(\mathcal{J})$$

and the points from this set consist of simple real eigenvalues which have no accumulation point in $\mathbb{R}\setminus\operatorname{der}(\lambda)$. Let us note the excluded set $\operatorname{der}(\lambda)$ is often empty or a one-point set since, in applications, we usually encounter Jacobi matrices whose diagonal is either an unbounded sequence of isolated points or a convergent sequence.

Moreover, the vector valued function $\xi:\mathbb{C}\setminus\operatorname{der}(\lambda)\to\mathbb{C}^{\infty}$, where

$$\xi_k(z):=\lim_{u\to z}(u-z)^{r(z)}\left(\prod_{l=1}^{k}\frac{w_{l-1}}{u-\lambda_l}\right)\mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l-u}\right\}_{l=k+1}^{\infty}\right),$$

for $k\in\mathbb{N}$, has the property that for $z\in\mathfrak{Z}(\mathcal{J})$, $\xi(z)$ is an eigenvector of J corresponding to the eigenvalue z. Similarly, by using $\mathfrak{F}$, we were able to find formulas for the ℓ^2-norm of the eigenvector $\xi(z)$ or the Green function of J (especially the Weyl m-function), see [33, Propositions 12 and 18].

As an application of the presented results, we provide a description of spectral properties of many concrete Jacobi operators in [33, 34]. In fact, in several examples of operators investigated in [34], we go even beyond the presented formalism since the condition (12) is violated. Nevertheless, one can proceed in a similar way to obtain desired results. In all cases, spectral properties of Jacobi operators have been described in terms of special functions such as Bessel functions, q-Bessel functions, Coulomb wave functions, confluent hypergeometric functions, etc. These special functions can be defined as hypergeometric series or their q-analogues, see [1, 13].

Example 4: We provide an illustrative example of a Jacobi matrix whose diagonal depends linearly on the index and parallels to the diagonal remain constant, i.e., we set $\lambda_n=n$ and $w_n=w\in\mathbb{R}\setminus\{0\}$, for $n\in\mathbb{N}$. The corresponding Jacobi matrix determines an unbounded self-adjoint operator J with discrete spectrum.

Condition (12) is clearly fulfilled, for instance, with $z_0=0$. Further, one has $\operatorname{der}(\lambda)=\emptyset$ and $\operatorname{spec}(J)=\mathfrak{Z}(\mathcal{J})$. Using (7) one derives that

$$\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{k-z}\right\}_{k=r+1}^{\infty}\right)=\mathfrak{F}\left(\left\{\frac{w}{k-z}\right\}_{k=r+1}^{\infty}\right)=w^{z-r}\Gamma(1+r-z)J_{r-z}(2w)$$

for $r\in\mathbb{Z}_+$. It follows that

$$\operatorname{spec}(J)=\{z\in\mathbb{R}\mid\ J_{-z}(2w)=0\}.$$

Components of corresponding eigenvectors $v(z)$ can be chosen as

$$v_k(z) = (-1)^k J_{k-z}(2w),\ k \in \mathbb{N}.$$

Moreover, formula [33, Eq. 34] for the Green function

$$G(z; i, j) := \langle e_i, (J - z)^{-1} e_j \rangle,\ i, j \in \mathbb{N},$$

provides us with an explicit expression for the matrix elements of the resolvent operator $(J - z)^{-1}$ in terms of Bessel functions and Lommel polynomials $R_{n,\nu}$ (see [36, Sec. 4], references therein, or Example 9). Indeed, taking into account [36, Eq. 34], one finds

$$G(z; i, j) = (-1)^{i+j} \frac{R_{i-1,1-z}(2w)\, J_{j-z}(2w)}{w J_{-z}(2w)},\ 1 \le i \le j,$$

for all $z \in \mathbb{C}$ for which $J_{-z}(2w) \neq 0$. If $i > j$ it suffices to interchange indices i and j in the last formula due to the symmetry property of the Green function: $G(z; i, j) = G(z; j, i)$.

For a more detailed analysis of the zeros of the Bessel function considered as a function of the order based on their relation with the spectrum of the Jacobi operator J, see [33, Subsec. 4.3].

Let us conclude this section with a short comment on the relation of the characteristic function $F_{\mathcal{J}}$ with the theory of regularized determinants, see [14] or [31]. Recall a compact operator A defined on a Hilbert space $\mathcal{H}$ belongs to the p-th Schatten-von Neumann class $\mathscr{I}_p(\mathcal{H})$, for $1 \le p \le \infty$, if the set of singular values of A forms a sequence from $\ell^p(\mathbb{N})$. In particular, $\mathscr{I}_1(\mathcal{H})$, $\mathscr{I}_2(\mathcal{H})$, and $\mathscr{I}_\infty(\mathcal{H})$ are spaces of trace class, Hilbert-Schmidt, and compact operators, respectively. If $A \in \mathscr{I}_p(\mathcal{H})$, for $p \in \mathbb{N}$, the regularized determinant $\det_p(I - zA)$ is a well defined entire function of $z \in \mathbb{C}$, whose zeros are reciprocal values of the eigenvalues of A.

Let us temporarily assume $\lambda, w \in \ell^2(\mathbb{N})$. Then (12) holds (for any $z_0 \neq 0$ not being in $\operatorname{Ran} \lambda$), the Jacobi matrix J is a Hilbert-Schmidt operator on $\ell^2(\mathbb{N})$, and for the regularized determinant of $I - zJ$ one has

$$\det{}_2(I - zJ) = \left(\prod_{k=1}^{\infty} (1 - z\lambda_k) e^{z\lambda_k} \right) F_{\mathcal{J}}(z^{-1}).$$

Indeed, by rewriting (10) into the form

$$\det(I_n - zJ_n) = \left(\prod_{k=1}^{\infty} (1 - z\lambda_k) \right) \mathfrak{F}\left(\frac{\gamma_1^2}{\lambda_1 - z^{-1}}, \frac{\gamma_2^2}{\lambda_2 - z^{-1}}, \ldots, \frac{\gamma_n^2}{\lambda_n - z^{-1}} \right),$$

multiplying both sides by $\exp(z \operatorname{Tr} J_n)$, and using the well known identity

$$\det \exp(A) = \exp(\operatorname{Tr}(A)), \quad \text{for } A \in \mathbb{C}^{n,n},$$

one arrives at the formula

$$\det\left[(I_n - zJ_n) \exp(zJ_n)\right] = \left(\prod_{k=1}^{n} (1 - z\lambda_k) e^{z\lambda_k} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_k^2}{\lambda_k - z^{-1}} \right\}_{k=1}^{n} \right).$$

Now, it suffices to send $n \to \infty$ on both sides of the last identity. The LHS tends to $\det_2(I - zJ)$ since the regularized determinant $\det_2$ is continuous functional on $\mathscr{I}_2(\ell^2(\mathbb{N}))$, see [31, Thm. 9.2.(c)], and $J_n \oplus 0 \to J$ in $\mathscr{I}_2(\ell^2(\mathbb{N}))$, as $n \to \infty$. The expression on the RHS converges too, by the second limit relation from (6) and since $\lambda \in \ell^2(\mathbb{N})$, see, for instance [28, Chp. 15].

Thus we see, function $F_{\mathcal{J}}$ can be expressed with the aid of the regularized determinant. However, this is true in some special cases only, for instance, if $\lambda, w \in \ell^2(\mathbb{N})$. On the other hand, $F_{\mathcal{J}}$ has been defined under the assumption (12) exclusively. This assumption does not imply J to belong to a p-th Schatten-von Neumann class (nor the compactness of the resolvent). Consequently, function $F_{\mathcal{J}}$ cannot be defined by using the theory of regularized determinants, in general. Another advantage of the presented construction is the characteristic function $F_{\mathcal{J}}$ has been introduced using only the sequences λ and w determining the Jacobi matrix (1) (although in a quite complicated manner). While the regularized determinant is usually introduced in terms of eigenvalues of the operator in question, see [31, Chps. 3,9].

I.4 Logarithm of $\mathfrak{F}$ and factorization of the characteristic function

There is another way how to interpret the definition of $\mathfrak{F}$ given in (2). The RHS of (2) can be viewed as an element of the ring of formal power series $\mathbb{C}[[x]]$ in countably many indeterminates $x = \{x_n\}_{n=1}^{\infty}$. So we are no longer restricted with x to the domain D, however, the RHS of (2) need not converge in $\mathbb{C}$.

It is not our intention to go into details of the theory of formal power series at all. One can do many things with formal power series at a purely algebraic level. Apart from multiplication and inversion of invertible elements ($\mathbb{C}[[x]]$ is a ring) one can differentiate them, there is a Taylor's formula for formal power series, and one can take an exponential or logarithm of certain elements of $\mathbb{C}[[x]]$. The ring $\mathbb{C}[[x]]$ is even equipped with canonical (product) topology. For the complete theory see [6, § 4.]. Another nice introduction to the theory of formal power series, however, in one indeterminate only, is given in [25].

A particular subset of invertible formal power series is composed of those of them having the constant term equal to 1. This is the case of $\mathfrak{F}(x)$ and the logarithm of $\mathfrak{F}(x)$ is a well defined element of $\mathbb{C}[[x]]$. In order to express the formal power series for $\log \mathfrak{F}(x)$ explicitly we need to introduce the following notation. For a multiindex $m \in \mathbb{N}^\ell$ denote by $d(m)$ its length, i.e. $d(m) = \ell$. Further, for a multiindex $m \in \mathbb{N}^\ell$ put

$$\beta(m) := \prod_{j=1}^{\ell-1} \binom{m_j + m_{j+1} - 1}{m_{j+1}}, \ \alpha(m) := \frac{\beta(m)}{m_1}\,. \tag{13}$$

Then in the ring of formal power series in the sequence of indeterminates $x = \{x_k\}_{k=1}^{\infty}$ one has

$$\log \mathfrak{F}(x) = -\sum_{\ell=1}^{\infty} \sum_{m \in \mathbb{N}^\ell} \alpha(m) \sum_{k=1}^{\infty} \prod_{j=1}^{d(m)} (x_{k+j-1}x_{k+j})^{m_j}\,. \tag{14}$$

In addition, if $x \in \mathbb{C}^\infty$ is such that $\sum_{k=1}^{\infty} |x_k x_{k+1}| < \log 2$, then the RHS of (14) converges in $\mathbb{C}$ and we have

$$|\text{the RHS of (14)}| \leq -\log\left(1 - \sum_{k=1}^{\infty} |x_k x_{k+1}|\right).$$

The proof of formula (14) is by no means trivial and it is the main result of paper [35].

Next, we recall two formulas derived again in [35] where it is shown the characteristic function admits Hadamard's infinite product factorization. The factorization can be made in two possible ways – either in the spectral parameter or in an auxiliary parameter which may be called the coupling constant.

First, in order to factorize the characteristic function in the spectral parameter, we regularize function $F_{\mathcal{J}}$ to obtain an entire function having the property that the set of its zeros coincides with the spectrum of the corresponding Jacobi operator. For this purpose we restrict ourself to real sequences λ and w and such that $\lim_{n\to\infty} \lambda_n = +\infty$ and $w_n \neq 0$, $\forall n \in \mathbb{N}$. In addition, without loss of generality, λ is assumed to be positive. Moreover, suppose that condition (12) is satisfied for $z_0 = 0$ and the sequence $\lambda^{-1} = \{1/\lambda_n\}_{n=1}^{\infty}$ belongs to $\ell^2(\mathbb{N})$.

Under these assumption, Jacobi operator J is self-adjoint with a compact (even Hilbert-Schmidt) resolvent, see [33, Cor. 13]. Thus J has discrete spectrum. Moreover, the characteristic function $F_{\mathcal{J}}$ can be regularized by the multiplication factor

$$\Phi_\lambda(z) := \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n}.$$

Since $\lambda^{-1} \in \ell^2(\mathbb{N})$, Φ_λ is a well defined entire function; see, for instance, [28, Chp. 15]. In this way one arrives at the entire function

$$H_J(z) := \Phi_\lambda(z) F_{\mathcal{J}}(z),$$

which is to be referred as the *regularized characteristic function* of the Jacobi operator J. One has

$$\operatorname{spec}(J) = \operatorname{spec}_p(J) = H_J^{-1}(\{0\}).$$

If J is invertible and $\lambda_n(J)$, $n \in \mathbb{N}$, stand for the eigenvalues of J, then H_J is an entire function of genus one and it admits the Hadamard's infinite product factorization of the form

$$H_J(z) = F_J(0)\, e^{bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n(J)}\right) e^{z/\lambda_n(J)} \tag{15}$$

where

$$b = \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_n(J)}\right) < \infty.$$

We illustrate the general formula (15) in the example with Bessel functions.

Example 5: Using (7) and the well known formula for the gamma function,

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n},$$

where γ is the Euler constant, one finds

$$H_J(z) = e^{\gamma z} w^z J_{-z}(2w).$$

J is the Jacobi operator introduced in Example 4. As a result one reveals the infinite product formula for a Bessel function considered as a function of its order. Assuming $J_0(2w) \neq 0$, the formula reads

$$\frac{w^z J_{-z}(2w)}{J_0(2w)} = e^{c(w)z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n(J)}\right) e^{z/\lambda_n(J)} \tag{16}$$

where

$$c(w) = \frac{1}{J_0(2w)} \sum_{k=0}^{\infty} (-1)^k \psi(k+1) \frac{w^{2k}}{(k!)^2}$$

and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the polygamma function, see [1, Sec. 6]. To derive the expression for $c(w)$ it suffices to compare the coefficients at z on both sides of (16).

Next, we present the factorization formula for the entire function

$$f(w) := \mathfrak{F}(wx), \ w \in \mathbb{C}.$$

Here we assume $x \in D$ such that $x_n \neq 0$, $\forall n \in \mathbb{N}$.

Clearly $F_{\mathcal{J}}(z) = f(z^{-1})$ where $\mathcal{J}$ is the Jacobi matrix (1) with $\lambda_n = 0$ and $w_n = \sqrt{x_n x_{n+1}}$ (any branch of the square root is suitable). The corresponding operator J represents a Hilbert-Schmidt operator on $\ell^2(\mathbb{N})$ and the set of nonzero eigenvalues of J coincides with the set of reciprocal values of zeros of f.

Since f is an even function its zeros can be arranged into sequences

$$\{\zeta_k\}_{k=1}^{N(f)} \cup \{-\zeta_k\}_{k=1}^{N(f)},$$

where each zero being repeated according to its multiplicity, and $N(f) \in \mathbb{Z}_+ \cup \{\infty\}$. The multiplicity of a zero z_0 of f coincides with the algebraic multiplicity of eigenvalue z_0^{-1} of J, see [35, Prop. 12]. Then one has

$$f(w) = \prod_{k=1}^{N(f)} \left(1 - \frac{w^2}{\zeta_k^2}\right). \tag{17}$$

By taking the logarithm of both sides, using formula (14), expanding the both sides into a power series at $w = 0$, and equating the coefficients at w^{2N} one derives

$$\sum_{\ell=1}^{N(f)} \frac{1}{\zeta_\ell^{2N}} = N \sum_{m \in \mathcal{M}(N)} \alpha(m) \sum_{k=1}^{\infty} \prod_{j=1}^{d(m)} (x_{k+j-1} x_{k+j})^{m_j}, \tag{18}$$

for $N \in \mathbb{N}$, where

$$\mathcal{M}(N) = \left\{ m \in \bigcup_{\ell=1}^{N} \mathbb{N}^\ell \;\middle|\; \sum_{j=1}^{\ell} m_j = N \right\}.$$

Consequently, (18) provides us with an explicit expression of the value of a spectral zeta function at an even integer. Recall, for an invertible operator A, with $A^{-1} \in \mathscr{I}_p(\mathcal{H})$, we call the function

$$\zeta_A(z) := \operatorname{Tr} A^{-z}$$

the *spectral zeta function* of an operator A, which is a well defined entire function on the half-plane $\{z \in \mathbb{C} \mid \operatorname{Re} z \geq p\}$ as it follows from Lidskii's theorem and properties of trace class operators, see [31, Chp. 3].

Let us recall the example with Bessel functions once more.

Example 6: Put $x_k = (\nu + k)^{-1}$, $\forall k \in \mathbb{N}$, where $\nu > -1$. Recalling (7) and putting $z = w/2$, one obtains the factorization of Bessel functions [1, 44],

$$\left(\frac{z}{2}\right)^{-\nu} \Gamma(\nu+1) J_\nu(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,k}^2}\right),$$

as a particular case of (17), where $j_{\nu,k}$ stands for the k-th positive zero of the Bessel function J_ν. The corresponding spectral zeta function

$$\sigma_\nu(s) = \sum_{k=1}^{\infty} \frac{1}{j_{\nu,k}{}^s}, \quad \operatorname{Re} s > 1,$$

is known as the Rayleigh function [19]. Rayleigh function generalizes the famous Riemann zeta function since $z^{1/2} J_{1/2}(z)$ is a constant multiple of $\sin(z)$, see [1, Eq. 10.1.11], hence $j_{1/2,k} = \pi k$, from which one deduces

$$\pi^s \sigma_{\frac{1}{2}}(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \operatorname{Re} s > 1.$$

Values $\sigma_\nu(2N)$ for $N \in \mathbb{N}$ are rational functions in ν and have originally been computed by Rayleigh for $1 \leq N \leq 5$ and by Cayley for $N = 8$ [44, § 15.51]. The particular case of (18) implies the equality

$$\sigma_\nu(2N) = 2^{-2N} N \sum_{k=1}^{\infty} \sum_{m \in \mathcal{M}(N)} \alpha(m) \prod_{j=1}^{d(m)} \left(\frac{1}{(j+k+\nu-1)(j+k+\nu)}\right)^{m_j}, \tag{19}$$

is valid for all $N \in \mathbb{N}$. Thus, one can determine the value $\sigma_\nu(2N)$ for any N by using (19). However, it seems a more efficient way to evaluate $\sigma_\nu(2N)$ is the recursive procedure using the identity

$$n = \sum_{k=1}^{n} (-1)^{k+1} 4^k (k!)^2 \binom{n}{k} \binom{\nu+n}{k} \sigma_\nu(2k),$$

which holds true for all $n \in \mathbb{N}$, see [19].

I.5 An application in the theory of continued fractions

For a given $x \in D$ such that $\mathfrak{F}(x) \neq 0$ let us introduce sequences $\{P_k\}_{k=0}^{\infty}$ and $\{Q_k\}_{k=0}^{\infty}$ by $P_0 = 0$ and $P_k = \mathfrak{F}(x_2, \ldots, x_k)$ for $k \geq 1$, $Q_k = \mathfrak{F}(x_1, \ldots, x_k)$ for $k \geq 0$. According to (5), the both sequences obey the difference equation

$$Y_{k+1} = Y_k - x_k x_{k+1} Y_{k-1}, \; k = 1, 2, 3, \ldots,$$

with the initial conditions $P_0 = 0$, $P_1 = 1$, $Q_0 = Q_1 = 1$, and define the infinite continued fraction

$$\frac{\mathfrak{F}(Tx)}{\mathfrak{F}(x)} = \lim_{k \to \infty} \frac{P_k}{Q_k} = \cfrac{1}{1 - \cfrac{x_1 x_2}{1 - \cfrac{x_2 x_3}{1 - \cfrac{x_3 x_4}{1 - \ldots}}}}. \tag{20}$$

For a sequence $a \in \mathbb{C}^\infty$, formal expression of the form

$$\cfrac{1}{1-\cfrac{a_1}{1-\cfrac{a_2}{1-\cfrac{a_3}{1-\dots}}}} \tag{21}$$

is referred to as formal *Stieltjes continued fraction*, or shortly as formal *S-fraction*. We have already mentioned $\mathfrak{F}(x)$ represents an invertible element of the ring $\mathbb{C}[[x]]$, so $\mathfrak{F}(x)^{-1} \in \mathbb{C}[[x]]$ and the LHS of equality (20) can be understood in a pure algebraic manner as

$$\mathfrak{F}(Tx)\mathfrak{F}(x)^{-1} \in \mathbb{C}[[x]], \tag{22}$$

where $Tx = \{x_{k+1}\}_{k=1}^\infty$ is a truncated sequence of indeterminates. Consequently, with any formal S-fraction (21) there is naturally associated a unique formal power series $f(a)$ in the indeterminates a which, under identification: $a_k = x_k x_{k+1}$, $\forall k$, equals (22). This power series expansion has been studied a long time ago, particularly in the case when the formal indeterminates a_k are replaced by $e_k x$ where e_k are fixed complex coefficients and x is a complex variable [27]; see also [43, Thm. 52.1]. Somewhat surprisingly, an explicit formula for $f(a)$ has been derived much later [11, 45]. Here we recall it while using the above introduced notation (13).

Theorem 7: The formal power series $f(a) \in \mathbb{C}[[a]]$ associated with the formal Stieltjes continued fraction (21) is given by the formula

$$f(a) = 1 + \sum_{\ell=1}^{\infty} \sum_{m \in \mathbb{N}^\ell} \beta(m) \prod_{j=1}^{\ell} a_j^{m_j}. \tag{23}$$

Alternatively to the proofs presented in [11, 45] we shall show that formula (23) can be derived in a straightforward manner from (14). In addition, we get as a byproduct another formula, this time for $\log f(a)$. Note that the logarithm of $f(a)$ actually makes good sense in $\mathbb{C}[[a]]$ (see [6]) since, even if unaware of (23), it is obvious from (21) as well as (22) that the constant term of $f(a)$ equals 1.

Proposition 8: Let $f(a) \in \mathbb{C}[[a]]$ be the formal power series expansion of the formal Stieltjes continued fraction (21). Then

$$\log f(a) = \sum_{\ell=1}^{\infty} \sum_{m \in \mathbb{N}^\ell} \alpha(m) \prod_{j=1}^{\ell} a_j^{m_j}. \tag{24}$$

Proof. First, note for $A, B \in \mathbb{C}[[x]]$ with constant terms being equal to 1, the well known identity

$$\log(AB) = \log A + \log B$$

remains true in $\mathbb{C}[[x]]]$, see [6, § 4.]. As a consequence, one verifies $\log A^{-1} = -\log A$ in $\mathbb{C}[[x]]$. Thus, under the identification $a_k = x_k x_{k+1}$, $\forall k \in \mathbb{N}$, one has the equalities

$$\log f(a) = \log\left(\mathfrak{F}(Tx)\mathfrak{F}(x)^{-1}\right) = \log \mathfrak{F}(Tx) - \log \mathfrak{F}(x).$$

By applying formula (14) to the RHS and making obvious cancellations one arrives at (24). $\square$

Proof of Theorem 7. Let $\tilde{f}(a)$ designate the RHS in (23) and $g(a)$ the RHS in (24). We have to show that $f(a) = \tilde{f}(a)$. From (13) it is obvious that $a_1 \partial g(a)/\partial a_1 = \tilde{f}(a) - 1$. On the other hand, it is as well clear from (21) that $f(a) = (1 - a_1 f(Ta))^{-1}$. From here one derives (using the common rules of differentiation which are known to be valid in $\mathbb{C}[[a]]$, too, see [6]) that

$$a_1 \frac{\partial}{\partial a_1} \log f(a) = f(a) - 1.$$

At the same time, Proposition 8 implies

$$a_1 \frac{\partial}{\partial a_1} \log f(a) = a_1 \frac{\partial}{\partial a_1} g(a) = \tilde{f}(a) - 1.$$

This shows (23). □

Part II

Orthogonal Polynomials and the Moment Problem

II.1 Function $\mathfrak{F}$ and orthogonal polynomials

Orthogonal Polynomials (=OPs) has been studied for a long time and the theory of OPs has been developed into a considerable depth [2, 8, 17, 41]. One defines a sequence of polynomials with real coefficients $\{P_n\}_{n=0}^{\infty}$, where $\deg P_n = n$, to be OPs by requiring the orthogonality relation

$$\int_{\mathbb{R}} P_m(x)P_n(x)\,\mathrm{d}\mu(x) = \delta_{mn}, \quad m,n \in \mathbb{Z}_+, \tag{25}$$

to hold where μ is a positive Borel measure on $\mathbb{R}$ with finite moments. Without loss of generality one may assume μ to be a probability measure, i.e. $\mu(\mathbb{R}) = 1$. As usual, μ is unambiguously determined by the distribution function $x \mapsto \mu((-\infty, x])$.

To avoid some exceptional situations we assume, in addition, that the distribution function of μ has an infinite number of points of increase. Then the set of monomials, $\{x^n \mid n \in \mathbb{Z}_+\}$, is linearly independent in $L^2(\mathbb{R}, \mathrm{d}\mu)$. One can arrive at the set of polynomials $\{P_n\}_{n=0}^{\infty}$ which is orthonormal with respect to μ by applying the Gram-Schmidt process on the set of monomials in the Hilbert space $L^2(\mathbb{R}, \mathrm{d}\mu)$. From the way the Gram-Schmidt process proceeds one deduces $\{P_n\}_{n=0}^{\infty}$ satisfies a three-term recurrence relation,

$$xP_0(x) = \lambda_0 P_0(x) + w_0 P_1(x), \quad xP_n(x) = w_{n-1}P_{n-1}(x) + \lambda_n P_n(x) + w_n P_{n+1}(x), \tag{26}$$

for $n \in \mathbb{N}$, where $\{\lambda_n\}_{n=0}^{\infty}$ is a real sequence and $\{w_n\}_{n=0}^{\infty}$ is a positive sequence, see [2, 8, 30]. Thus, any sequence of OPs forms a solution of the three-term recurrence (26).

On the other hand, due to the Favard's theorem, the opposite statement is also true. For any sequence of real polynomials, $\{P_n\}_{n=0}^{\infty}$, with $\deg P_n = n$, satisfying the recurrence relation (26) there exists a positive Borel measure making this sequence orthonormal.

Polynomials $\{P_n\}_{n=0}^{\infty}$ that are the solution of equations

$$xu_n = w_{n-1}u_{n-1} + \lambda_n u_n + w_n u_{n+1}, \quad n \in \mathbb{N},$$

satisfying the initial conditions $P_0(x) = 1$ and $P_1(x) = (x - \lambda_0)/w_0$ are called *OPs of the first kind*. The second linearly independent solution $\{Q_n\}_{n=0}^{\infty}$ of the same difference equation with initial conditions $Q_0(x) = 0$ and $Q_1(x) = 1/w_0$ is referred to as *OPs of the second kind*. These two polynomial sequences are related by the formula

$$Q_n(x) = \int_{\mathbb{R}} \frac{P_n(x) - P_n(y)}{x - y}\mathrm{d}\mu(y), \quad \forall n \in \mathbb{Z}_+,$$

where μ is the measure of orthogonality of OPs $\{P_n\}_{n=0}^{\infty}$, see, for example, [2, Chp. 1].

It is a usual situation in applications that a sequence of OPs is prescribed by the three-term recurrence rule (26). Then a natural question is: How the measure of orthogonality μ looks like?

This problem is very closely related to the spectral analysis of a Jacobi operator determined by the Jacobi matrix (1), now however, the defining sequences λ and w are coefficients from the recurrence (26) and hence they are indexed by $\mathbb{Z}_+$ (instead of $\mathbb{N}$). This choice of the set of indices is quite standard in the theory of OPs, we follow Akhiezer's monograph [2]. Let J be a Jacobi operator determined by sequences λ and w from the three-term recurrence (26). One easily verifies

$$P_n(J)e_0 = e_n, \ \forall n \in \mathbb{Z}_+, \tag{27}$$

where $\{e_n \mid n \in \mathbb{Z}_+\}$ stands for the standard basis of $\ell^2(\mathbb{Z}_+)$ and $\{P_n\}_{n=0}^{\infty}$ are OPs of the first kind. Assume the Jacobi matrix represents a unique self-adjoint operator J on $\ell^2(\mathbb{Z}_+)$ and let E_J denotes the projection-valued spectral measure of J. Then, by using (27) and the Spectral Theorem, one gets

$$\delta_{m,n} = \langle e_m, e_n \rangle = \langle P_m(J)e_0, P_n(J)e_0 \rangle = \int_{\mathbb{R}} P_m(x)P_n(x)\mathrm{d}\mu(x), \quad m,n \in \mathbb{Z}_+,$$

where we denote

$$\mu(.) := \langle e_0, E_J(.)e_0 \rangle.$$

Consequently, the measure of orthogonality μ is determined by the spectral measure of J. This measure is supported on $\mathrm{spec}(J)$, see [2, 30].

If the Jacobi matrix is in the limit circle case we can similarly construct the whole one-parameter family of measures of orthogonality μ_h of respective OPs, for $h \in \mathbb{R} \cup \{\infty\}$, using the spectral measure of J_h, a self-adjoint extension of $J_{\min}$. Measures μ_h are known as N-extremal measures referring to the corresponding Hamburger moment problem, see Section II.2. The interesting fact is that the N-extremal measures do not form the whole set of measures with respect to which OPs $\{P_n\}_{n=0}^{\infty}$ are orthogonal (see [2, 30] or Section II.2).

Let us remark the OPs of the first kind $\{P_n\}_{n=0}^{\infty}$ are related to $\mathfrak{F}$. Indeed, by using (5), one easily verifies

$$P_n(x) = \prod_{k=0}^{n-1} \left(\frac{x - \lambda_k}{w_k} \right) \mathfrak{F}\left(\left\{ \frac{\gamma_k^2}{\lambda_k - x} \right\}_{k=0}^{n-1} \right), \quad n \in \mathbb{Z}_+. \tag{28}$$

Recall $\{\gamma_k\}_{k=0}^{\infty}$ is any sequence satisfying equations $\gamma_k \gamma_{k+1} = w_k$, for $k \in \mathbb{Z}_+$.

Assuming a sequence of OPs is defined via the recurrence rule (26), i.e. via formula (28), we provide a description of the measure of orthogonality in terms of sequences λ and w using the advantage of $\mathfrak{F}$ once more. For the sake of simplicity we suppose a particular case with λ being a real sequence from $\ell^1(\mathbb{Z}_+)$, and w being a positive sequence from $\ell^2(\mathbb{Z}_+)$. The general result is given in [36, Thm. 1].

Under the above assumptions on λ and w the Jacobi operator J is compact and (12) holds for any $z_0 \neq 0$ not belonging to the range of λ. Moreover, the characteristic function of J can be regularized with the aid of the entire function

$$\phi_\lambda(z) := \prod_{n=0}^{\infty} (1 - z\lambda_n).$$

Thus we can introduce an entire function $\mathcal{G}_J$ by

$$\mathcal{G}_J(z) := \begin{cases} \phi_\lambda(z) F_J(z^{-1}) & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases} \tag{29}$$

The measure of orthogonality μ for the corresponding sequence of OPs defined in (26) fulfills

$$\mathrm{supp}(\mu) \setminus \{0\} = \{z^{-1} \mid \mathcal{G}_J(z) = 0\}$$

where the RHS is a bounded discrete subset of $\mathbb{R}$ with 0 as the only accumulation point. Moreover, for $x \in \mathrm{supp}(\mu) \setminus \{0\}$ one has

$$\mu(\{x\}) = -x \frac{\mathcal{G}_{J^{(1)}}(x^{-1})}{\mathcal{G}_J'(x^{-1})}.$$

Here $J^{(1)}$ denotes the Jacobi operator determined by the diagonal sequence $T\lambda = \{\lambda_{n+1}\}_{n=0}^{\infty}$ and the weight sequence $Tw = \{w_{n+1}\}_{n=0}^{\infty}$, see [36, Thm. 3].

The orthogonality relation for $\{P_n\}_{n=0}^{\infty}$ now reads

$$\mu(\{0\})P_m(0)P_n(0) - \sum_{k=1}^{\infty} \frac{\mathcal{G}_{J^{(1)}}(\mu_k)}{\mu_k \mathcal{G}'_J(\mu_k)} P_m(\mu_k)P_n(\mu_k) = \delta_{m,n}, \quad m, n \in \mathbb{Z}_+, \tag{30}$$

where $\{\mu_k \mid k \in \mathbb{N}\}$ stands for the set of all zeros of $\mathcal{G}_J$. Number $\mu(\{0\})$ vanishes if and only if 0 is not an eigenvalue of J. If $0 \in \operatorname{spec}_p(J)$ then $\{P_n(0)\}_{n=0}^{\infty}$ is the corresponding eigenvector of J and

$$\mu(\{0\}) = \langle e_0, E_J(\{0\})e_0 \rangle = \left(\sum_{n=0}^{\infty} (P_n(0))^2 \right)^{-1}.$$

Example 9: Let us illustrate the presented method of finding the measure of orthogonality on a concrete example with Lommel polynomials, well known from the theory of Bessel functions (see, for example [44, § 9.6-9.73] or [10, Chp. VII]). Lommel polynomials can be written explicitly in the form

$$R_{n,\nu}(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} \frac{\Gamma(\nu+n-k)}{\Gamma(\nu+k)} \left(\frac{2}{x}\right)^{n-2k}$$

where $n \in \mathbb{Z}_+$, $\nu \in \mathbb{C}$, $-\nu \notin \mathbb{Z}_+$ and $x \in \mathbb{C}\backslash\{0\}$. Here we use the traditional terminology though, obviously, $R_{n,\nu}(x)$ is a polynomial in the variable x^{-1} rather than in x. One readily verifies the Lommel polynomials obey the recurrence

$$R_{n+1,\nu}(x) = \frac{2\,(n+\nu)}{x} R_{n,\nu}(x) - R_{n-1,\nu}(x), \quad n \in \mathbb{Z}_+, \tag{31}$$

with the initial conditions $R_{-1,\nu}(x) = 0$, $R_{0,\nu}(x) = 1$.

For $\nu > -1$ and $n \in \mathbb{Z}_+$, set temporarily

$$\lambda_n = 0 \quad \text{and} \quad w_n = 1/\sqrt{(\nu+n+1)(\nu+n+2)}\,.$$

Then the solution $\{P_n\}_{n=0}^{\infty}$ of recurrence (26) with this particular choice of λ and w is related with Lommel polynomials by the relation

$$R_{n,\nu+1}(x) = \sqrt{\frac{\nu+1}{\nu+n+1}}\, P_n\!\left(\frac{2}{x}\right), \tag{32}$$

as it follows from (31). Since obviously $\lambda \in \ell^1(\mathbb{Z}_+)$ and $w \in \ell^2(\mathbb{Z}_+)$ is positive for $\nu > -1$, we can use formula (30) to derive the orthogonality relation for Lommel polynomials.

First, recalling (7), by (29) one has

$$\mathcal{G}_J(z) = \mathcal{F}_J(z^{-1}) = \Gamma(\nu+1)\, z^{-\nu} J_\nu(2z)$$

and similarly

$$\mathcal{G}_{J^{(1)}}(z) = \Gamma(\nu+2)\, z^{-\nu-1} J_{\nu+1}(2z).$$

Second, 0 is not an eigenvalue of the corresponding Jacobi operator J. In fact, invertibility of J can be verified straightforwardly by solving the formal eigenvalue equation for 0. At last, note $x^{-\nu}J_\nu(x)$ is an even function, denote by $j_{k,\nu}$ the k-th positive zero of $J_\nu(x)$ and put $j_{-k,\nu} = -j_{k,\nu}$ for $k \in \mathbb{N}$. Then formula (30) tells us that the orthogonality relation takes the form

$$-2(\nu+1)\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{J_{\nu+1}(j_{k,\nu})}{j_{k,\nu}^2\, J_\nu'(j_{k,\nu})}\, P_m\!\left(\frac{2}{j_{k,\nu}}\right) P_n\!\left(\frac{2}{j_{k,\nu}}\right) = \delta_{mn}$$

where $J_\nu'(x)$ denotes the partial derivative of $J_\nu(x)$ with respect to x. Finally, by using (32) together with well known identity [1, Eq. 9.1.27],

$$\partial_x J_\nu(x) = \frac{\nu}{x}\, J_\nu(x) - J_{\nu+1}(x),$$

the orthogonality relation simplifies to a nice formula

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} j_{k,\nu}^{-2}\, R_{n,\nu+1}(j_{k,\nu})R_{m,\nu+1}(j_{k,\nu}) = \frac{1}{2(n+\nu+1)}\,\delta_{mn}, \tag{33}$$

valid for $\nu > -1$ and $m, n \in \mathbb{Z}_+$. Most likely, the orthogonality relation (33) has been originally derived by Dickinson in [9].

Let us remark the so called *Askey scheme* [20] is a comprehensive list of today's well known OPs. OPs from the Askey scheme are defined as terminating hypergeometric or q-hypergeometric series. The orthogonality relations of OPs as well as many other properties are listed in the Askey scheme. However, Lommel polynomials are not involved although there is a relation for them in terms of hypergeometric series, see [9]. The reason for that is maybe the orthogonality relation for Lommel polynomials is not fully explicit due to the presents of zeros $j_{k,\nu}$, in contrast to orthogonality relations of all OPs from the Askey scheme.

In view of Example 9, one can say Lommel polynomials are associated with Bessel function J_ν since this function plays a crucial role in the orthogonality relation. Apart from the orthogonality, asymptotic behavior of Lommel polynomials $R_{n,\nu}$, as $n \to \infty$, can be expressed in terms of Bessel function $J_{\nu-1}$, as it follows from Hurwitz' limit relation [44, § 9.65]. In [36] we have introduced a new family of OPs associated with regular Coulomb wave function, see [1, Chp. 14]. This family generalizes Lommel polynomials in one additional parameter. The introduced method of finding the measure of orthogonality is applicable in this case. Surprisingly, the orthogonality relation has almost the same form as in the case of Lommel polynomials, see [36, Thm. 14].

II.2 The moment problem

In this section, we explain what it means to solve a moment problem. We focus on the so called Hamburger moment problem (=Hmp) in the indeterminate case. In particular, we stress the importance of the entire functions A, B, C and D from the Nevanlinna parametrization since having them at hand one can describe any solution of the corresponding indeterminate Hmp in a systematic way.

In [38], we have found explicit expressions for the Nevanlinna functions in the moment problem related with a q-analogue of Lommel polynomials. These polynomials are orthogonal and they have been introduced and studied by Koelink and others in

[21, 22, 23]. Especially paper [21] served as a strong motivation for work [38], Koelink pointed out that it would be of interest to determine functions from the Nevanlinna parametrization corresponding to the Hmp in question. He derived one exceptional solution to the Hmp (N-extremal, see below) and found a corresponding relation of orthogonality for q-Lommel polynomials. All other N-extremal measures of orthogonality are described in [38]. Moreover, in [37] we provide a detailed spectral analysis of a certain Jacobi operator. This operator is chosen suitably so that its spectral properties allow us to reproduce, in a quite straightforward but alternative way, some results concerning the so called Hahn-Exton q-Bessel functions originally derived in [21, 22].

Suppose a real sequence $m = \{m_n\}_{n=0}^{\infty}$ is given. To solve a Hmp means to answer the following 3 questions. Is there a positive Borel measure μ on $\mathbb{R}$ whose n-th moment is equal to m_n, i.e.,

$$\int_{\mathbb{R}} x^n \mathrm{d}\mu(x) = m_n, \tag{34}$$

for all $n \in \mathbb{Z}_+$? If so, is the measure μ determined uniquely by the sequence m? If this is not the case, how all the measures with the moment sequence m can be described?

If μ fulfills (34) we say that μ is a *solution* to the Hmp. If the solution is unique the Hmp is called *determinate*. Otherwise the Hmp is said to be *indeterminate*. A comprehensive treatise on the classical results from the theory of the moment problem is given in [2, 29].

The systematic study of the moment problem has been initiated by Stieltjes in his memoir [39] from 1894-95, though he restricted himself to measures supported on $[0, \infty)$. Hamburger continued his work in the series of papers [15] from 1920-21, dealing with measures supported on the whole real line. Hamburger's theorem concerns the existence of a solution of Hmp. It says the Hmp has a solution if and only if the sequence m is positive definite which holds true if and only if the $(n+1) \times (n+1)$ Hankel matrices

$$\Delta_n(m) := \begin{pmatrix} m_0 & m_1 & m_2 & \dots & m_n \\ m_1 & m_2 & m_3 & \dots & m_{n+1} \\ m_2 & m_3 & m_4 & \dots & m_{n+2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ m_n & m_{n+1} & m_{n+2} & \dots & m_{2n} \end{pmatrix}$$

are positive definite for all $n \in \mathbb{Z}_+$. The problem of determinacy of a Hmp has been treated in terms of Hankel matrices by Hamburger, too. He showed the Hmp is determinate if and only if

$$\lim_{n \to \infty} \frac{\det \Delta_n(m)}{\det \Delta_{n-1}(T^4 m)} = 0$$

where T stands for the shift operator defined in connection with the identity (3). A simple criterion for the Hmp to be determinate, which is however a sufficient condition only, is due to Carleman [7]. He proved that if

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt[2n]{m_{2n}}} = \infty$$

then the Hmp is determinate. So if the moment sequence does not grow too rapidly, the Hmp is determinate. The opposite, however, is not true. By using the Carleman's criterion one can easily prove, for example, the probability measure of a normally distributed random variable is uniquely determined by its moments.

Let a positive measure μ with a moment sequence m be given. Without loss of generality we can assume the moment sequence is normalized so that $m_0 = 1$. OPs $P_n(x)$ satisfying orthogonality relation (25) are determined by the moment sequence m. Indeed, the explicit formula reads

$$P_n(x) = \frac{1}{\sqrt{\det\left[\Delta_{n-1}(m)\Delta_n(m)\right]}} \begin{vmatrix} m_0 & m_1 & \dots & m_n \\ m_1 & m_2 & \dots & m_{n+1} \\ \vdots & \vdots & & \vdots \\ m_{n-1} & m_n & \dots & m_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix},$$

for $n \in \mathbb{Z}_+$, where one has to set $\det \Delta_{-1}(m) := 1$, see [2, Eq. 1.4].

Let us draw our attention to the Hmp in the indeterminate case. The Hmp is indeterminate if and only if the Jacobi operator $J_{\min}$, determined by diagonal sequences $\{\lambda_n\}_{n=0}^{\infty}$ and off-diagonal sequence $\{w_n\}_{n=0}^{\infty}$ from recurrence (26), has deficiency indices $(1,1)$. In terms of OPs, the indeterminacy of the Hmp is equivalent to the case when both sequences $\{P_n(x)\}_{n=0}^{\infty}$ and $\{Q_n(x)\}_{n=0}^{\infty}$ belong to $\ell^2(\mathbb{Z}_+)$ for at least one $x \in \mathbb{R}$. It is even necessary and sufficient that there exists $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$, such that either $\{P_n(z)\}_{n=0}^{\infty}$ or $\{Q_n(z)\}_{n=0}^{\infty}$ belongs to $\ell^2(\mathbb{Z}_+)$. In this case, series

$$\sum_{n=0}^{\infty} |P_n(z)|^2 \quad \text{and} \quad \sum_{n=0}^{\infty} |Q_n(z)|^2$$

converge locally uniformly on $\mathbb{C}$. All proofs of mentioned statements can be found, for instance, in [2, 30].

Recall one way of the definition of the *Nevanlinna functions* A, B, C and D is the following:

$$A(z) = z\sum_{n=0}^{\infty} Q_n(0)Q_n(z), \qquad B(z) = -1 + z\sum_{n=0}^{\infty} Q_n(0)P_n(z),$$
$$C(z) = 1 + z\sum_{n=0}^{\infty} P_n(0)Q_n(z), \qquad D(z) = z\sum_{n=0}^{\infty} P_n(0)P_n(z),$$

where P_n and Q_n are OPs of the first and second kind, respectively. Since all the series converge locally uniformly on $\mathbb{C}$ all Nevanlinna functions are entire. In addition, the *Nevanlinna matrix*

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

has determinant equal to one, i.e.,

$$A(z)D(z) - B(z)C(z) = 1, \quad \forall z \in \mathbb{C}. \tag{35}$$

Furthermore, functions A, B, C and D share many properties as entire complex functions. For instance, they are of the same order less or equal to 1, the same type (minimal exponential) and have the same Phragmén-Lindelöf indicator function which is non-negative, see [4].

The description of all solutions of indeterminate Hmp is due to Nevanlinna [24]. The parameter space is the one-point compactification of the set $\mathscr{P}$ of Pick functions, which are holomorphic functions in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ with nonnegative imaginary part. The *Nevanlinna parametrization* is established via the homeomorphism $\varphi \mapsto \mu_\varphi$ of $\mathscr{P} \cup \{\infty\}$ onto the set of solutions of the indeterminate Hmp given by the formula

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu_\varphi(x)}{x-z} = -\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{36}$$

which expresses that the Stieltjes (or Cauchy) transform of any solution μ of indeterminate Hmp is given by the RHS of (36) for a unique $\varphi \in \mathscr{P} \cup \{\infty\}$.

Strictly speaking it is not the set of solutions which is parametrized but the set of their Stieltjes transforms which are holomorphic functions in the cut plane $\mathbb{C} \setminus \mathbb{R}$. This is in principal as good, since the Stieltjes transform is a one-to-one mapping from the set of finite complex Borel measures to the set of holomorphic functions on $\mathbb{C} \setminus \mathbb{R}$. The inverse mapping is given by the Perron-Stieltjes inversion formula which states μ is the weak limit for $\epsilon \to 0+$ of measures with density

$$\rho_\epsilon(x) := \frac{1}{2\pi i}\left(\int_{\mathbb{R}} \frac{\mathrm{d}\mu(u)}{u-x-i\epsilon} - \int_{\mathbb{R}} \frac{\mathrm{d}\mu(u)}{u-x+i\epsilon}\right)$$

with respect to the Lebesgue measure, see [2, Chp. 3].

Thus, to solve the indeterminate Hmp means, in certain sense, to find expressions for Nevanlinna functions A, B, C and D. The formulas in concrete cases are usually in terms of special functions. In particular functions B and D play an important role.

A particular subset of the set of solutions of the indeterminate Hmp is formed by so called *Nevanlinna extremal*, or shortly *N-extremal* measures. N-extremal measures can be parametrized by a parameter $t \in \mathbb{R} \cup \{\infty\}$ since these measures correspond to the choice

$$\varphi(z) = t, \quad \operatorname{Im} z \neq 0,\, t \in \mathbb{R} \cup \{\infty\},$$

for the Pick function φ in (36). Thus, one has

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu_t(x)}{z-x} = \frac{A(z)t - C(z)}{B(z)t - D(z)}, \text{ for } t \in \mathbb{R}, \text{ or } \int_{\mathbb{R}} \frac{\mathrm{d}\mu_\infty(x)}{z-x} = \frac{A(z)}{B(z)}. \tag{37}$$

Measures μ_t are purely discrete and they can be characterized by at least two different propositions. First one is due to Riezs, see [26] or [29, p. 62], stating N-extremal measures are the only measures (among solutions of the Hmp) for which polynomials $\mathbb{C}[x]$ are dense in $L^2(\mathbb{R}, \mathrm{d}\mu)$.

At the same time, $\mu_t = \langle e_0, E_{J_t} e_0 \rangle$ where e_0 is the first vector of the standard basis of $\ell^2(\mathbb{Z}_+)$ and E_{J_t} is the projection-valued spectral measure of a self-adjoint extension J_t of $J_{\min}$. Thus, N-extremal measures correspond to spectral measures of all the self-adjoint Jacobi operators determined by the Jacobi matrix whose diagonal and off-diagonal sequences are defined by the three-term recurrence of the corresponding OPs.

The support of μ_t is the set of poles of meromorphic functions on the RHSs in formulas in (37). Since the zeros of functions

$$z \mapsto A(z)t - C(z) \quad \text{and} \quad z \mapsto B(z)t - D(z)$$

are real, simple and interlace, see [2, Chp. 2, Sec. 4], one gets μ_t is supported on the set

$$\mathfrak{Z}_t := \{x \in \mathbb{R} \mid B(x)t - D(x) = 0\}, \quad \text{for } t \in \mathbb{R},$$

or

$$\mathfrak{Z}_\infty := \{x \in \mathbb{R} \mid B(x) = 0\}.$$

Consequently, the first formula in (37) is, in fact, the Mittag-Leffler expansion of the meromorphic function on the RHS, cf. [2, footnote at p. 55],

$$\sum_{x \in \mathfrak{Z}_t} \frac{\mu_t(\{x\})}{z - x} = \frac{A(z)t - C(z)}{B(z)t - D(z)},$$

from which one deduces

$$\mu_t(\{x\}) = \operatorname{Res}\left(\frac{A(.)t - C(.)}{B(.)t - D(.)}, x\right) = \frac{A(x)t - C(x)}{B'(x)t - D'(x)},$$

for $x \in \mathfrak{Z}_t$. We treat the case $t \in \mathbb{R}$, if $t = \infty$ one proceeds in a similar way. The last identity can be slightly rewritten. Note if $x \in \mathfrak{Z}_t$ then $B(x)t = D(x)$ and taking into account identity (35) one observes

$$\frac{A(x)t - C(x)}{B'(x)t - D'(x)} = \frac{1}{B'(x)D(x) - B(x)D'(x)}.$$

Consequently, the magnitude of jumps of the distribution function $x \mapsto \mu_t((-\infty, x])$ is independent of t.

Altogether, N-extremal measures are completely determined by functions B and D and they can be written in the form

$$\mu_t = \sum_{x \in \mathfrak{Z}_t} \rho(x)\delta_x,$$

where we put

$$\rho(x) := \frac{1}{B'(x)D(x) - B(x)D'(x)},$$

$t \in \mathbb{R} \cup \{\infty\}$, and δ_x stands for the Dirac measure supported on $\{x\}$. Recall also for the function ρ it holds

$$\sum_{n=0}^{\infty} |P_n(z)|^2 = \frac{1}{\rho(z)}$$

and $0 < \rho(z) < 1$, for all $z \in \mathbb{C}$, see again [2, Chp. 2, Sec. 4].

Besides the N-extremal measures, by setting $\varphi(z) = t + i\gamma$, with $z \in \mathbb{C}_+$, $t \in \mathbb{R}$, $\gamma > 0$, for the Pick function in (36), one arrives at a two-parametric family

$$\{\mu_{t,\gamma} \mid t \in \mathbb{R}, \gamma > 0\}$$

of solutions of Hmp. Berg and Valent [5] proved measures $\mu_{t,\gamma}$ are all absolutely continuous with respect to the Lebesgue measure. Their density can be expressed in terms of functions B and D as follows

$$\frac{\mathrm{d}\mu_{t,\gamma}}{\mathrm{d}x} = \frac{\gamma\pi^{-1}}{(tB(x) - D(x))^2 + (\gamma B(x))^2}, \quad x \in \mathbb{R}.$$

It was Krein [2, p. 87] who proved the following criterion: If there exists a solution μ of the Hmp whose absolutely continuous part w_μ is such that

$$\int_{\mathbb{R}} \frac{\log w_\mu(x)}{1+x^2} \mathrm{d}x > -\infty \tag{38}$$

then the Hmp is indeterminate. Measures $\mu_{t,\gamma}$ are interesting since the solution $\mu_{0,1}$ is the one that maximizes the entropy integral from (38) among all the densities of solutions of the Hmp, see [12] for even more general result.

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Part III

Reprints to the Part I

Linear Algebra and its Applications 434 (2011) 1336–1353

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

On the eigenvalue problem for a particular class of finite Jacobi matrices

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ARTICLE INFO

Article history:
Received 18 October 2010
Accepted 8 November 2010
Available online 3 December 2010

Submitted by R.A. Brualdi

AMS classification:
47B36
15A18
33C10

Keywords:
Tridiagonal matrix
Finite Jacobi matrix
Eigenvalue problem
Characteristic function

ABSTRACT

A function $\mathfrak{F}$ with simple and nice algebraic properties is defined on a subset of the space of complex sequences. Some special functions are expressible in terms of $\mathfrak{F}$, first of all the Bessel functions of first kind. A compact formula in terms of the function $\mathfrak{F}$ is given for the determinant of a Jacobi matrix. Further we focus on the particular class of Jacobi matrices of odd dimension whose parallels to the diagonal are constant and whose diagonal depends linearly on the index. A formula is derived for the characteristic function. Yet another formula is presented in which the characteristic function is expressed in terms of the function $\mathfrak{F}$ in a simple and compact manner. A special basis is constructed in which the Jacobi matrix becomes a sum of a diagonal matrix and a rank-one matrix operator. A vector-valued function on the complex plain is constructed having the property that its values on spectral points of the Jacobi matrix are equal to corresponding eigenvectors.

1. Introduction

The results of the current paper are related to the eigenvalue problem for finite-dimensional symmetric tridiagonal (Jacobi) matrices. Notably, the eigenvalue problem for finite Jacobi matrices is solvable explicitly in terms of generalized hypergeometric series [7]. Here we focus on a very particular class of Jacobi matrices which makes it possible to derive some expressions in a comparatively simple and compact form. We do not aim at all, however, at a complete solution of the eigenvalue problem. We restrict ourselves to derivation of several explicit formulas, first of all that for the characteristic function, as explained in more detail below. We also develop some auxiliary notions which may be, to our opinion, of independent interest.

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doi:10.1016/j.laa.2010.11.010

First, we introduce a function, called $\mathfrak{F}$, defined on a subset of the space of complex sequences. In the remainder of the paper it is intensively used in various formulas. The function $\mathfrak{F}$ has remarkably simple and nice algebraic properties. Among others, with the aid of $\mathfrak{F}$ one can relate an infinite continued fraction to any sequence from the definition domain on which $\mathfrak{F}$ takes a nonzero value. This may be compared to the fact that there exists a correspondence between infinite Jacobi matrices and infinite continued fractions, as explained in [2, Chapter 1]. Let us also note that some special functions are expressible in terms of $\mathfrak{F}$. First of all this concerns the Bessel functions of first kind. We examine the relationship between $\mathfrak{F}$ and the Bessel functions and provide some supplementary details on it.

Further we introduce an infinite antisymmetric matrix, with entries indexed by integers, such that its every row or column obeys a second-order difference equation which is very well known from the theory of Bessel functions. With the aid of function $\mathfrak{F}$ one derives a general formula for entries of this matrix. The matrix also plays an essential role in the remainder of the paper.

As an application we present a comparatively simple formula for the determinant of a Jacobi matrix of odd dimension under the assumption that the neighboring parallels to the diagonal are constant. As far as the determinant is concerned this condition is not very restrictive since a Jacobi matrix can be written as a product of another Jacobi matrix with all units on the neighboring parallels which is sandwiched with two diagonal matrices. The formula further simplifies in the particular case when the diagonal is antisymmetric (with respect to its center). In that case zero is always an eigenvalue and we give an explicit formula for the corresponding eigenvector.

Finally we focus on the rather particular class of Jacobi matrices of odd dimension whose parallels to the diagonal are constant and whose diagonal depends linearly on the index. Within this class it suffices to consider matrices whose diagonal is, in addition, antisymmetric. In this case we derive a formula for the characteristic function. Yet another formula is presented in which the characteristic function is expressed in terms of the function $\mathfrak{F}$ in a very simple and compact manner. Moreover, we construct a basis in which the Jacobi matrix becomes a sum of a diagonal matrix and a rank-one matrix operator. This form is rather suitable for various computations. Particularly, one can readily derive a formula for the resolvent. In addition, a vector-valued function on the complex plain is constructed having the property that its values on spectral points of the Jacobi matrix are equal to corresponding eigenvectors.

2. The function $\mathfrak{F}$

We introduce a function $\mathfrak{F}$ defined on a subset of the linear space formed by all complex sequences $x = \{x_k\}_{k=1}^{\infty}$.

Definition 1. Define $\mathfrak{F} : D \to \mathbb{C}$,

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \cdots x_{k_m} x_{k_m+1} \tag{1}$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables we identify $\mathfrak{F}(x_1, x_2, \ldots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$. By convention, we also put $\mathfrak{F}(\emptyset) = 1$ where $\emptyset$ is the empty sequence.

Remark 2. Note that the domain D is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$. To see that the series on the RHS of (1) converges absolutely whenever $x \in D$ observe that the absolute value of the mth summand is majorized by the expression

$$\sum_{\substack{k \in \mathbb{N}^m \\ k_1 < k_2 < \cdots < k_m}} |x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \cdots x_{k_m} x_{k_m+1}| \leqslant \frac{1}{m!} \left(\sum_{j=1}^{\infty} |x_j x_{j+1}| \right)^m .$$

Obviously, if all but finitely many elements of a sequence x are zeroes then $\mathfrak{F}(x)$ reduces to a finite sum. Thus

$$\mathfrak{F}(x_1) = 1,\ \mathfrak{F}(x_1, x_2) = 1 - x_1x_2,\ \mathfrak{F}(x_1, x_2, x_3) = 1 - x_1x_2 - x_2x_3,$$
$$\mathfrak{F}(x_1, x_2, x_3, x_4) = 1 - x_1x_2 - x_2x_3 - x_3x_4 + x_1x_2x_3x_4,\ \text{etc.}$$

Let T denote the truncation operator from the left defined on the space of all sequences:

$$T\left(\{x_k\}_{k=1}^{\infty}\right) = \{x_{k+1}\}_{k=1}^{\infty}.$$

T^n, $n = 0, 1, 2, \ldots$, stands for a power of T. Hence $T^n(\{x_k\}_{k=1}^{\infty}) = \{x_{k+n}\}_{k=1}^{\infty}$.

The proof of the following proposition is immediate.

Proposition 3. *For all $x \in D$ one has*

$$\mathfrak{F}(x) = \mathfrak{F}(Tx) - x_1x_2\,\mathfrak{F}\left(T^2x\right). \tag{2}$$

Particularly, if $n \geqslant 2$ then

$$\mathfrak{F}(x_1, x_2, x_3, \ldots, x_n) = \mathfrak{F}(x_2, x_3, \ldots, x_n) - x_1x_2\,\mathfrak{F}(x_3, \ldots, x_n). \tag{3}$$

Remark 4. Clearly, given that $\mathfrak{F}(\emptyset) = \mathfrak{F}(x_1) = 1$, relation (3) determines recursively and unambiguously $\mathfrak{F}(x_1, \ldots, x_n)$ for any finite number of variables $n \in \mathbb{Z}_+$ (including $n = 0$).

Remark 5. One readily verifies that

$$\mathfrak{F}(x_1, x_2, \ldots, x_n) = \mathfrak{F}(x_n, \ldots, x_2, x_1). \tag{4}$$

Hence equality (3) implies, again for $n \geqslant 2$,

$$\mathfrak{F}(x_1, \ldots, x_{n-2}, x_{n-1}, x_n) = \mathfrak{F}(x_1, \ldots, x_{n-2}, x_{n-1}) - x_{n-1}x_n\,\mathfrak{F}(x_1, \ldots, x_{n-2}). \tag{5}$$

Remark 6. For a given $x \in D$ such that $\mathfrak{F}(x) \neq 0$ let us introduce sequences $\{P_k\}_{k=0}^{\infty}$ and $\{Q_k\}_{k=0}^{\infty}$ by $P_0 = 0$ and $P_k = \mathfrak{F}(x_2, \ldots, x_k)$ for $k \geqslant 1$, $Q_k = \mathfrak{F}(x_1, \ldots, x_k)$ for $k \geqslant 0$. According to (5), the both sequences obey the difference equation

$$Y_{k+1} = Y_k - x_kx_{k+1}Y_{k-1}, \quad k = 1, 2, 3, \ldots,$$

with the initial conditions $P_0 = 0$, $P_1 = 1$, $Q_0 = Q_1 = 1$, and define the infinite continued fraction

$$\frac{\mathfrak{F}(Tx)}{\mathfrak{F}(x)} = \lim_{k\to\infty}\frac{P_k}{Q_k} = \cfrac{1}{1 - \cfrac{x_1x_2}{1 - \cfrac{x_2x_3}{1 - \cfrac{x_3x_4}{1 - \cdots}}}}.$$

Proposition 3 admits a generalization.

Proposition 7. *For every $x \in D$ and $k \in \mathbb{N}$ one has*

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \ldots, x_k)\,\mathfrak{F}(T^kx) - \mathfrak{F}(x_1, \ldots, x_{k-1})x_kx_{k+1}\,\mathfrak{F}(T^{k+1}x). \tag{6}$$

Proof. Let us proceed by induction in k. For $k = 1$, equality (6) coincides with (2). Suppose (6) is true for $k \in \mathbb{N}$. Applying Proposition 3 to the sequence T^kx and using (5) one finds that the RHS of (6) equals

$$\begin{aligned}&\mathfrak{F}(x_1, \ldots, x_k)\,\mathfrak{F}(T^{k+1}x) - \mathfrak{F}(x_1, \ldots, x_k)x_{k+1}x_{k+2}\,\mathfrak{F}(T^{k+2}x)\\ &\quad - \mathfrak{F}(x_1, \ldots, x_{k-1})x_kx_{k+1}\,\mathfrak{F}(T^{k+1}x)\\ &= \mathfrak{F}(x_1, \ldots, x_k, x_{k+1})\,\mathfrak{F}(T^{k+1}x) - \mathfrak{F}(x_1, \ldots, x_k)x_{k+1}x_{k+2}\,\mathfrak{F}(T^{k+2}x).\end{aligned}$$

This concludes the verification. □

Remark 8. With the aid of Proposition 3 one can rewrite equality (6) as follows

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \ldots, x_k)\, \mathfrak{F}\left(\frac{\mathfrak{F}(x_1, \ldots, x_{k-1})}{\mathfrak{F}(x_1, \ldots, x_k)}\, x_k, x_{k+1}, x_{k+2}, x_{k+3}, \ldots \right). \tag{7}$$

Later on, we shall also need the following identity.

Lemma 9. *For any $n \in \mathbb{N}$ one has*

$$\begin{aligned} &u_1 \mathfrak{F}(u_2, u_3, \ldots, u_n)\mathfrak{F}(v_1, v_2, v_3, \ldots, v_n) - v_1 \mathfrak{F}(u_1, u_2, u_3, \ldots, u_n)\mathfrak{F}(v_2, v_3, \ldots, v_n) \\ &= \sum_{j=1}^{n} \left(\prod_{k=1}^{j-1} u_k v_k \right) (u_j - v_j)\, \mathfrak{F}(u_{j+1}, u_{j+2}, \ldots, u_n)\mathfrak{F}(v_{j+1}, v_{j+2}, \ldots, v_n). \end{aligned} \tag{8}$$

Proof. The equality can be readily proved by induction in n with the aid of (3). □

Example 10. For $t, w \in \mathbb{C}$, $|t| < 1$, a simple computation leads to the equality

$$\mathfrak{F}\left(\left\{ t^{k-1} w \right\}_{k=1}^{\infty} \right) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{t^{m(2m-1)} w^{2m}}{(1-t^2)(1-t^4)\cdots(1-t^{2m})}. \tag{9}$$

This function can be identified with a basic hypergeometric series (also called$q$-hypergeometric series) defined by

$${}_r\phi_s(a; b; q, z) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_s; q)_k} \left((-1)^k q^{\frac{1}{2}k(k-1)} \right)^{1+s-r} \frac{z^k}{(q; q)_k}$$

where $r, s \in \mathbb{Z}_+$ (nonnegative integers) and

$$(\alpha; q)_k = \prod_{j=0}^{k-1} \left(1 - \alpha q^j \right), \quad k = 0, 1, 2, \ldots,$$

see [5]. In fact, the RHS in (9) equals ${}_0\phi_1(; 0; t^2, -tw^2)$ where

$${}_0\phi_1(; 0; q, z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{(q; q)_k} z^k = \sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{(1-q)(1-q^2)\cdots(1-q^k)} z^k,$$

with $q, z \in \mathbb{C}$, $|q| < 1$, and the recursive rule (2) takes the form

$${}_0\phi_1(; 0; q, z) = {}_0\phi_1(; 0; q, qz) + z\, {}_0\phi_1(; 0; q, q^2 z). \tag{10}$$

Put $e(q; z) = {}_0\phi_1(; 0; q, (1-q)z)$. Then $\lim_{q\uparrow 1} e(q; z) = \exp(z)$. Hence $e(q; z)$ can be regarded as a q-deformed exponential function though this is not the standard choice (compare with [5] or [6] and references therein). Equality (10) can be interpreted as the discrete derivative

$$\frac{e(q; z) - e(q; qz)}{(1-q)z} = e\left(q; q^2 z \right).$$

Moreover, in view of Remark 6, one has

$$\cfrac{1}{1 + \cfrac{z}{1 + \cfrac{qz}{1 + \cfrac{q^2 z}{1 + \cdots}}}} = \frac{{}_0\phi_1(; 0; q, qz)}{{}_0\phi_1(; 0; q, z)}.$$

This equality is related to the Rogers–Ramanujan identities, see the discussion in [3, Chapter 7].

Example 11. The Bessel functions of the first kind can be expressed in terms of function $\mathfrak{F}$. More precisely, for $\nu \notin -\mathbb{N}$, one has

$$J_\nu(2w) = \frac{w^\nu}{\Gamma(\nu+1)}\,\mathfrak{F}\left(\left\{\frac{w}{\nu+k}\right\}_{k=1}^{\infty}\right). \tag{11}$$

The recurrence relation (2) transforms to the well known identity

$$zJ_\nu(z) - 2(\nu+1)J_{\nu+1}(z) + zJ_{\nu+2}(z) = 0.$$

To prove (11) one can proceed by induction in $j = 0, 1, \ldots, m-1$, to show that

$$\begin{aligned}
&\sum_{k_1=1}^{\infty}\sum_{k_2=k_1+2}^{\infty}\cdots\sum_{k_m=k_{m-1}+2}^{\infty}\\
&\quad\times\frac{1}{(\nu+k_1)(\nu+k_1+1)(\nu+k_2)(\nu+k_2+1)\cdots(\nu+k_m)(\nu+k_m+1)}\\
&=\frac{1}{j!}\sum_{k_1=1}^{\infty}\sum_{k_2=k_1+2}^{\infty}\cdots\sum_{k_{m-j}=k_{m-j-1}+2}^{\infty}\\
&\quad\times\frac{1}{(\nu+k_1)(\nu+k_1+1)(\nu+k_2)(\nu+k_2+1)\cdots(\nu+k_{m-j})(\nu+k_{m-j}+1)}\\
&\quad\times\frac{1}{(\nu+k_{m-j}+2)(\nu+k_{m-j}+3)\cdots(\nu+k_{m-j}+j+1)}.
\end{aligned}$$

In particular, for $j = m-1$, the RHS equals

$$\begin{aligned}
&\frac{1}{(m-1)!}\sum_{k_1=1}^{\infty}\frac{1}{(\nu+k_1)(\nu+k_1+1)(\nu+k_1+2)\cdots(\nu+k_1+m)}\\
&=\frac{1}{m!\,(\nu+1)(\nu+2)\cdots(\nu+m)}=\frac{\Gamma(\nu+1)}{m!\,\Gamma(\nu+m+1)}
\end{aligned}$$

and so

$$\frac{w^\nu}{\Gamma(\nu+1)}\,\mathfrak{F}\left(\left\{\frac{w}{\nu+k}\right\}_{k=1}^{\infty}\right) = \sum_{m=0}^{\infty}(-1)^m\,\frac{w^{2m+\nu}}{m!\,\Gamma(\nu+m+1)},$$

as claimed. Furthermore, Remark 6 provides us with the infinite fraction

$$\frac{\nu+1}{w}\,\frac{J_{\nu+1}(2w)}{J_\nu(2w)} = \cfrac{1}{1-\cfrac{\cfrac{w^2}{(\nu+1)(\nu+2)}}{1-\cfrac{\cfrac{w^2}{(\nu+2)(\nu+3)}}{1-\cfrac{\cfrac{w^2}{(\nu+3)(\nu+4)}}{1-\cdots}}}}.$$

This can be rewritten as

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \cfrac{z}{2(\nu+1) - \cfrac{z^2}{2(\nu+2) - \cfrac{z^2}{2(\nu+3) - \cfrac{z^2}{2(\nu+4) - \cdots}}}}.$$

Comparing to Example 11, one can also find the value of $\mathfrak{F}$ on the truncated sequence $\{w/(\nu+k)\}_{k=1}^{n}$.

Proposition 12. *For $n \in \mathbb{Z}_+$ and $\nu \in \mathbb{C} \setminus \{-n, -n+1, \ldots, -1\}$ one has*

$$\mathfrak{F}\left(\frac{w}{\nu+1}, \frac{w}{\nu+2}, \ldots, \frac{w}{\nu+n}\right) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+n+1)} \sum_{s=0}^{[n/2]} (-1)^s \frac{(n-s)!}{s!\,(n-2s)!} w^{2s} \prod_{j=s}^{n-1-s} (\nu+n-j). \tag{12}$$

In particular, for $m, n \in \mathbb{Z}_+$, $m \leqslant n$, one has

$$\mathfrak{F}\left(\frac{w}{m+1}, \frac{w}{m+2}, \ldots, \frac{w}{n}\right) = \frac{m!}{n!} \sum_{s=0}^{[(n-m)/2]} (-1)^s \frac{(n-s)!\,(n-m-s)!}{s!\,(m+s)!\,(n-m-2s)!} w^{2s}. \tag{13}$$

Proof. Firstly, the equality

$$\begin{aligned}&\sum_{k=1}^{n} \frac{(n+1-k)(n+2-k)\cdots(n+s-1-k)}{(\nu+k)(\nu+k+1)\cdots(\nu+k+s)}\\ &= \frac{n\,(n+1)\cdots(n+s-1)}{s\,(\nu+n+s)\,(\nu+1)(\nu+2)\cdots(\nu+s)}\end{aligned} \tag{14}$$

holds for all $n \in \mathbb{Z}_+$, $\nu \in \mathbb{C}$, $\nu \notin -\mathbb{N}$, and $s \in \mathbb{N}$. To show (14) one can proceed by induction in s. The case $s = 1$ is easy to verify. For the induction step from $s-1$ to s, with $s > 1$, let us denote the LHS of (14) by $Y_s(\nu, n)$. One observes that

$$Y_s(\nu, n) = \frac{\nu+n+s-1}{s} Y_{s-1}(\nu, n) - \frac{\nu+n+2s-1}{s} Y_{s-1}(\nu+1, n).$$

Applying the induction hypothesis the equality readily follows.

Next one shows that

$$\begin{aligned}&\sum_{k_1=1}^{n-2s+2} \sum_{k_2=k_1+2}^{n-2s+4} \cdots \sum_{k_s=k_{s-1}+2}^{n}\\ &\times \frac{1}{(\nu+k_1)(\nu+k_1+1)(\nu+k_2)(\nu+k_2+1)\cdots(\nu+k_s)(\nu+k_s+1)}\\ &= \frac{(n-2s+2)(n-2s+3)\cdots(n-s+1)}{s!\,(\nu+1)(\nu+2)\cdots(\nu+s)\,(\nu+n-s+2)(\nu+n-s+3)\cdots(\nu+n+1)}\end{aligned} \tag{15}$$

holds for all $n \in \mathbb{Z}_+$, $s \in \mathbb{N}$, $2s \leqslant n+2$. To this end, we again proceed by induction in s. The case $s = 1$ is easy to verify. In the induction step from $s-1$ to s, with $s > 1$, one applies the induction hypothesis to the LHS of (15) and arrives at the expression

$$\sum_{k=1}^{n-2s+2} \frac{1}{(\nu+k)(\nu+k+1)\,(s-1)!} \times \frac{(n-k-2s+3)(n-k-2s+4)\cdots(n-k-s+1)}{(\nu+k+2)(\nu+k+3)\cdots(\nu+k+s)\,(\nu+n-s+3)(\nu+n-s+4)\cdots(\nu+n+1)}.$$

Using (14) one obtains the RHS of (15), as claimed.

Finally, to conclude the proof, it suffices to notice that

$$\mathfrak{F}\left(\frac{w}{\nu+1}, \frac{w}{\nu+2}, \dots, \frac{w}{\nu+n}\right) = 1 + \sum_{s=1}^{[n/2]} (-1)^s \sum_{k_1=1}^{n-2s+1} \sum_{k_2=k_1+2}^{n-2s+3} \cdots \sum_{k_s=k_{s-1}+2}^{n-1} \times \frac{w^{2s}}{(\nu+k_1)(\nu+k_1+1)(\nu+k_2)(\nu+k_2+1)\cdots(\nu+k_s)(\nu+k_s+1)}$$

and to use equality (15). □

One can complete Proposition 12 with another relation to Bessel functions.

Proposition 13. *For $m, n \in \mathbb{Z}_+$, $m \leqslant n$, one has*

$$\begin{aligned}\pi J_m(2w)Y_{n+1}(2w) &= -\frac{n!}{m!}\, w^{m-n-1}\, \mathfrak{F}\left(\frac{w}{m+1}, \frac{w}{m+2}, \dots, \frac{w}{n}\right) \\ &\quad - \sum_{s=0}^{m-1} \frac{(m-s-1)!\,(n-m+2s+1)!}{s!\,(n+s+1)!\,(n-m+s+1)!}\, w^{n-m+2s+1} + O\big(w^{m+n+1}\log(w)\big).\end{aligned} \tag{16}$$

Proof. Recall the following two facts from the theory of Bessel functions (see, for instance, [4, Chapter VII]). Firstly, for $\mu, \nu \notin -\mathbb{N}$, one has

$$J_\mu(z)J_\nu(z) = \sum_{s=0}^{\infty} (-1)^s \frac{(s+\mu+\nu+1)_s}{s!\,\Gamma(\mu+s+1)\Gamma(\nu+s+1)} \left(\frac{z}{2}\right)^{\mu+\nu+2s}$$

where $(a)_s = a(a+1)\cdots(a+s-1)$ is the Pochhammer symbol. Secondly, for $n \in \mathbb{Z}_+$,

$$\pi Y_n(z) = \frac{\partial}{\partial\nu}\left(J_\nu(z) - (-1)^n J_{-\nu}(z)\right)\bigg|_{\nu=n}.$$

For $m, n \in \mathbb{Z}_+$, $m \leqslant n$, a straightforward computation based on these facts yields

$$\begin{aligned}\pi J_m(z)Y_n(z) &= -\sum_{s=0}^{[(n-m-1)/2]} (-1)^s \frac{(n-s-1)!\,(n-m-s-1)!}{s!\,(m+s)!\,(n-m-2s-1)!}\left(\frac{z}{2}\right)^{m-n+2s} \\ &\quad - \sum_{s=0}^{m-1} \frac{(m-s-1)!\,(n-m+2s)!}{s!\,(n+s)!\,(n-m+s)!}\left(\frac{z}{2}\right)^{n-m+2s} + 2J_m(z)J_n(z)\log\left(\frac{z}{2}\right) \\ &\quad + \sum_{s=0}^{\infty} (-1)^s \frac{(m+n+2s)!}{s!\,(m+s)!\,(n+s)!\,(m+n+s)!}\left(\frac{z}{2}\right)^{m+n+2s} \Big(2\psi(m+n+2s+1) \\ &\quad - \psi(m+s+1) - \psi(n+s+1) - \psi(m+n+s+1) - \psi(s+1)\Big)\end{aligned} \tag{17}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. The proposition follows from (17) and (13). □

Remark 14. Note that the first term on the RHS of (16) contains only negative powers of w. One can extend (16) to the case $n = m - 1$. Then

$$\pi J_m(2w)Y_m(2w) = -\sum_{s=0}^{m-1} \frac{(m-s-1)!\,(2s)!}{(s!)^2\,(m+s)!}\, w^{2s} + O(w^{2m}\log(w)).$$

3. The matrix $\mathfrak{J}$

In this section we introduce an infinite matrix $\mathfrak{J}$ that is basically determined by two simple properties – it is antisymmetric and its every row satisfies a second-order difference equation known from the theory of Bessel functions. Of course, in that case every column of the matrix satisfies the difference equation as well.

Lemma 15. *Suppose $w \in \mathbb{C} \setminus \{0\}$. The dimension of the vector space formed by infinite-dimensional matrices $A = \{A(m,n)\}_{m,n\in\mathbb{Z}}$ satisfying, for all $m, n \in \mathbb{Z}$,*

$$wA(m, n-1) - nA(m,n) + wA(m, n+1) = 0 \tag{18}$$

and

$$A(n,m) = -A(m,n), \tag{19}$$

equals 1. *Every such a matrix is unambiguously determined by the value $A(0,1)$, and one has*

$$\forall n \in \mathbb{Z},\ A(n, n+1) = A(0,1). \tag{20}$$

Proof. Suppose A solves (18) and (19). Then $A(m,m) = 0$. Equating $m = n$ in (18) and using (19) one finds that $A(n, n+1) = -A(n, n-1) = A(n-1, n)$. Hence (20) is fulfilled. Clearly, the matrix A is unambiguously determined by the second-order difference equation (18) in n and by the initial conditions $A(m,m) = 0, A(m, m+1) = A(0,1)$, when m runs through $\mathbb{Z}$.

Conversely, choose $\lambda \in \mathbb{C}, \lambda \neq 0$. Let A be the unique matrix determined by (18) and the initial conditions $A(m,m) = 0, A(m, m+1) = \lambda$. It suffices to show that A satisfies (19) as well. Note that $A(m, m-1) = -\lambda$. Furthermore,

$$\begin{aligned} & wA(m-1, m+1) - mA(m, m+1) + wA(m+1, m+1) \\ &= wA(m-1, m+1) - mA(m-1, m) + wA(m-1, m-1) \\ &= 0. \end{aligned}$$

From (18) and the initial conditions it follows that $A(m, m+2) = (m+1)\lambda/w$, and so $mA(m, m+2) = (m+1)A(m-1, m+1)$. Consequently,

$$\begin{aligned} & wA(m-1, m+2) - mA(m, m+2) + wA(m+1, m+2) \\ &= wA(m-1, m+2) - (m+1)A(m-1, m+1) + wA(m-1, m) \\ &= 0. \end{aligned}$$

One observes that, for a given $m \in \mathbb{Z}$, the sequence

$$x_n = -A(m-1, n) + \frac{m}{w} A(m,n),\ n \in \mathbb{Z},$$

solves the difference equation

$$wx_{n-1} - nx_n + wx_{n+1} = 0 \tag{21}$$

with the initial conditions $x_{m+1} = A(m+1, m+1)$, $x_{m+2} = A(m+1, m+2)$. By the uniqueness, $x_n = A(m+1, n)$. This means that, for all $m, n \in \mathbb{Z}$,

$$wA(m-1, n) - mA(m, n) + wA(m+1, n) = 0.$$

Put $B(m, n) = -A(n, m)$. Then B fulfills (18) and $B(m, m) = 0$, $B(m, m+1) = \lambda$. Whence $B = A$. □

Lemma 16. *Suppose $w \in \mathbb{C} \setminus \{0\}$. If a matrix $A = \{A(m, n)\}_{m,n\in\mathbb{Z}}$ satisfies (18) and (19) then*

$$\forall m, n \in \mathbb{Z},\ A(m, -n) = (-1)^n A(m, n),\ A(-m, n) = (-1)^m A(m, n). \tag{22}$$

Proof. For any sequence $\{x_n\}_{n\in\mathbb{Z}}$ satisfying the difference equation (21) one can verify, by mathematical induction, that $x_{-n} = (-1)^n x_n$, $n = 0, 1, 2, \ldots$. □

Definition 17. For a given parameter $w \in \mathbb{C} \setminus \{0\}$ let $\mathfrak{J} = \{\mathfrak{J}(m, n)\}_{m,n\in\mathbb{Z}}$ denote the unique matrix satisfying (18), (19) and $\mathfrak{J}(m, m+1) = 1$, $\forall m \in \mathbb{Z}$.

Remark 18. Here are several particular entries of the matrix $\mathfrak{J}$,

$$\mathfrak{J}(m, m) = 0,\ \mathfrak{J}(m, m+1) = 1,\ \mathfrak{J}(m, m+2) = \frac{m+1}{w},\ \mathfrak{J}(m, m+3) = \frac{(m+1)(m+2)}{w^2} - 1,$$

with $m \in \mathbb{Z}$. Some other particular values follow from (19) and (22). Below, in Proposition 22, we derive a general formula for $\mathfrak{J}(m, n)$.

Lemma 19. *For $0 \leqslant m < n$ one has (with the convention $\mathfrak{F}(\emptyset)$=1)*

$$\mathfrak{J}(m, n) = \frac{(n-1)!}{m!}\, w^{m-n+1}\, \mathfrak{F}\left(\frac{w}{m+1}, \frac{w}{m+2}, \ldots, \frac{w}{n-1}\right). \tag{23}$$

Proof. The RHS of (23) equals 1 for $n = m+1$, and $(m+1)/w$ for $n = m+2$. Moreover, in view of (5), the RHS satisfies the difference equation (21) in the index n. □

Remark 20. From (23) and (11) it follows that

$$\forall m \in \mathbb{Z},\ \lim_{n\to\infty} \frac{w^{n-1}}{(n-1)!}\, \mathfrak{J}(m, n) = J_m(2w).$$

This is in agreement with the well known fact that, for any $w \in \mathbb{C}$, the sequence $\{J_n(2w)\}_{n\in\mathbb{Z}}$ fulfills the second-order difference equation (21).

Remark 21. Rephrasing Proposition 13 and Remark 14 one has, for $m, n \in \mathbb{Z}_+$, $m \leqslant n$,

$$\pi J_m(2w) Y_n(2w) = -w^{-1}\mathfrak{J}(m, n) - \sum_{s=0}^{m-1} \frac{(m-s-1)!\,(n-m+2s)!}{s!\,(n+s)!\,(n-m+s)!}\, w^{n-m+2s} + O(w^{m+n}\log(w)).$$

Since, by definition, the matrix $\mathfrak{J}$ is antisymmetric it suffices to determine the values $\mathfrak{J}(m, n)$ for $m \leqslant n$, $m, n \in \mathbb{Z}$. In the derivations to follow as well as in the remainder of the paper we use the Newton symbol in the usual sense, i.e. for any $z \in \mathbb{C}$ and $n \in \mathbb{Z}_+$ we put

$$\binom{z}{n} = \frac{z(z-1)\cdots(z-n+1)}{n!}.$$

Proposition 22. *For* $m, n \in \mathbb{Z}$, $m \leqslant n$, *one has*

$$\mathfrak{J}(m,n) = \sum_{s=0}^{[(n-m-1)/2]} (-1)^s \binom{n-s-1}{n-m-2s-1} \frac{(n-m-s-1)!}{s!}\, w^{m-n+2s+1}. \tag{24}$$

Proof. We distinguish several cases. First, consider the case $0 \leqslant m < n$. Then (24) follows from (23) and (13). Observe also that for $m = n, m, n \in \mathbb{Z}$, the RHS of (24) is an empty sum and so the both sides in (24) are equal to 0.

Second, consider the case $m \leqslant 0 \leqslant n$. Put $m = -k$, $k \in \mathbb{Z}_+$. The RHS of (24) becomes

$$\sum_{s=0}^{[(n+k-1)/2]} (-1)^s \binom{n-s-1}{n+k-2s-1} \frac{(n+k-s-1)!}{s!}\, w^{-k-n+2s+1}. \tag{25}$$

Suppose $k \leqslant n$. Then the summands in (25) vanish for $s = 0, 1, \ldots, k-1$, and so the sum equals

$$\sum_{s=0}^{[(n-k-1)/2]} (-1)^{s+k} \frac{(n-k-s-1)!}{(n-k-2s-1)!\, s!}\, \frac{(n-s-1)!}{(s+k)!}\, w^{k-n+2s+1}.$$

By the first step, this expression is equal to $(-1)^k \mathfrak{J}(k,n) = \mathfrak{J}(-k,n)$ (see Lemma 16). Further, suppose $k \geqslant n$. Then the summands in (25) vanish for $s = 0, 1, \ldots, n-1$, and so the sum equals

$$\sum_{s=0}^{[(k-n-1)/2]} (-1)^{n+s} \binom{-s-1}{k-n-2s-1} \frac{(k-s-1)!}{(n+s)!}\, w^{n-k+2s+1}.$$

Using once more the first step, this expression is readily seen to be equal to $(-1)^{k+1}\mathfrak{J}(n,k) = \mathfrak{J}(-k,n)$.

Finally, consider the case $m \leqslant n \leqslant 0$. Put $m = -k$, $n = -\ell$, $k, \ell \in \mathbb{Z}_+$. Hence $0 \leqslant \ell \leqslant k$. The RHS of (24) becomes

$$\sum_{s=0}^{[(k-\ell-1)/2]} (-1)^s \binom{-\ell-s-1}{k-\ell-2s-1} \frac{(k-\ell-s-1)!}{s!}\, w^{\ell-k+2s+1}.$$

Using again the first step, this expression is readily seen to be equal to $(-1)^{k+\ell+1}\mathfrak{J}(\ell,k) = \mathfrak{J}(-k,-\ell)$. $\square$

4. The characteristic function for the antisymmetric diagonal

For a given $d \in \mathbb{Z}_+$ let $E_\pm$ denote the $(2d+1) \times (2d+1)$ matrix with units on the upper (lower) parallel to the diagonal and with all other entries equal to zero. Hence

$$(E_+)_{j,k} = \delta_{j+1,k},\ (E_-)_{j,k} = \delta_{j,k+1},\ j,k = -d, -d+1, -d+2, \ldots, d.$$

For $y = (y_{-d}, y_{-d+1}, y_{-d+2}, \ldots, y_d) \in \mathbb{C}^{2d+1}$ let $\operatorname{diag}(y)$ denote the diagonal $(2d+1) \times (2d+1)$ matrix with the sequence y on the diagonal. Everywhere in what follows, I stands for a unit matrix.

First a formula is presented for the determinant of a Jacobi matrix with a general diagonal but with constant neighboring parallels to the diagonal. As explained in the subsequent remark, however, this formula can be extended to the general case with the aid of a simple decomposition of the Jacobi matrix in question.

Proposition 23. *For* $d \in \mathbb{N}$, $w \in \mathbb{C}$ *and* $y = (y_{-d}, y_{-d+1}, y_{-d+2}, \ldots, y_d) \in \mathbb{C}^{2d+1}$, $\prod_{k=1}^{d} y_k y_{-k} \neq 0$, *one has*

$$\begin{aligned}\det(\operatorname{diag}(y)+wE_{+}+wE_{-}) = {} & \left(\prod_{k=1}^{d} y_k y_{-k}\right)\Bigg[y_0\,\mathfrak{F}\left(\frac{w}{y_1},\dots,\frac{w}{y_d}\right)\mathfrak{F}\left(\frac{w}{y_{-1}},\dots,\frac{w}{y_{-d}}\right)\\ & -\frac{w^2}{y_1}\,\mathfrak{F}\left(\frac{w}{y_2},\dots,\frac{w}{y_d}\right)\mathfrak{F}\left(\frac{w}{y_{-1}},\dots,\frac{w}{y_{-d}}\right)\\ & -\frac{w^2}{y_{-1}}\,\mathfrak{F}\left(\frac{w}{y_1},\dots,\frac{w}{y_d}\right)\mathfrak{F}\left(\frac{w}{y_{-2}},\dots,\frac{w}{y_{-d}}\right)\Bigg].\end{aligned} \tag{26}$$

Proof. Let us proceed by induction in d. The case $d=1$ is easy to verify. Put $\mathcal{N}_d(w;y)=\det(\operatorname{diag}(y)+wE_{+}+wE_{-})$. Suppose (26) is true for some $d\geqslant 1$. For given $w\in\mathbb{C}$ and $y\in\mathbb{C}^{2d+3}$ consider the quantity $\mathcal{N}_{d+1}(w;y)$. Let us split the corresponding $(2d+3)\times(2d+3)$ Jacobi matrix into four blocks by splitting the set of indices into two disjoint sets $\{-d-1,d+1\}$ and $\{-d,-d+1,-d+2,\dots,d\}$. Applying the rule

$$\det\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\det(A)\det(D-CA^{-1}B)$$

one derives the recurrence relation

$$\begin{aligned}&\mathcal{N}_{d+1}(w;y_{-d-1},y_{-d},y_{-d+1},\dots,y_d,y_{d+1})\\ &=y_{d+1}y_{-d-1}\mathcal{N}_d(w;y'_{-d},y_{-d+1},y_{-d+2},\dots,y_{d-1},y'_d)\end{aligned}$$

where

$$y'_d=\left(1-\frac{w^2}{y_d y_{d+1}}\right)y_d,\quad y'_{-d}=\left(1-\frac{w^2}{y_{-d}y_{-d-1}}\right)y_{-d}.$$

Now it is sufficient to use the induction hypothesis jointly with the equality

$$(1-x_{n-1}x_n)\,\mathfrak{F}\left(x_1,x_2,\dots,x_{n-2},\frac{x_{n-1}}{1-x_{n-1}x_n}\right)=\mathfrak{F}(x_1,x_2,\dots,x_{n-1},x_n)$$

which is valid for $n\geqslant 2$ and which follows from relations (4) and (7), with $k=2$. □

Remark 24. Let us consider a general finite symmetric Jacobi matrix J of the form

$$J=\begin{pmatrix}\lambda_1 & w_1 & & & & \\ w_1 & \lambda_2 & w_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & w_{n-2} & \lambda_{n-1} & w_{n-1}\\ & & & & w_{n-1} & \lambda_n\end{pmatrix}$$

such that $\prod_{k=1}^{n-1}w_k\neq 0$. The Jacobi matrix can be decomposed into the product

$$J=G\tilde{J}G \tag{27}$$

where $G=\operatorname{diag}(\gamma_1,\gamma_2,\dots,\gamma_n)$ is a diagonal matrix and $\tilde{J}$ is a Jacobi matrix with all units on the neighboring parallels to the diagonal,

$$\tilde{J} = \begin{pmatrix} \tilde{\lambda}_1 & 1 & & & & \\ 1 & \tilde{\lambda}_2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & \tilde{\lambda}_{n-1} & 1 \\ & & & & 1 & \tilde{\lambda}_n \end{pmatrix}.$$

Hence $\det(J) = \left(\prod_{k=1}^{n} {\gamma_k}^2\right) \det(\tilde{J})$, and one can employ formula (26) to evaluate $\det(\tilde{J})$ (in the case of odd dimension). In more detail, one can put

$$\gamma_{2k-1} = \prod_{j=1}^{k-1} \frac{w_{2j}}{w_{2j-1}}, \quad \gamma_{2k} = w_1 \prod_{j=1}^{k-1} \frac{w_{2j+1}}{w_{2j}}, \quad k = 1, 2, 3, \ldots.$$

Alternatively, the sequence $\{\gamma_k\}_{k=1}^{n}$ is defined recursively by $\gamma_1 = 1$, $\gamma_{k+1} = w_k/\gamma_k$. Furthermore, $\tilde{\lambda}_k = \lambda_k/{\gamma_k}^2$. With this choice, (27) is clearly true.

Next we aim to derive a formula for the characteristic function of a Jacobi matrix with an antisymmetric diagonal. Suppose $\lambda = (\lambda_{-d}, \lambda_{-d+1}, \lambda_{-d+2}, \ldots, \lambda_d) \in \mathbb{C}^{2d+1}$ and $\lambda_{-k} = -\lambda_k$ for $-d \leqslant k \leqslant d$; in particular, $\lambda_0 = 0$. We consider the Jacobi matrix $K = \operatorname{diag}(\lambda) + wE_+ + wE_-$. Let us denote, temporarily, by S the diagonal matrix with alternating signs on the diagonal, $S = \operatorname{diag}(1, -1, 1, \ldots, 1)$, and by Q the permutation matrix with the entries $Q_{j,k} = \delta_{j+k,0}$ for $-d \leqslant j, k \leqslant d$. The commutation relations

$$SQKQS = -K, \quad S^2 = Q^2 = I,$$

imply

$$\det(K - zI) = \det\big(SQ(K - zI)QS\big) = -\det(K + zI).$$

Hence the characteristic function of K is an odd polynomial in the variable z. This can be also seen from the explicit formula (28) derived below.

Proposition 25. *Suppose $d \in \mathbb{N}$, $w \in \mathbb{C}$, $\lambda \in \mathbb{C}^{2d+1}$ and $\lambda_{-k} = -\lambda_k$ for $k = -d, -d+1, -d+2, \ldots, d$. Then*

$$\begin{aligned} &\frac{(-1)^{d+1}}{z} \det\big(\operatorname{diag}(\lambda) + wE_+ + wE_- - zI\big) \qquad (28) \\ &= \left(\prod_{k=1}^{d} \left(\lambda_k^2 - z^2\right)\right) \mathfrak{F}\left(\frac{w}{\lambda_1 - z}, \ldots, \frac{w}{\lambda_d - z}\right) \mathfrak{F}\left(\frac{w}{\lambda_1 + z}, \ldots, \frac{w}{\lambda_d + z}\right) \\ &\quad + 2\sum_{j=1}^{d} w^{2j} \left(\prod_{k=j+1}^{d} \left(\lambda_k^2 - z^2\right)\right) \mathfrak{F}\left(\frac{w}{\lambda_{j+1} - z}, \ldots, \frac{w}{\lambda_d - z}\right) \mathfrak{F}\left(\frac{w}{\lambda_{j+1} + z}, \ldots, \frac{w}{\lambda_d + z}\right). \end{aligned}$$

Proof. This is a particular case of (26) where one has to set $y_k = \lambda_k - z$ for $k > 0$, $y_0 = -z$, $y_k = -(\lambda_{-k} + z)$ for $k < 0$. To complete the proof it suffices to verify that

$$\frac{w^2}{z(\lambda_1 - z)}\,\mathfrak{F}\left(\frac{w}{\lambda_2 - z}, \ldots, \frac{w}{\lambda_d - z}\right)\mathfrak{F}\left(\frac{w}{\lambda_1 + z}, \frac{w}{\lambda_2 + z}, \ldots, \frac{w}{\lambda_d + z}\right)$$
$$-\frac{w^2}{z(\lambda_1 + z)}\,\mathfrak{F}\left(\frac{w}{\lambda_1 - z}, \frac{w}{\lambda_2 - z}, \ldots, \frac{w}{\lambda_d - z}\right)\mathfrak{F}\left(\frac{w}{\lambda_2 + z}, \ldots, \frac{w}{\lambda_d + z}\right)$$
$$= 2\sum_{j=1}^{d} w^{2j}\left(\prod_{k=1}^{j}\frac{1}{\lambda_k^2 - z^2}\right)\mathfrak{F}\left(\frac{w}{\lambda_{j+1} - z}, \ldots, \frac{w}{\lambda_d - z}\right)\mathfrak{F}\left(\frac{w}{\lambda_{j+1} + z}, \ldots, \frac{w}{\lambda_d + z}\right).$$

To this end, one can apply (8), with $n = d$, $u_k = w/(\lambda_k - z)$, $v_k = w/(\lambda_k + z)$. Note that $u_j - v_j = 2zu_jv_j/w$. □

Zero always belongs to spectrum of the Jacobi matrix K for the characteristic function is odd. Moreover, as is well known and as it simply follows from the analysis of the eigenvalue equation, if $w \neq 0$ then to every eigenvalue of K there belongs exactly one linearly independent eigenvector.

Proposition 26. *Suppose $w \in \mathbb{C}$, $\lambda \in \mathbb{C}^{2d+1}$, $\lambda_{-k} = -\lambda_k$ for $-d \leqslant k \leqslant d$, and $\prod_{k=1}^{d}\lambda_k \neq 0$. Then the vector $v \in \mathbb{C}^{2d+1}$, $v^T = (\theta_{-d}, \theta_{-d+1}, \theta_{-d+2}, \ldots, \theta_d)$, with the entries*

$$\theta_k = (-1)^k w^k\left(\prod_{j=k+1}^{d}\lambda_j\right)\mathfrak{F}\left(\frac{w}{\lambda_{k+1}}, \frac{w}{\lambda_{k+2}}, \ldots, \frac{w}{\lambda_d}\right) \quad \text{for } k = 0, 1, 2, \ldots, d, \tag{29}$$

$\theta_{-k} = (-1)^k\,\theta_k$ for $-d \leqslant k \leqslant d$, belongs to the kernel of the Jacobi matrix $\operatorname{diag}(\lambda) + wE_+ + wE_-$. *In particular, $\theta_0 = \lambda_1\lambda_2\ldots\lambda_d\,\mathfrak{F}(w/\lambda_1, w/\lambda_2, \ldots, w/\lambda_d)$, $\theta_d = (-1)^d w^d$, and so $v \neq 0$.*

Remark. Clearly, formulas (29) can be extended to the case $\prod_{k=1}^{d}\lambda_k = 0$ as well provided one makes the obvious cancellations.

Proof. One has to show that

$$w\theta_{k-1} + \lambda_k\theta_k + w\theta_{k+1} = 0, \quad k = -d+1, -d+2, \ldots, d-1,$$

and $\lambda_{-d}\theta_{-d} + w\theta_{-d+1} = 0$, $w\theta_{d-1} + \lambda_d\theta_d = 0$. Owing to the symmetries $\lambda_{-k} = -\lambda_k$, $\theta_{-k} = (-1)^k\theta_k$, it suffices to verify the equalities only for indices $0 \leqslant k \leqslant d$. This can be readily carried out using the explicit formulas (29) and the rule (3). □

5. Jacobi matrices with a linear diagonal

Finally we focus on finite-dimensional Jacobi matrices of odd dimension whose diagonal depends linearly on the index and whose parallels to the diagonal are constant. Without loss of generality one can assume that the diagonal equals
$(-d, -d+1, -d+2, \ldots, d)$, $d \in \mathbb{Z}_+$. For $w \in \mathbb{C}$ put

$$K_0 = \operatorname{diag}(-d, -d+1, -d+2, \ldots, d), \quad K(w) = K_0 + wE_+ + wE_-.$$

Concerning the characteristic function $\chi(z) = \det(K(w) - z)$, we know that this is an odd function. Put

$$\chi_{\text{red}}(z) = \frac{(-1)^{d+1}}{z}\,\det(K(w) - z).$$

Hence $\chi_{\text{red}}(z)$ is an even polynomial of degree $2d$. Further, denote by $\{e_{-d}, e_{-d+1}, e_{-d+2}, \ldots, e_d\}$ the standard basis in $\mathbb{C}^{2d+1}$.

Suppose $w \neq 0$. Let us consider a family of column vectors $x_{s,n} \in \mathbb{C}^{2d+1}$ depending on the parameters $s, n \in \mathbb{Z}$ and defined by

$$x_{s,n}^T = (\mathfrak{J}(s+d,n), \mathfrak{J}(s+d-1,n), \mathfrak{J}(s+d-2,n), \ldots, \mathfrak{J}(s-d,n)).$$

From the fact that the matrix $\mathfrak{J}$ obeys (18), (19) one derives that

$$\forall s, n \in \mathbb{Z},\ K(w)x_{s,n} = s\,x_{s,n} - w\,\mathfrak{J}(s+d+1,n)e_{-d} - w\,\mathfrak{J}(s-d-1,n)e_d.$$

Put

$$v_s = x_{s,s+d+1},\ s \in \mathbb{Z}.$$

Recalling that $\mathfrak{J}(m,m) = \mathfrak{J}(-m,m) = 0$ one has

$$K(w)v_s = s\,v_s - w\,\mathfrak{J}(s-d-1,s+d+1)e_d. \tag{30}$$

Remark 27. Putting $s = 0$ one gets $K(w)v_0 = 0$, and so v_0 spans the kernel of $K(w)$.

Lemma 28. *For every $\ell = -d, -d+1, -d+2, \ldots, d$, one has*

$$w^{d+\ell} \sum_{s=-d}^{\ell} \frac{(-1)^{\ell+s}}{(d+s)!\,(\ell-s)!}\, v_s \in e_\ell + \operatorname{span}\{e_{\ell+1}, e_{\ell+2}, \ldots, e_d\}.$$

In particular,

$$e_d = w^{2d} \sum_{s=-d}^{d} \frac{(-1)^{d+s}}{(d+s)!\,(d-s)!}\, v_s. \tag{31}$$

Consequently, $\mathcal{V} = \{v_{-d}, v_{-d+1}, v_{-d+2}, \ldots, v_d\}$ is a basis in $\mathbb{C}^{2d+1}$.

Proof. One has to show that

$$w^{d+\ell} \sum_{s=-d}^{\ell} \frac{(-1)^{\ell+s}}{(d+s)!\,(\ell-s)!}\, \mathfrak{J}(s-k, s+d+1) = \delta_{\ell,k} \text{ for } -d \leqslant k \leqslant \ell.$$

Note that for any $a \in \mathbb{C}$ and $n \in \mathbb{Z}_+$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{a+k}{r} = 0, \quad r = 0, 1, 2, \ldots, n-1, \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{a+k}{n} = (-1)^n.$$

Using these equalities and (24) one can readily show, more generally, that

$$\sum_{s=-d}^{\ell} \frac{(-1)^{\ell+s}}{(d+s)!\,(\ell-s)!}\, \mathfrak{J}(m+s, n+s) = 0 \text{ for } m, n \in \mathbb{Z}, m \leqslant n \leqslant m+d+\ell,$$

and

$$\sum_{s=-d}^{\ell} \frac{(-1)^{\ell+s}}{(d+s)!\,(\ell-s)!}\, \mathfrak{J}(m+s, m+d+\ell+s+1) = w^{-d-\ell}.$$

This proves the lemma. □

Denote by $\tilde{K}(w)$ the matrix of $K(w)$ in the basis $\mathcal{V}$ introduced in Lemma 28. Let $a, b \in \mathbb{C}^{2d+1}$ be the column vectors defined by $a^T = (\alpha_{-d}, \alpha_{-d+1}, \alpha_{-d+2}, \ldots \alpha_d)$, $b^T = (\beta_{-d}, \beta_{-d+1}, \beta_{-d+2}, \ldots \beta_d)$,

$$\alpha_s = \mathfrak{J}(s-d-1, s+d+1), \quad \beta_s = \frac{(-1)^{d+s}\, w^{2d+1}}{(d+s)!\,(d-s)!}, \quad s = -d, -d+1, -d+2, \ldots, d. \tag{32}$$

Note that

$$\alpha_{-s} = -\alpha_s, \quad \beta_{-s} = \beta_s. \tag{33}$$

The former equality follows from (22) and (19). From (30) and (31) one deduces that

$$\tilde{K}(w) = K_0 - ba^T. \tag{34}$$

Note, however, that the components of the vectors a and b depend on w, too, though not indicated in the notation.

According to (34), $\tilde{K}(w)$ differs from the diagonal matrix K_0 by a rank-one correction. This form is suitable for various computations. Particularly, one can express the resolvent of $\tilde{K}(w)$ explicitly,

$$(\tilde{K}(w) - z)^{-1} = (K_0 - z)^{-1} + \frac{1}{1 - a^T(K_0 - z)^{-1}b}\,(K_0 - z)^{-1}ba^T(K_0 - z)^{-1}.$$

The equality holds for any $z \in \mathbb{C}$ such that $z \notin \operatorname{spec}\{K_0\} = \{-d, -d+1, -d+2, \ldots, d\}$ and $1 - a^T(K_0 - z)^{-1}b \neq 0$. Clearly, this set of excluded values of z is finite.

Let us proceed to derivation of a formula for the characteristic function of $K(w)$. Proposition 25 is applicable to $K(w)$ and so

$$\begin{aligned}\chi_{\mathrm{red}}(z) = {} & \left(\prod_{k=1}^{d}\left(k^2 - z^2\right)\right)\mathfrak{F}\left(\frac{w}{1-z}, \ldots, \frac{w}{d-z}\right)\mathfrak{F}\left(\frac{w}{1+z}, \ldots, \frac{w}{d+z}\right) \\ & + 2\sum_{j=1}^{d} w^{2j}\left(\prod_{k=j+1}^{d}\left(k^2 - z^2\right)\right)\mathfrak{F}\left(\frac{w}{j+1-z}, \ldots, \frac{w}{d-z}\right)\mathfrak{F}\left(\frac{w}{j+1+z}, \ldots, \frac{w}{d+z}\right).\end{aligned} \tag{35}$$

Below we derive a more convenient formula for $\chi_{\mathrm{red}}(z)$.

Lemma 29. *One has*

$$\chi_{\mathrm{red}}(0) = \sum_{s=0}^{d} \frac{((d-s)!)^2\,(2d-s+1)!}{s!\,(2d-2s+1)!}\, w^{2s} \tag{36}$$

and

$$\chi_{\mathrm{red}}(n) = \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2k+1)!\binom{n+k}{2k+1}\binom{d+k+1}{2k+1}w^{2d-2k} \tag{37}$$

for $n = 1, 2, \ldots, d$.

Proof. Let us first verify the formula for $\chi_{\mathrm{red}}(0)$. From (35) it follows that

$$\chi_{\mathrm{red}}(0) = (d!)^2\,\mathfrak{F}\left(w, \frac{w}{2}, \ldots, \frac{w}{d}\right)^2 + 2\sum_{j=1}^{d} w^{2j}\left(\frac{d!}{j!}\right)^2\mathfrak{F}\left(\frac{w}{j+1}, \frac{w}{j+2}, \ldots, \frac{w}{d}\right)^2.$$

By Proposition 13,

$$\chi_{\rm red}(0) = \pi^2 w^{2d+2}\, Y_{d+1}(2w)^2 \left(J_0(2w)^2 + 2\sum_{j=1}^{d} J_j(2w)^2 \right) + O\left(w^{2d+2}\log(w)\right).$$

Further we need some basic facts concerning Bessel functions; see, for instance, [1, Chapter 9]. Recall that

$$J_0(z)^2 + 2\sum_{j=1}^{\infty} J_j(z)^2 = 1.$$

Hence

$$\begin{aligned}\chi_{\rm red}(0) &= \pi^2 w^{2d+2}\, Y_{d+1}(2w)^2 + O\left(w^{2d+2}\log(w)\right)\\ &= \left(\sum_{k=0}^{d} \frac{(d-k)!}{k!}\, w^{2k}\right)^2 + O\left(w^{2d+2}\log(w)\right).\end{aligned}$$

Note that $\chi_{\rm red}(0)$ is a polynomial in the variable w of degree $2d$, and so

$$\chi_{\rm red}(0) = \sum_{s=0}^{d}\sum_{k=0}^{s} \frac{(d-k)!\,(d-s+k)!}{k!\,(s-k)!}\, w^{2s}.$$

Using the identity

$$\begin{aligned}\sum_{k=0}^{s} \frac{(d-k)!\,(d-s+k)!}{k!\,(s-k)!} &= ((d-s)!)^2 \sum_{k=0}^{s} \binom{d-k}{d-s}\binom{d-s+k}{d-s}\\ &= \frac{((d-s)!)^2\,(2d-s+1)!}{s!\,(2d-2s+1)!}\end{aligned}$$

one arrives at (36).

To show (37) one can make use of (34). One has

$$\chi_{\rm red}(z) = \frac{(-1)^{d+1}}{z}\,\det(\tilde{K}(w)-z) = \frac{(-1)^{d+1}}{z}\,\det(K_0-z)\det\left(I-(K_0-z)^{-1}ba^T\right).$$

Note that $\det(I+ba^T) = 1 + a^T b$. Hence, in view of (33),

$$\chi_{\rm red}(z) = \prod_{k=1}^{d}\left(k^2-z^2\right)\left(1-\sum_{s=-d}^{d}\frac{\beta_s\alpha_s}{s-z}\right) = \prod_{k=1}^{d}\left(k^2-z^2\right)\left(1-2\sum_{s=1}^{d}\frac{s\beta_s\alpha_s}{s^2-z^2}\right).$$

Using (32) one gets

$$\chi_{\rm red}(n) = -2n\beta_n\alpha_n \prod_{\substack{k=1\\k\neq n}}^{d}\left(k^2-n^2\right) = \frac{(-1)^d}{n}\, w^{2d+1}\,\mathfrak{J}(n-d-1, n+d+1).$$

Formula (37) then follows from (24). □

Proposition 30. *For every $d \in \mathbb{Z}_+$ one has*

$$\chi_{\text{red}}(z) = \sum_{s=0}^{d} \binom{2d-s+1}{s} w^{2s} \prod_{k=1}^{d-s} \left(k^2 - z^2\right). \tag{38}$$

Proof. Since $\chi_{\text{red}}(z)$ is an even polynomial in z of degree $2d$ it is enough to check that the RHS of (38) coincides, for $z = 0, 1, 2, \ldots, d$, with $\chi_{\text{red}}(0), \chi_{\text{red}}(1), \chi_{\text{red}}(2), \ldots, \chi_{\text{red}}(d)$. With the knowledge of values (36) and (37), this is a matter of straightforward computation. □

Remark 31. Using (38) it is not difficult to check that formula (37) is valid for any $n \in \mathbb{N}$, including $n > d$ (the summation index k runs from 1 to $\min\{n-1, d\}$).

Remark 32. If $w \in \mathbb{R}$, $w \neq 0$, then the spectrum of the Jacobi matrix $K(w)$ is real and simple, and formula (38) implies that the interval $[-1, 1]$ contains no other eigenvalue except of 0.

Eigenvectors of $K(w)$ can be expressed in terms of the function $\mathfrak{F}$, too. Suppose $w \neq 0$. Let us introduce the vector-valued function $x(z) \in \mathbb{C}^{2d+1}$ depending on $z \in \mathbb{C}$, $x(z)^T = (\xi_{-d}(z), \xi_{-d+1}(z), \xi_{-d+2}(z), \ldots, \xi_d(z))$,

$$\xi_k(z) = w^{-d-k} \frac{\Gamma(z+d+1)}{\Gamma(z-k+1)} \mathfrak{F}\left(\frac{w}{z-k+1}, \frac{w}{z-k+2}, \ldots, \frac{w}{z+d}\right), \quad -d \leqslant k \leqslant d.$$

With the aid of (3) one derives the equality

$$(K(w)-z)x(z) = -w^{-2d} \frac{\Gamma(z+d+1)}{\Gamma(z-d)} \mathfrak{F}\left(\frac{w}{z-d}, \frac{w}{z-d+1}, \ldots, \frac{w}{z+d}\right) e_d. \tag{39}$$

Remark 33. According to (12),

$$\xi_k(z) = w^{-d-k} \sum_{s=0}^{[(d+k)/2]} (-1)^s \frac{(d+k-s)!}{s!\,(d+k-2s)!} w^{2s} \prod_{j=s}^{d+k-s-1} (z+d-j).$$

Hence $\xi_k(z)$ is a polynomial in z of degree $d+k$. In particular, $\xi_{-d}(z) = 1$, and so $x(z) \neq 0$.

Proposition 34. *One has*

$$\chi(z) = -z \left(\prod_{k=1}^{d} \left(z^2 - k^2\right)\right) \mathfrak{F}\left(\frac{w}{z-d}, \frac{w}{z-d+1}, \frac{w}{z-d+2}, \ldots, \frac{w}{z+d}\right). \tag{40}$$

If $w \in \mathbb{C}$, $w \neq 0$, then for every eigenvalue $\lambda \in \operatorname{spec}(K(w))$, $x(\lambda)$ is an eigenvector corresponding to λ.

Proof. Denote by $P(z)$ the RHS of (40). By (39), if $P(\lambda) = 0$ then $x(\lambda)$ is an eigenvector of $K(w)$. Thus it suffices to verify (40). The both sides depend on w polynomially and so it is enough to prove the equality for $w \in \mathbb{R} \setminus \{0\}$. Note that $P(z)$ is a polynomial in z of degree $2d+1$, and the coefficient standing at z^{2d+1} equals -1. The set of roots of $P(z)$ is contained in $\operatorname{spec}(K(w))$. One can show that $P(z)$ has no multiple roots. In fact, suppose $P(\lambda) = P'(\lambda) = 0$ for some $\lambda \in \mathbb{R}$. From (39) one deduces that $(K(w)-\lambda)x(\lambda) = 0$, $(K(w)-\lambda)x'(\lambda) = x(\lambda)$ (here $x'(z)$ is the derivative of $x(z)$). Hence

$$(K(w)-\lambda)^2 x(\lambda) = (K(w)-\lambda)^2 x'(\lambda) = 0.$$

Note that $x'(\lambda) \neq 0$, and $x'(\lambda)$ differs from a multiple of $x(\lambda)$ for $\xi_{-d}(z) = 1$. This contradicts the fact, however, that the spectrum of $K(w)$ is simple. One concludes that the set of roots of $P(z)$ coincides with $\operatorname{spec}(K(w))$. Necessarily, $P(z)$ is equal to the characteristic function of $K(w)$. □

Remark 35. With the aid of (40) one can rederive equality (37). For $1 \leqslant n \leqslant d$, a straightforward computation gives

$$\chi(n) = (-1)^{d+n} w^{2d+1}\,\mathfrak{J}(d-n+1, d+n+1).$$

Equality (37) then follows from (24).

Acknowledgments

The authors wish to acknowledge gratefully partial support from Grant No. 201/09/0811 of the Czech Science Foundation (P.Š.) and from Grant No. LC06002 of the Ministry of Education of the Czech Republic (F.Š.).

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Linear Algebra and its Applications 438 (2013) 4130–4155

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

The characteristic function for Jacobi matrices with applications

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ARTICLE INFO

Article history:
Received 27 December 2011
Accepted 14 January 2013
Available online 13 March 2013

Submitted by R.A. Brualdi

AMS classification:
47B36
15A18
33C99

Keywords:
Infinite Jacobi matrix
Spectral problem
Characteristic function

ABSTRACT

We introduce a class of Jacobi operators with discrete spectra which is characterized by a simple convergence condition. With any operator J from this class we associate a characteristic function as an analytic function on a suitable domain, and show that its zero set actually coincides with the set of eigenvalues of J in that domain. As an application we construct several examples of Jacobi matrices for which the characteristic function can be expressed in terms of special functions. In more detail we study the example where the diagonal sequence of J is linear while the neighboring parallels to the diagonal are constant.

1. Introduction

In this paper we introduce and study a class of infinite symmetric but in general complex Jacobi matrices $\mathcal{J}$ characterized by a simple convergence condition. This class is also distinguished by the discrete character of spectra of the corresponding Jacobi operators. Doing so we extend and generalize an approach to Jacobi matrices which was originally initiated, under much more restricted circumstances, in [14]. We refer to [11] for a rather general analysis of how the character of spectrum of a Jacobi operator may depend on the asymptotic behavior of weights.

For a given Jacobi matrix $\mathcal{J}$, one constructs a characteristic function $F_{\mathcal{J}}(z)$ as an analytic function on the domain $\mathbb{C}_0^{\lambda}$ obtained by excluding from the complex plane the closure of the range of the diagonal sequence λ of $\mathcal{J}$. Under some comparatively simple additional assumptions, like requiring the real

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http://dx.doi.org/10.1016/j.laa.2013.01.021

part of λ to be semibounded or $\mathcal{J}$ to be real, one can show that $\mathcal{J}$ determines a unique closed Jacobi operator J on $\ell^2(\mathbb{N})$. Moreover, the spectrum of J in the domain $\mathbb{C}_0^\lambda$ is discrete and coincides with the zero set of $F_{\mathcal{J}}(z)$. When establishing this relationship one may also treat the poles of $F_{\mathcal{J}}(z)$ which occur at the points from the range of the sequence λ not belonging to the set of accumulation points, however. In addition, as an important step of the proof, one makes use of an explicit formula for the Green function associated with J.

The characteristic function as well as numerous formulas throughout the paper are expressed in terms of a function, called $\mathfrak{F}$, defined on a subset of the space of complex sequences. In the introductory part we recall from [14] the definition of $\mathfrak{F}$ and its basic properties which are then completed by various additional facts. On the other hand, we conclude the paper with some applications of the derived results. We present several examples of Jacobi matrices for which the characteristic function can be expressed in terms of special functions (the Bessel functions or the basic hypergeometric series). A particular attention is paid to the example where the diagonal sequence λ is linear while the neighboring parallels to the diagonal are constant. In this case the characteristic equation in the variable z reads $J_{-z}(2w) = 0$, with w being a parameter, and our main concern is how the spectrum of the Jacobi operator depends on w.

2. The function $\mathfrak{F}$

2.1. Definition and basic properties

Let us recall from [14] some basic definitions and properties concerning a function $\mathfrak{F}$ defined on a subset of the linear space formed by all complex sequences $x = \{x_k\}_{k=1}^\infty$. Moreover, we complete this brief overview by a few additional facts.

Definition 1. Define $\mathfrak{F} : D \to \mathbb{C}$,

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \cdots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \cdots x_{k_m} x_{k_m+1}, \tag{1}$$

where

$$D = \left\{ \{x_k\}_{k=1}^\infty \subset \mathbb{C};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}.$$

For a finite number of complex variables we identify $\mathfrak{F}(x_1, x_2, \ldots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$. By convention, we also put $\mathfrak{F}(\emptyset) = 1$ where $\emptyset$ is the empty sequence.

Let us remark that the value of $\mathfrak{F}$ on a finite complex sequence can be expressed as the determinant of a finite Jacobi matrix. Using some basic linear algebra it is easy to show that, for $n \in \mathbb{N}$ and $\{x_j\}_{j=1}^n \subset \mathbb{C}$, one has

$$\mathfrak{F}(x_1, x_2, \ldots, x_n) = \det X_n, \tag{2}$$

where

$$X_n = \begin{pmatrix} 1 & x_1 & & & & \\ x_2 & 1 & x_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & x_{n-1} & 1 & x_{n-1} \\ & & & & x_n & 1 \end{pmatrix}.$$

Note that the domain D is not a linear space. One has, however, $\ell^2(\mathbb{N}) \subset D$. In fact, the absolute value of the mth summand on the RHS of (1) is majorized by the expression

$$\sum_{\substack{k\in\mathbb{N}^m \\ k_1<k_2<\cdots<k_m}} |x_{k_1}x_{k_1+1}x_{k_2}x_{k_2+1}\cdots x_{k_m}x_{k_m+1}| \leqslant \frac{1}{m!}\left(\sum_{j=1}^{\infty}|x_jx_{j+1}|\right)^m .$$

Hence for $x \in D$ one has the estimate

$$|\mathfrak{F}(x)| \leqslant \exp\left(\sum_{k=1}^{\infty}|x_kx_{k+1}|\right). \tag{3}$$

Furthermore, $\mathfrak{F}$ satisfies the relation

$$\mathfrak{F}(x) = \mathfrak{F}(x_1,\ldots,x_k)\,\mathfrak{F}(T^kx) - \mathfrak{F}(x_1,\ldots,x_{k-1})x_kx_{k+1}\mathfrak{F}(T^{k+1}x), \quad k=1,2,\ldots, \tag{4}$$

where $x \in D$ and T denotes the truncation operator from the left defined on the space of all sequences, $T(\{x_n\}_{n=1}^{\infty}) = \{x_{n+1}\}_{n=1}^{\infty}$. In particular, for $k=1$ one gets the rule

$$\mathfrak{F}(x) = \mathfrak{F}(Tx) - x_1x_2\mathfrak{F}(T^2x). \tag{5}$$

In addition, one has the symmetry property

$$\mathfrak{F}(x_1,x_2,\ldots,x_{k-1},x_k) = \mathfrak{F}(x_k,x_{k-1},\ldots,x_2,x_1).$$

If combined with (4), one gets

$$\mathfrak{F}(x_1,x_2,\ldots,x_{k+1}) = \mathfrak{F}(x_1,x_2,\ldots,x_k) - x_kx_{k+1}\,\mathfrak{F}(x_1,x_2,\ldots,x_{k-1}). \tag{6}$$

Lemma 2. *For $x \in D$ one has*

$$\lim_{n\to\infty}\mathfrak{F}(T^nx) = 1 \tag{7}$$

and

$$\lim_{n\to\infty}\mathfrak{F}(x_1,x_2,\ldots,x_n) = \mathfrak{F}(x). \tag{8}$$

Proof. First, similarly as in (3), one gets the estimate

$$|\mathfrak{F}(T^nx) - 1| \leqslant \exp\left(\sum_{k=n+1}^{\infty}|x_kx_{k+1}|\right) - 1.$$

This shows (7).

Second, in view of (4), the difference $|\mathfrak{F}(x) - \mathfrak{F}(x_1,x_2,\ldots,x_n)|$ can be majorized by the expression

$$|1-\mathfrak{F}(T^nx)|\exp\left(\sum_{k=1}^{\infty}|x_kx_{k+1}|\right) + |x_nx_{n+1}|\exp\left(2\sum_{k=1}^{\infty}|x_kx_{k+1}|\right).$$

From here one derives the (rather rough) estimate

$$|\mathfrak{F}(x) - \mathfrak{F}(x_1,x_2,\ldots,x_n)| \leqslant 2\exp\left(2\sum_{k=1}^{\infty}|x_kx_{k+1}|\right)\sum_{k=n}^{\infty}|x_kx_{k+1}|. \tag{9}$$

This shows (8). □

Proposition 3. *The function $\mathfrak{F}$ is continuous on $\ell^2(\mathbb{N})$.*

Proof. If $x \in \ell^2(\mathbb{N}) \subset D$ then from (9) one derives that, for any $n \in \mathbb{N}$,

$$|\mathfrak{F}(x) - \mathfrak{F}(x_1, x_2, \ldots, x_n)| \leqslant 2 \,\exp\!\left(2\,\|x\|^2\right)\, \|(I - P_{n-1})x\|^2$$

where P_m stands for the orthogonal projection on $\ell^2(\mathbb{N})$ onto the subspace spanned by the first m vectors of the canonical basis. From this estimate and from the fact that $\mathfrak{F}(x_1, x_2, \ldots, x_n)$ is a polynomial function the proposition readily follows. □

2.2. Jacobi matrices

Let us denote by $\mathcal{J}$ an infinite Jacobi matrix of the form

$$\mathcal{J} = \begin{pmatrix} \lambda_1 & w_1 & & \\ v_1 & \lambda_2 & w_2 & \\ & v_2 & \lambda_3 & w_3 \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

where $w = \{w_n\}_{n=1}^{\infty}$, $v = \{v_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$ and $\lambda = \{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$. Provided any of the sequences is unbounded it is reasonable to distinguish in the notation between $\mathcal{J}$ and an operator represented by this matrix. Such an operator J need not be unique, as discussed in Section 3.2. Further, by J_n we denote the nth truncation of $\mathcal{J}$, i.e.

$$J_n = \begin{pmatrix} \lambda_1 & w_1 & & & \\ v_1 & \lambda_2 & w_2 & & \\ & \ddots & \ddots & \ddots & \\ & & v_{n-2} & \lambda_{n-1} & w_{n-1} \\ & & & v_{n-1} & \lambda_n \end{pmatrix}. \tag{10}$$

As is well known and in fact quite obvious, any solution $\{x_k\}$ of the formal eigenvalue equation

$$\lambda_1 x_1 + w_1 x_2 = z x_1, \quad v_{k-1} x_{k-1} + \lambda_k x_k + w_k x_{k+1} = z x_k \ \text{ for } k \geqslant 2, \tag{11}$$

with $z \in \mathbb{C}$, is unambiguously determined by its first component x_1. Consequently, any operator J whose matrix equals $\mathcal{J}$ may have only simple eigenvalues.

We wish to show that the characteristic function of the finite Jacobi matrix J_n can be expressed in terms of $\mathfrak{F}$. To this end, let us introduce the sequences $\{\gamma_k^{\pm}\}_{k=1}^{n}$ defined recursively by

$$\gamma_1^{\pm} = 1, \quad \gamma_{k+1}^{+} = w_k/\gamma_k^{-} \ \text{ and } \ \gamma_{k+1}^{-} = v_k/\gamma_k^{+}, \quad k \geqslant 1. \tag{12}$$

More explicitly, the sequence $\{\gamma_k^{-}\}_{k=1}^{n}$ can be expressed as

$$\gamma_{2k-1}^{-} = \prod_{j=1}^{k-1} \frac{v_{2j}}{w_{2j-1}}, \quad \gamma_{2k}^{-} = v_1 \prod_{j=1}^{k-1} \frac{v_{2j+1}}{w_{2j}}, \quad k = 1, 2, 3, \ldots.$$

As for the sequence $\{\gamma_k^{+}\}_{k=1}^{n}$, the corresponding expressions are of the same form but with w being replaced by v and vice versa. Note that if $v_k = w_k$ for all $k = 1, 2, \ldots, n-1$, then $\gamma_k^{-} = \gamma_k^{+}$ for all $k = 1, 2, \ldots, n$.

Proposition 4. *Let $\{\gamma_k^{\pm}\}_{k=1}^{n}$ be the sequences defined in (12). Then the equality*

$$\det(J_n - zI_n) = \left(\prod_{k=1}^{n}(\lambda_k - z)\right)\mathfrak{F}\!\left(\frac{\gamma_1^-\gamma_1^+}{\lambda_1 - z}, \frac{\gamma_2^-\gamma_2^+}{\lambda_2 - z}, \ldots, \frac{\gamma_n^-\gamma_n^+}{\lambda_n - z}\right) \tag{13}$$

holds for all $z \in \mathbb{C}$ (after obvious cancellations, the RHS is well defined even for $z = \lambda_k$).

Proof. Put $\tilde{\lambda}_k = \lambda_k/\gamma_k^-\gamma_k^+$. As remarked in [14, Remark 24], the Jacobi matrix J_n can be decomposed into the product $J_n = G_n^-\tilde{J}_n G_n^+$ where $G_n^{\pm} = \operatorname{diag}(\gamma_1^{\pm}, \gamma_2^{\pm}, \ldots, \gamma_n^{\pm})$ are diagonal matrices, and $\tilde{J}_n$ is a Jacobi matrix whose diagonal equals the sequence $(\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n)$ and which has all units on the neighboring parallels to the diagonal. The proposition now readily follows from this decomposition combined with (2). □

Moreover, with the aid of (13) and using some basic calculus from linear algebra one can derive the following formula for the resolvent.

Proposition 5. *The matrix entries of the resolvent $R_n(z) = (J_n - zI_n)^{-1}$, with $z \in \mathbb{C} \setminus \operatorname{spec}(J_n)$, may be expressed as ($1 \leqslant i, j \leqslant n$)*

$$R_n(z)_{i,j} = -\,\Omega(i,j)\left(\prod_{l=\min(i,j)}^{\max(i,j)}(z - \lambda_l)\right)^{-1} \times \mathfrak{F}\!\left(\left\{\frac{\gamma_l^-\gamma_l^+}{\lambda_l - z}\right\}_{l=1}^{\min(i,j)-1}\right)\mathfrak{F}\!\left(\left\{\frac{\gamma_l^-\gamma_l^+}{\lambda_l - z}\right\}_{l=\max(i,j)+1}^{n}\right)\mathfrak{F}\!\left(\left\{\frac{\gamma_l^-\gamma_l^+}{\lambda_l - z}\right\}_{l=1}^{n}\right)^{-1} \tag{14}$$

where

$$\Omega(i,j) = \begin{cases} \prod_{l=i}^{j-1} w_l, & \text{if } i < j,\\ 1, & \text{if } i = j,\\ \prod_{l=j}^{i-1} v_l, & \text{if } i > j. \end{cases}$$

In the remainder of the paper we concentrate, however, on symmetric Jacobi matrices with $v = w$, i.e. we put

$$\mathcal{J} = \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where $\lambda = \{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$ and $w = \{w_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$. In that case some definitions introduced above simplify. First of all, one has $\gamma_k^- = \gamma_k^+ = \gamma_k$ where

$$\gamma_{2k-1} = \prod_{j=1}^{k-1}\frac{w_{2j}}{w_{2j-1}}, \quad \gamma_{2k} = w_1\prod_{j=1}^{k-1}\frac{w_{2j+1}}{w_{2j}}, \quad k = 1, 2, 3, \ldots.$$

Then $\gamma_k\gamma_{k+1} = w_k$.

2.3. More on the function $\mathfrak{F}$

In [14] one can find two examples of special functions expressed in terms of $\mathfrak{F}$. The first example is concerned with the Bessel functions of the first kind. In more detail, for $w, \nu \in \mathbb{C}$, $\nu \notin -\mathbb{N}$, one has

$$J_\nu(2w) = \frac{w^\nu}{\Gamma(\nu+1)}\,\mathfrak{F}\left(\left\{\frac{w}{\nu+k}\right\}_{k=1}^{\infty}\right). \tag{15}$$

Notice that jointly with (3) this implies

$$|J_\nu(2w)| \leqslant \left|\frac{w^\nu}{\Gamma(\nu+1)}\right| \exp\left(\sum_{k=1}^{\infty}\left|\frac{w^2}{(\nu+k)(\nu+k+1)}\right|\right). \tag{16}$$

In the second example one shows that the formula

$$\mathfrak{F}\left(\left\{t^{k-1}w\right\}_{k=1}^{\infty}\right) = 1 + \sum_{m=1}^{\infty}(-1)^m \frac{t^{m(2m-1)}w^{2m}}{(1-t^2)(1-t^4)\cdots(1-t^{2m})} = {}_0\phi_1(;0;t^2,-tw^2) \tag{17}$$

holds for $t, w \in \mathbb{C}$, $|t| < 1$. Here ${}_0\phi_1$ is the basic hypergeometric series (also called q-hypergeometric series) being defined by

$${}_0\phi_1(;b;q,z) = \sum_{k=0}^{\infty}\frac{q^{k(k-1)}}{(q;q)_k(b;q)_k}\,z^k,$$

and

$$(a;q)_k = \prod_{j=0}^{k-1}\left(1-aq^j\right), \quad k = 0, 1, 2, \ldots,$$

is the q-Pochhammer symbol; see [8] for more details.

In this connection let us recall one more identity proved in [14, Lemma 9], namely

$$\begin{aligned} &u_1\,\mathfrak{F}(u_2,u_3,\ldots,u_n)\,\mathfrak{F}(v_1,v_2,\ldots,v_n) - v_1\,\mathfrak{F}(u_1,u_2,\ldots,u_n)\,\mathfrak{F}(v_2,v_3,\ldots,v_n)\\ &= \sum_{j=1}^{n}\left(\prod_{k=1}^{j-1}u_kv_k\right)(u_j-v_j)\,\mathfrak{F}(u_{j+1},u_{j+2},\ldots,u_n)\,\mathfrak{F}(v_{j+1},v_{j+2},\ldots,v_n)\,. \end{aligned}$$

For the particular choice

$$u_k = \frac{w}{\mu+k}, \quad v_k = \frac{w}{\nu+k}, \quad 1 \leqslant k \leqslant n,$$

one can consider the limit $n \to \infty$. Using (15) and (16) one arrives at the equation

$$J_\mu(2w)J_{\nu+1}(2w) - J_{\mu+1}(2w)J_\nu(2w) = \frac{\mu-\nu}{w}\sum_{j=1}^{\infty}J_{\mu+j}(2w)J_{\nu+j}(2w). \tag{18}$$

Definition (1) can naturally be extended to more general ranges of indices. For any sequence $\{x_n\}_{n=N_1}^{N_2}$, $N_1, N_2 \in \mathbb{Z}\cup\{-\infty,+\infty\}$, $N_1 \leqslant N_2+1$, (if $N_1 = N_2+1 \in \mathbb{Z}$ then the sequence is considered as empty) such that

$$\sum_{k=N_1}^{N_2-1}|x_kx_{k+1}| < \infty$$

one can define

$$\mathfrak{F}\left(\{x_k\}_{k=N_1}^{N_2}\right) = 1 + \sum_{m=1}^{\infty}(-1)^m \sum_{k\in\mathcal{I}(N_1,N_2,m)} x_{k_1}x_{k_1+1}x_{k_2}x_{k_2+1}\ldots x_{k_m}x_{k_m+1},$$

where

$$\mathcal{I}(N_1, N_2, m) = \left\{k \in \mathbb{Z}^m;\ k_j + 2 \leqslant k_{j+1} \text{ for } 1 \leqslant j \leqslant m-1,\ N_1 \leqslant k_1,\ k_m < N_2\right\}.$$

With this definition one can generalize the rule (4). Now one has

$$\mathfrak{F}\left(\{x_k\}_{k=N_1}^{N_2}\right) = \mathfrak{F}\left(\{x_k\}_{k=N_1}^{n}\right)\mathfrak{F}\left(\{x_k\}_{k=n+1}^{N_2}\right) - x_n x_{n+1}\,\mathfrak{F}\left(\{x_k\}_{k=N_1}^{n-1}\right)\mathfrak{F}\left(\{x_k\}_{k=n+2}^{N_2}\right) \tag{19}$$

provided $n \in \mathbb{Z}$ satisfies $N_1 \leqslant n < N_2$.

This extension also opens a way for applications of the function $\mathfrak{F}$ to bilateral difference equations. Suppose that sequences $\{w_n\}_{n=-\infty}^{\infty}$ and $\{\zeta_n\}_{n=-\infty}^{\infty}$ are such that $w_n \neq 0$, $\zeta_n \neq 0$ for all n, and

$$\sum_{k=-\infty}^{\infty} \left|\frac{w_k^2}{\zeta_k \zeta_{k+1}}\right| < \infty.$$

Consider the difference equation

$$w_n u_{n+1} - \zeta_n u_n + w_{n-1} u_{n-1} = 0, \quad n \in \mathbb{Z}. \tag{20}$$

Define the sequence $\{\mathcal{P}_n\}_{n\in\mathbb{Z}}$ by $\mathcal{P}_0 = 1$ and $\mathcal{P}_{n+1} = (w_n/\zeta_{n+1})\mathcal{P}_n$ for all n. Hence

$$\mathcal{P}_n = \prod_{k=1}^{n} \frac{w_{k-1}}{\zeta_k} \ \text{ for } n > 0, \quad \mathcal{P}_0 = 1, \quad \mathcal{P}_n = \prod_{k=n+1}^{0} \frac{\zeta_k}{w_{k-1}} \ \text{ for } n < 0.$$

The sequence $\{\gamma_n\}_{n\in\mathbb{Z}}$ is again defined so that $\gamma_1 = 1$ and $\gamma_n\gamma_{n+1} = w_n$ for all $n \in \mathbb{Z}$. Hence

$$\gamma_{2k-1} = \prod_{j=1}^{k-1} \frac{w_{2j}}{w_{2j-1}}, \quad \gamma_{2k} = w_1 \prod_{j=1}^{k-1} \frac{w_{2j+1}}{w_{2j}}, \quad \text{for } k = 1, 2, 3, \ldots,$$

and

$$\gamma_{2k-1} = \prod_{j=k}^{0} \frac{w_{2j-1}}{w_{2j}}, \quad \gamma_{2k} = w_1 \prod_{j=k}^{0} \frac{w_{2j}}{w_{2j+1}}, \quad \text{for } k = 0, -1, -2, \ldots.$$

Then the sequences $\{f_n\}_{n\in\mathbb{Z}}$ and $\{g_n\}_{n\in\mathbb{Z}}$,

$$f_n = \mathcal{P}_n\,\mathfrak{F}\left(\left\{\frac{\gamma_k^{\,2}}{\zeta_k}\right\}_{k=n+1}^{\infty}\right), \quad g_n = \frac{1}{w_{n-1}\mathcal{P}_{n-1}}\,\mathfrak{F}\left(\left\{\frac{\gamma_k^{\,2}}{\zeta_k}\right\}_{k=-\infty}^{n-1}\right), \tag{21}$$

represent two solutions of the bilateral difference equation (20).

For two solutions $u = \{u_n\}_{n\in\mathbb{Z}}$ and $v = \{v_n\}_{n\in\mathbb{Z}}$ of (20) the Wronskian is introduced as

$$\mathcal{W}(u, v) = w_n\left(u_n v_{n+1} - u_{n+1} v_n\right).$$

As is well known, this is a constant independent of the index n. Moreover, two solutions are linearly dependent iff their Wronskian vanishes. For the solutions f and g given in (21) one can use (19) to evaluate their Wronskian getting

$$\mathcal{W}(f, g) = \mathfrak{F}\left(\left\{\frac{\gamma_n^{\,2}}{\zeta_n}\right\}_{n=-\infty}^{\infty}\right).$$

One may also consider an application of a discrete analog of Green's formula to the solutions (21) [2]. In general, suppose that sequences $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ solve respectively the difference equations

$$w_n u_{n+1} - \zeta_n^{(1)} u_n + w_{n-1} u_{n-1} = 0, \quad w_n v_{n+1} - \zeta_n^{(2)} v_n + w_{n-1} v_{n-1} = 0, \quad n \in \mathbb{N}. \tag{22}$$

In that case it is well known and easy to check that

$$\sum_{j=1}^{n}\left(\zeta_j^{(1)}-\zeta_j^{(2)}\right)u_jv_j = w_0\left(u_0v_1-u_1v_0\right)-w_n\left(u_nv_{n+1}-u_{n+1}v_n\right). \tag{23}$$

Proposition 6. *Suppose that the convergence condition*

$$\sum_{k=1}^{\infty}\left|\frac{w_k^2}{\zeta_k\zeta_{k+1}}\right|<\infty$$

is satisfied for the both difference equations in (22). Moreover, assume that

$$\sup_{n\geqslant 1}\left|\frac{w_n^2}{\zeta_n^{(1)}\zeta_{n+1}^{(2)}}\right|<\infty \quad \textit{and} \quad \sup_{n\geqslant 1}\left|\frac{w_n^2}{\zeta_n^{(2)}\zeta_{n+1}^{(1)}}\right|<\infty.$$

Then the corresponding solutions $f^{(1)}, f^{(2)}$ from (21) fulfill

$$\sum_{j=1}^{\infty}\left(\zeta_j^{(1)}-\zeta_j^{(2)}\right)f_j^{(1)}f_j^{(2)} = w_0\left(f_0^{(1)}f_1^{(2)}-f_1^{(1)}f_0^{(2)}\right). \tag{24}$$

Proof. In view of (23) it suffices to show that

$$\lim_{n\to\infty} w_nf_n^{(1)}f_{n+1}^{(2)} = \lim_{n\to\infty} w_nf_{n+1}^{(1)}f_n^{(2)} = 0. \tag{25}$$

By the convergence assumption, for all $n > n_0$ one has

$$|w_n| \leqslant \frac{1}{2}\sqrt{|\zeta_n^{(1)}||\zeta_{n+1}^{(1)}|},\quad |w_n| \leqslant \frac{1}{2}\sqrt{|\zeta_n^{(2)}||\zeta_{n+1}^{(2)}|}.$$

Using (3), after some straightforward manipulations one gets the estimate

$$\left|w_nf_n^{(1)}f_{n+1}^{(2)}\right| \leqslant 2^{-2(n-n_0)}\exp\left(\sum_{k=1}^{\infty}\left|\frac{w_k^2}{\zeta_k^{(1)}\zeta_{k+1}^{(1)}}\right|+\left|\frac{w_k^2}{\zeta_k^{(2)}\zeta_{k+1}^{(2)}}\right|\right)\prod_{k=1}^{n_0}\left|\frac{w_{k-1}^2}{\zeta_k^{(1)}\zeta_k^{(2)}}\right|$$
$$\times|\zeta_{n_0}^{(1)}\zeta_{n_0}^{(2)}|^{1/2}\,\frac{|w_n|}{\left|\zeta_n^{(1)}\zeta_{n+1}^{(2)}\right|^{1/2}}.$$

This implies (25). □

In the literature on Jacobi matrices one encounters a construction of an infinite matrix associated with the bilateral difference equation (20) [15, Section 1.1], [9, Theorem 1.2]. Let us define the matrix $\mathfrak{J}$ with entries $\mathfrak{J}(m,n)$, $m,n\in\mathbb{Z}$, so that for every fixed m, the sequence $u_n=\mathfrak{J}(m,n)$, $n\in\mathbb{Z}$, solves (20) with the initial conditions $\mathfrak{J}(m,m)=0$, $\mathfrak{J}(m,m+1)=1/w_m$.

Using (6) one verifies that, for $m<n$,

$$\mathfrak{J}(m,n)=\frac{1}{w_m}\left(\prod_{j=m+1}^{n-1}\frac{\zeta_j}{w_j}\right)\mathfrak{F}\!\left(\frac{\gamma_{m+1}^2}{\zeta_{m+1}},\frac{\gamma_{m+2}^2}{\zeta_{m+2}},\ldots,\frac{\gamma_{n-1}^2}{\zeta_{n-1}}\right).$$

Moreover, it is quite obvious that, for all $m,n\in\mathbb{Z}$,

$$\mathfrak{J}(m,n)=\frac{1}{\mathcal{W}(u,v)}\left(u_mv_n-v_mu_n\right),$$

where $\{u_n\}$, $\{v_n\}$ is any couple of independent solutions of (20). Hence the matrix $\mathfrak{J}$ is antisymmetric. It also follows that, $\forall m, n, k, \ell \in \mathbb{Z}$,

$$\mathfrak{J}(m,k)\mathfrak{J}(n,\ell) - \mathfrak{J}(m,\ell)\mathfrak{J}(n,k) = \mathfrak{J}(m,n)\mathfrak{J}(k,\ell).$$

Example 7. As an example let us again have a look at the particular case where $w_n = w$, $\zeta_n = \nu + n$ for all $n \in \mathbb{Z}$ and some $w, \nu \in \mathbb{C}$, $w \neq 0$, $\nu \notin \mathbb{Z}$. One finds, with the aid of (15), that the solutions (21) now read

$$f_n = \Gamma(\nu+1)\, w^{-\nu} J_{\nu+n}(2w), \quad g_n = \frac{(-1)^n \pi}{\sin(\pi\nu)\Gamma(\nu+1)}\, w^{\nu} J_{-\nu-n}(2w).$$

Hence the Wronskian equals

$$\mathcal{W}(f,g) = \frac{\pi w}{\sin(\pi\nu)}\left(-J_\nu(2w)J_{-\nu-1}(2w) - J_{\nu+1}(2w)J_{-\nu}(2w)\right) = \mathfrak{F}\left(\left\{\frac{w}{\nu+n}\right\}_{n=-\infty}^{\infty}\right).$$

Recalling once more (15) we note that the RHS equals

$$\lim_{N\to\infty} \mathfrak{F}\left(\left\{\frac{w}{\nu-N+n}\right\}_{n=1}^{\infty}\right) = \lim_{N\to\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\,\frac{\Gamma(\nu-N+1)}{\Gamma(\nu-N+n+1)}\, w^{2n} = 1.$$

Thus one gets the well known relation [1, Eq. (9.1.15)]

$$J_{\nu+1}(2w)J_{-\nu}(2w) + J_\nu(2w)J_{-\nu-1}(2w) = -\frac{\sin(\pi\nu)}{\pi w}\,. \tag{26}$$

Concerning the matrix $\mathfrak{J}$, this particular choice brings us to the case discussed in [14, Proposition 22]. Then the Bessel functions $Y_{n+\nu}(2w)$ and $J_{n+\nu}(2w)$, depending on the index $n \in \mathbb{Z}$, represent other two linearly independent solutions of (20). Since [1, Eq. (9.1.16)]

$$J_{\nu+1}(z)Y_\nu(z) - J_\nu(z)Y_{\nu+1}(z) = \frac{2}{\pi z}$$

one finds that

$$\mathfrak{J}(m,n) = \pi\left(Y_{m+\nu}(2w)J_{n+\nu}(2w) - J_{m+\nu}(2w)Y_{n+\nu}(2w)\right).$$

Moreover, for $\sigma = m + \mu$ and $k = n - m > 0$ one has

$$J_{\sigma+k}(2w)Y_\sigma(2w) - J_\sigma(2w)Y_{\sigma+k}(2w) = \frac{\Gamma(\sigma+k)}{\pi w^k\,\Gamma(\sigma+1)}\,\mathfrak{F}\left(\left\{\frac{w}{\sigma+j}\right\}_{j=1}^{k-1}\right).$$

Finally, putting $\zeta_n^{(1)} = \mu + n$, $\zeta_n^{(2)} = \nu + n$ and $w_n = w$, $\forall n \in \mathbb{N}$, in Eq. (22), one verifies that (24) holds true and reveals this way once more the identity (18).

3. A class of Jacobi operators with point spectra

3.1. The characteristic function

Being inspired by Proposition 4 and notably by Eq. (13), we introduce the (renormalized) characteristic function associated with a Jacobi matrix $\mathcal{J}$ as

$$F_{\mathcal{J}}(z) := \mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - z}\right\}_{n=1}^{\infty}\right). \tag{27}$$

It is treated as a complex function of a complex variable z and is well defined provided the sequence in the argument of $\mathfrak{F}$ belongs to the domain D. Let us show that this is guaranteed under the assumption that there exists $z_0 \in \mathbb{C}$ such that

$$\sum_{n=1}^{\infty} \left| \frac{w_n^2}{(\lambda_n - z_0)(\lambda_{n+1} - z_0)} \right| < \infty. \tag{28}$$

For $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ let us denote

$$\mathbb{C}_0^{\lambda} := \mathbb{C} \setminus \overline{\{\lambda_n;\ n \in \mathbb{N}\}}.$$

Clearly,

$$\overline{\{\lambda_n;\ n \in \mathbb{N}\}} = \{\lambda_n;\ n \in \mathbb{N}\} \cup \operatorname{der}(\lambda)$$

where $\operatorname{der}(\lambda)$ stands for the set of all finite accumulation points of the sequence λ (i.e. $\operatorname{der}(\lambda)$ is equal to the set of limit points of all possible convergent subsequences of λ).

Lemma 8. *Let condition (28) be fulfilled for at least one $z_0 \in \mathbb{C}_0^{\lambda}$. Then the series*

$$\sum_{n=1}^{\infty} \frac{w_n^2}{(\lambda_n - z)(\lambda_{n+1} - z)} \tag{29}$$

converges absolutely and locally uniformly in z on $\mathbb{C}_0^{\lambda}$. Moreover,

$$\forall z \in \mathbb{C}_0^{\lambda},\ \left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=1}^{\infty} \in D \quad and \quad \lim_{n\to\infty} \mathfrak{F}\left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=1}^{n} \right) = F_{\mathcal{J}}(z), \tag{30}$$

and the convergence is locally uniform on $\mathbb{C}_0^{\lambda}$. Consequently, $F_{\mathcal{J}}(z)$ is a well defined analytic function on $\mathbb{C}_0^{\lambda}$.

Proof. Let $K \subset \mathbb{C}_0^{\lambda}$ be a compact subset. Then the ratio

$$\frac{|\lambda_n - z_0|}{|\lambda_n - z|} \leqslant 1 + \frac{|z - z_0|}{|\lambda_n - z|}$$

admits an upper bound, uniform in $z \in K$ and $n \in \mathbb{N}$. The uniform convergence on K of the series (29) thus becomes obvious. Moreover, the absolute convergence of (29) means nothing but $\left\{\gamma_k^2/(\lambda_k - z)\right\}_{k=1}^{\infty} \in D$.

The limit (30) follows from Lemma 2. Moreover, using (30) and also (6), (3) one has

$$\begin{aligned} \left| \mathfrak{F}\left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=1}^{n} \right) - F_{\mathcal{J}}(z) \right| &\leqslant \sum_{l=n}^{\infty} \left| \mathfrak{F}\left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=1}^{l} \right) - \mathfrak{F}\left(\left\{ \frac{\gamma_k^2}{\lambda_k - z} \right\}_{k=1}^{l+1} \right) \right| \\ &\leqslant \sum_{l=n}^{\infty} \left| \frac{w_l^2}{(\lambda_l - z)(\lambda_{l+1} - z)} \right| \exp\left(\sum_{k=1}^{\infty} \left| \frac{w_k^2}{(\lambda_k - z)(\lambda_{k+1} - z)} \right| \right). \end{aligned}$$

From this estimate and the locally uniform convergence of the series (29) one deduces the locally uniform convergence of the sequence of functions (30). □

By a closer inspection one finds that, under the assumptions of Lemma 8, the function $F_{\mathcal{J}}(z)$ is meromorphic on $\mathbb{C} \setminus \operatorname{der}(\lambda)$ with poles at the points $z = \lambda_n$ for some $n \in \mathbb{N}$ (not belonging to $\operatorname{der}(\lambda)$, however). For any such z, the order of the pole is less than or equal to $r(z)$ where

$$r(z) := \sum_{k=1}^{\infty} \delta_{z,\lambda_k}$$

is the number of members of the sequence λ coinciding with z (hence $r(z) = 0$ for $z \in \mathbb{C}_0^\lambda$). To see this, suppose that $r(z) \geqslant 1$ and let M be the maximal index such that $\lambda_M = z$. Using (4) one derives that, for $u \in \mathbb{C}_0^\lambda$,

$$\begin{aligned} F_{\mathcal{J}}(u) \, &= \, \mathfrak{F}\left(\left\{\frac{{\gamma_n}^2}{\lambda_n - u}\right\}_{n=1}^{M}\right) \mathfrak{F}\left(\left\{\frac{{\gamma_n}^2}{\lambda_n - u}\right\}_{n=M+1}^{\infty}\right) \\ &\quad + \mathfrak{F}\left(\left\{\frac{{\gamma_n}^2}{\lambda_n - u}\right\}_{n=1}^{M-1}\right) \frac{\gamma_M^2\gamma_{M+1}^2}{(u-z)(\lambda_{M+1}-u)} \, \mathfrak{F}\left(\left\{\frac{{\gamma_n}^2}{\lambda_n - u}\right\}_{n=M+2}^{\infty}\right). \end{aligned}$$

The RHS clearly has a pole at the point $u = z$ of order at most $r(z)$.

3.2. The Jacobi operator J

Our goal is to investigate spectral properties of a closed operator J on $\ell^2(\mathbb{N})$ whose matrix in the canonical basis coincides with $\mathcal{J}$. Unless the Jacobi matrix determines a bounded operator, there need not be a unique way how to introduce J, however. But among all admissible operators one may distinguish two particular cases which may respectively be regarded, in a natural way, as the minimal and the maximal operator with the required properties; see, for instance, [3].

Definition 9. The operator $J_{\max}$ is defined so that

$$\mathrm{Dom}(J_{\max}) = \{y \in \ell^2(\mathbb{N});\ \mathcal{J}y \in \ell^2(\mathbb{N})\},$$

and one sets $J_{\max}y = \mathcal{J}y$, $\forall y \in \mathrm{Dom}\, J_{\max}$. Here and in what follows $\mathcal{J}y$ is understood as the formal matrix product while treating y as a column vector. To define the operator $J_{\min}$ one first introduces the operator $\dot{J}$ so that $\mathrm{Dom}(\dot{J})$ is the linear hull of the canonical basis, and again $\dot{J}y = \mathcal{J}y$ for all $y \in \mathrm{Dom}(\dot{J})$. $\dot{J}$ is known to be closable [3], and $J_{\min}$ is defined as the closure of $\dot{J}$.

One has the following relations between the operators $J_{\min}$, $J_{\max}$ and their adjoint operators [3, Lemma 2.1]. Let $\mathcal{J}^H$ designates the Jacobi matrix obtained from $\mathcal{J}$ by taking the complex conjugate of each entry. Then $J_{\min}^* = J_{\max}^H$, $J_{\max}^* = J_{\min}^H$. In particular, the maximal operator $J_{\max}$ is a closed extension of $J_{\min}$. It is even true that any closed operator J whose domain contains the canonical basis and whose matrix in this basis equals $\mathcal{J}$ fulfills $J_{\min} \subset J \subset J_{\max}$. Moreover, if $\mathcal{J}$ is Hermitian, i.e. $\mathcal{J} = \mathcal{J}^H$ (which means nothing but $\mathcal{J}$ is real), then $J_{\min}^* = J_{\max} \supset J_{\min}$. Hence $J_{\min}$ is symmetric with the deficiency indices either (0, 0) or (1, 1).

We are primarily interested in the situation where $J_{\min} = J_{\max}$ since then there exists a unique closed operator J defined by the Jacobi matrix $\mathcal{J}$, and it turns out that the spectrum of J is determined in a well defined sense by the characteristic function $F_{\mathcal{J}}(z)$. If this happens $\mathcal{J}$ is sometimes called proper [3].

Let us recall more details on this property. We remind the reader that the orthogonal polynomials of the first kind, $p_n(z)$, $n \in \mathbb{Z}_+$, are defined by the recurrence

$$w_{n-1}p_{n-2}(z) + (\lambda_n - z)p_{n-1}(z) + w_n p_n(z) = 0, \quad n = 2, 3, 4, \ldots,$$

with the initial conditions $p_0(z) = 1$, $p_1(z) = (z - \lambda_1)/w_1$. The orthogonal polynomials of the second kind, $q_n(z)$, obey the same recurrence but the initial conditions are $q_0(z) = 0$, $q_1(z) = 1/w_1$; see [2,4]. It is not difficult to verify that these polynomials are expressible in terms of the function $\mathfrak{F}$ as follows:

$$p_n(z) = \left(\prod_{k=1}^{n} \frac{z - \lambda_k}{w_k}\right) \mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l - z}\right\}_{l=1}^{n}\right), \quad n = 0, 1, 2 \ldots,$$

and

$$q_n(z)=\frac{1}{w_1}\left(\prod_{k=2}^{n}\frac{z-\lambda_k}{w_k}\right)\mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l-z}\right\}_{l=2}^{n}\right),\quad n=1,2,3\ldots.$$

The complex Jacobi matrix $\mathcal{J}$ is called determinate if at least one of the sequences $p(0)=\{p_{n-1}(0)\}_{n=1}^{\infty}$ or $q(0)=\{q_{n-1}(0)\}_{n=1}^{\infty}$ is not an element of $\ell^2(\mathbb{N})$. For real Jacobi matrices there exits a parallel terminology. Instead of determinate one calls $\mathcal{J}$ limit point at $+\infty$, and instead of indeterminate one calls $\mathcal{J}$ limit circle at $+\infty$, see [15, Section 2.6]. According to [16, Theorem 22.1], $\mathcal{J}$ is indeterminate if both $p(z)$ and $q(z)$ are elements of ℓ^2 for at least one $z\in\mathbb{C}$, and in this case they are elements of ℓ^2 for all $z\in\mathbb{C}$. For a real Jacobi matrix $\mathcal{J}$ one can prove that it is proper if and only if it is determinate (or, in another terminology, limit point), and this happens if and only if $p(z)\notin\ell^2(\mathbb{N})$ for some and hence any $z\in\mathbb{C}\setminus\mathbb{R}$; see [2, Section 4.1] or [15, Lemma 2.16].

For complex Jacobi matrices one can also specify assumptions under which $J_{\min}=J_{\max}$. In what follows, $\rho(A)$ designates the resolvent set of a closed operator A. Concerning the essential spectrum, one observes that $\text{spec}_{ess}(J_{\min})=\text{spec}_{ess}(J_{\max})$ [3, Eq. (2.10)]. Hence if $\rho(J_{\max})\neq\emptyset$ then $\text{spec}_{ess}(J_{\min})\neq\mathbb{C}$. Moreover, in that case $\mathcal{J}$ is determinate [3, Theorem 2.11(a)] and proper [3, Theorem 2.6(a)]. This way one extracts from [3] the following result.

Theorem 10. *If $\rho(J_{max})\neq\emptyset$ then $J_{min}=J_{max}$.*

3.3. The spectrum and the zero set of the characteristic function

Let us define

$$\mathfrak{Z}(\mathcal{J}):=\left\{z\in\mathbb{C}\setminus\text{der}(\lambda);\ \lim_{u\to z}(u-z)^{r(z)}F_{\mathcal{J}}(u)=0\right\}.\tag{31}$$

Of course, $\mathfrak{Z}(\mathcal{J})\cap\mathbb{C}_0^{\lambda}$ is nothing but the set of zeros of $F_{\mathcal{J}}(z)$. Further, for $k\in\mathbb{Z}_+$ and $z\in\mathbb{C}\setminus\text{der}(\lambda)$ we put

$$\xi_k(z):=\lim_{u\to z}(u-z)^{r(z)}\left(\prod_{l=1}^{k}\frac{w_{l-1}}{u-\lambda_l}\right)\mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l-u}\right\}_{l=k+1}^{\infty}\right),\tag{32}$$

where one sets $w_0=1$. Either denote by $M=M_z$ the maximal index, if any, such that $z=\lambda_M$ or put $M=0$ otherwise. One observes that for $k\geqslant M$,

$$\xi_k(z)=\prod_{l=1}^{k}w_{l-1}\left(\prod_{\substack{l=1\\ \lambda_l\neq z}}^{k}(z-\lambda_l)\right)^{-1}\mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l-z}\right\}_{l=k+1}^{\infty}\right).\tag{33}$$

Proposition 11. *Let condition (28) be fulfilled for at least one $z_0\in\mathbb{C}_0^{\lambda}$. If*

$$\xi_0(z)\equiv\lim_{u\to z}(u-z)^{r(z)}F_{\mathcal{J}}(u)=0$$

for some $z\in\mathbb{C}\setminus\text{der}(\lambda)$, then z is an eigenvalue of J_{max} and

$$\xi(z):=(\xi_1(z),\xi_2(z),\xi_3(z),\ldots)$$

is a corresponding eigenvector.

Proof. Using (5) one verifies that if $\xi_0(z)=0$ then the column vector $\xi(z)$ solves the matrix equation $\mathcal{J}\xi(z)=z\xi(z)$. To complete the proof one has to show that $\xi(z)$ does not vanish and belongs to $\ell^2(\mathbb{N})$.

First, we claim $\xi_1(z) \neq 0$. Suppose, on the contrary, that $\xi_1(z) = 0$. Then the formal eigenvalue equation (which is a second order recurrence) implies $\xi(z) = 0$. From (33) it follows that

$$\mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l - z}\right\}_{l=k+1}^{\infty}\right) = 0$$

for all $k \geqslant M = M_z$. This equality is in contradiction with (7), however.

Second, suppose $z \notin \operatorname{der}(\lambda)$ is fixed. By Lemma 8, there exists $N \in \mathbb{N}$, $N > M$, such that

$$|w_n^2| \leqslant |\lambda_n - z||\lambda_{n+1} - z|/2, \ \forall n \geqslant N.$$

Let us denote

$$C = \prod_{l=1}^{N} |w_{l-1}|^2 \prod_{\substack{l=1 \\ \lambda_l \neq z}}^{N} |z - \lambda_l|^{-2}.$$

Using also (3) one can estimate

$$\begin{aligned}\sum_{k=N}^{\infty} |\xi_k(z)|^2 &= \sum_{k=N}^{\infty} \prod_{l=1}^{k} |w_{l-1}|^2 \prod_{\substack{l=1 \\ \lambda_l \neq z}}^{k} |z - \lambda_l|^{-2} \left|\mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l - z}\right\}_{l=k+1}^{\infty}\right)\right|^2 \\ &\leqslant C \exp\left(2\sum_{k=N+1}^{\infty}\left|\frac{w_k^2}{(\lambda_k - z)(\lambda_{k+1} - z)}\right|\right) \sum_{k=N}^{\infty} \prod_{l=N+1}^{k}\left(\frac{1}{2}\left|\frac{z - \lambda_{l-1}}{z - \lambda_l}\right|\right).\end{aligned}$$

Since $|\lambda_k - z| \geqslant \tau$ for all $k > M$ and some $\tau > 0$, the RHS is finite. □

Further we wish to prove a statement converse to Proposition 11. Our approach is based on a formula for the Green function generalizing a similar result known for the finite-dimensional case; see (14).

Proposition 12. *Let condition (28) be fulfilled for at least one $z_0 \in \mathbb{C}_0^{\lambda}$. If $z \in \mathbb{C} \setminus \operatorname{der}(\lambda)$ does not belong to the zero set $\mathfrak{Z}(\mathcal{J})$ then $z \in \rho(J_{max})$ and the Green function for the spectral parameter z,*

$$G(z; i, j) := \langle e_i, (J_{max} - z)^{-1} e_j \rangle, \quad i, j \in \mathbb{N},$$

(a matrix in the canonical basis) is given by the formula

$$\begin{aligned}G(z; i, j) = &-\frac{1}{w_{\max(i,j)}}\left(\prod_{l=\min(i,j)}^{\max(i,j)} \frac{w_l}{z - \lambda_l}\right) \\ &\times \mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l - z}\right\}_{l=1}^{\min(i,j)-1}\right) \mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l - z}\right\}_{l=\max(i,j)+1}^{\infty}\right) \mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l - z}\right\}_{l=1}^{\infty}\right)^{-1}.\end{aligned} \tag{34}$$

In particular, for the Weyl m-function one has

$$m(z) := G(z; 1, 1) = \frac{1}{\lambda_1 - z} \mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l - z}\right\}_{l=2}^{\infty}\right) \mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l - z}\right\}_{l=1}^{\infty}\right)^{-1}. \tag{35}$$

Proof. Denote by $R(z)_{i,j}$ the RHS of (34). Thus $R(z)$ is an infinite matrix provided its entries $R(z)_{i,j}$, $i, j \in \mathbb{N}$, make good sense. Suppose that a complex number z does not belong to $\mathfrak{Z}(J) \cup \operatorname{der}(\lambda)$. By Lemma 8, in that case the RHS of (34) is well defined. By inspection of the expression one finds that this is so even if z happens to coincide with a member λ_k of the sequence λ not belonging to $\operatorname{der}(\lambda)$, i.e. the seeming singularity at $z = \lambda_k$ is removable. For the sake of simplicity we assume in the remainder of the proof, however, that z does not belong to the range of the sequence λ. The only purpose of this assumption is just to simplify the discussion and to avoid more complex expressions but otherwise it is not essential for the result.

First let us show that there exists a constant C, possibly depending on z but independent of the indices i, j, such that

$$|R(z)_{i,j}| \leqslant C\, 2^{-|i-j|}, \quad \forall i, j \in \mathbb{N}. \tag{36}$$

To this end, denote

$$\tau = \inf\{|z - \lambda_n|;\ n \in \mathbb{N}\} > 0.$$

Assuming (28), one can choose $n_0 \in \mathbb{N}$ so that, for all $n \geqslant n_0$,

$$|w_n|^2 \leqslant |\lambda_n - z|\, |\lambda_{n+1} - z|/4. \tag{37}$$

Let us assume, for the sake of definiteness, that $i \leqslant j$. Again by (28) and (3),

$$\left| \mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l - z}\right\}_{l=1}^{i-1}\right) \mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l - z}\right\}_{l=j+1}^{\infty}\right) \mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l - z}\right\}_{l=1}^{\infty}\right)^{-1} \right| \leqslant C_1,$$

for all i, j. It remains to estimate the expression

$$\frac{1}{|\lambda_j - z|} \left| \prod_{l=i}^{j-1} \frac{w_l}{\lambda_l - z} \right|. \tag{38}$$

We distinguish three cases. For the finite set of couples i, j, $i \leqslant j \leqslant n_0$, (38) is bounded from above by a constant C_2. Using (37), if $i \leqslant n_0 \leqslant j$ then (38) is majorized by

$$C_2 \left| \frac{\lambda_{n_0} - z}{\lambda_j - z} \prod_{l=n_0}^{j-1} \frac{w_l}{\lambda_l - z} \right| \leqslant C_2 \tau^{-1/2} |\lambda_{n_0} - z|^{1/2}\, 2^{-j+n_0}.$$

Similarly, if $n_0 \leqslant i \leqslant j$ then (38) is majorized by $\tau^{-1} 2^{-j+i}$. From these partial upper bounds the estimate (36) readily follows.

From (36) one deduces that the matrix $R(z)$ represents a bounded operator on $\ell^2(\mathbb{N})$. In fact, one can write $R(z)$ as a countable sum,

$$R(z) = \sum_{s \in \mathbb{Z}} R(z; s), \tag{39}$$

where the matrix elements of the summands are $R(z; s)_{i,j} = R(z)_{i,j}$ if $i - j = s$ and $R(z; s)_{i,j} = 0$ otherwise. Thus $R(z; s)$ has nonvanishing elements on only one parallel to the diagonal and

$$\|R(z; s)\| = \sup\{|R(z)_{i,j}|;\ i - j = s\} \leqslant C\, 2^{-|s|}.$$

Hence the series (39) converges in the operator norm. With some abuse of notation, we shall denote the corresponding bounded operator again by the symbol $R(z)$.

Further one observes that, on the level of formal matrix products,

$$(\mathfrak{J} - z)R(z) = R(z)(\mathfrak{J} - z) = I.$$

The both equalities are in fact equivalent to the countable system of equations (with $w_0 = 0$)

$$w_{k-1} G(z; i, k-1) + (\lambda_k - z) G(z; i, k) + w_k G(z; i, k+1) = \delta_{i,k}, \quad i, k \in \mathbb{N}.$$

This can be verified, in a straightforward manner, with the aid of the rule (4) or some of its particular cases (5) and (6). By inspection of the domains one then readily shows that the operators $J_{\max} - z$ and $R(z)$ are mutually inverse and so $z \in \rho(J_{\max})$. $\square$

Corollary 13. *Suppose, in addition to the assumptions of Proposition 12, that* $|\lambda_n| \to \infty$ *as* $n \to \infty$. *Then* $\operatorname{der}(\lambda) = \emptyset$ *and for every* $z \in \mathbb{C} \setminus \mathfrak{Z}(\mathcal{J})$, *the resolvent* $(J_{max} - z)^{-1}$ *is compact.*

Proof. Keeping the notation from the proof of Proposition 12, suppose $|\lambda_n| \to \infty$ as $n \to \infty$, and $z \in \mathbb{C} \setminus \mathfrak{Z}(\mathcal{J})$. Then it readily turns out that the estimates may be somewhat refined. In particular, (38) is majorized by

$$|\lambda_i - z|^{-1/2} |\lambda_j - z|^{-1/2} \, 2^{-j+i}$$

for $n_0 \leqslant i \leqslant j$. But this implies that $R(z;s)_{i,j} \to 0$ as $i, j \to \infty$, with $i - j = s$ being constant. It follows that the operators $R(z;s)$ are compact. Since the series (39) converges in the operator norm, $R(z)$ is compact as well. $\square$

Corollary 14. *If condition (28) is fulfilled for at least one* $z_0 \in \mathbb{C}_0^\lambda$ *then*

$$\operatorname{spec}(J_{max}) \setminus \operatorname{der}(\lambda) = \operatorname{spec}_p(J_{max}) \setminus \operatorname{der}(\lambda) = \mathfrak{Z}(\mathcal{J}).$$

Proof. Propositions 11 and12 respectively imply the inclusions

$$\mathfrak{Z}(\mathcal{J}) \subset \operatorname{spec}_p(J_{\max}) \setminus \operatorname{der}(\lambda), \quad \operatorname{spec}(J_{\max}) \setminus \operatorname{der}(\lambda) \subset \mathfrak{Z}(\mathcal{J}).$$

This shows the equality. $\square$

Theorem 15. *Suppose that the convergence condition (28) is fulfilled for at least one* $z_0 \in \mathbb{C}_0^\lambda$ *and the function* $F_{\mathcal{J}}(z)$ *does not vanish identically on* $\mathbb{C}_0^\lambda$. *Then* $J_{min} = J_{max} =: J$ *and*

$$\operatorname{spec}(J) \setminus \operatorname{der}(\lambda) = \operatorname{spec}_p(J) \setminus \operatorname{der}(\lambda) = \mathfrak{Z}(\mathcal{J}). \tag{40}$$

Suppose, in addition, that the set $\mathbb{C} \setminus \operatorname{der}(\lambda)$ *is connected. Then* $\operatorname{spec}(J) \setminus \operatorname{der}(\lambda)$ *consists of simple eigenvalues which have no accumulation points in* $\mathbb{C} \setminus \operatorname{der}(\lambda)$.

Proof. By the assumptions, $\mathbb{C} \setminus (\operatorname{der}(\lambda) \cup \mathfrak{Z}(\mathcal{J})) \neq \emptyset$. From Proposition 12 one infers that $\rho(J_{\max}) \neq \emptyset$. According to Theorem 10, one has $J_{\min} = J_{\max}$. Then (40) becomes a particular case of Corollary 14.

Let us assume that $\mathbb{C} \setminus \operatorname{der}(\lambda)$ is connected. Then the set $\mathbb{C}_0^\lambda$ is clearly connected as well. Suppose on the contrary that the point spectrum of J has an accumulation point in $\mathbb{C} \setminus \operatorname{der}(\lambda)$. Then, by equality (40), the set of zeros of the analytic function $F_{\mathcal{J}}(z)$ has an accumulation point in $\mathbb{C} \setminus \operatorname{der}(\lambda)$. This accumulation point may happen to be a member λ_n of the sequence λ, but then one knows that $F_{\mathcal{J}}(z)$ has a pole of finite order at λ_n. In any case, taking into account that $\mathbb{C}_0^\lambda$ is connected one comes to the conclusion that $F_{\mathcal{J}}(z) = 0$ everywhere on $\mathbb{C}_0^\lambda$, a contradiction. $\square$

Remark 16. Theorem 15 is derived under two assumptions:

(i) The convergence condition (28) is fulfilled for at least one $z_0 \in \mathbb{C}_0^\lambda$.
(ii) The function $F_{\mathcal{J}}(z)$ does not vanish identically on $\mathbb{C}_0^\lambda$.

But let us point out that assumption (ii) is automatically fulfilled if (i) is true and the range of the sequence λ is contained in a halfplane. This happens, for example, if the sequence λ is real or the sequence $\{\operatorname{Re} \lambda_n\}_{n=1}^\infty$ is semibounded. In fact, let us for definiteness consider the latter case and suppose that $\operatorname{Re} \lambda_n \geqslant c$, $\forall n \in \mathbb{N}$. Then $(-\infty, c) \subset \mathbb{C}_0^\lambda$ and $1/|\lambda_n - z|$ tends to 0 monotonically for all n as $z \to -\infty$. Similarly as in (3) one derives the estimate

$$|F_{\mathcal{J}}(z) - 1| \leqslant \exp\left(\sum_{n=1}^\infty \left| \frac{w_n^2}{(\lambda_n - z)(\lambda_{n+1} - z)} \right| \right) - 1.$$

It follows that $\lim_{z\to-\infty} F_{\mathcal{J}}(z) = 1$. Notice that in the real case, the function $F_{\mathcal{J}}(z)$ can identically vanish neither on the upper nor on the lower halfplane.

Corollary 17. *Let $\mathcal{J}$ be real and suppose (28) is fulfilled for at least one $z_0 \in \mathbb{C}_0^\lambda$. Then $J_{min} = J_{max} = J$ is self-adjoint and* $\operatorname{spec}(J) \setminus \operatorname{der}(\lambda) = \mathfrak{Z}(\mathcal{J})$ *consists of simple real eigenvalues which have no accumulation points in $\mathbb{R} \setminus \operatorname{der}(\lambda)$.*

Proof. Some assumptions in Theorem 15 become superfluous if $\mathcal{J}$ is real. As observed in Remark 16, assuming the convergence condition the function $F_{\mathcal{J}}(z)$ cannot vanish identically on $\mathbb{C}_0^\lambda$. The operator J is self-adjoint and may have only real eigenvalues. The set $\mathbb{C} \setminus \operatorname{der}(\lambda)$ may happen to be disconnected only if the range of the sequence λ is dense in $\mathbb{R}$, i.e. $\operatorname{der}(\lambda) = \mathbb{R}$. But even then the conclusion of the theorem remains trivially true. □

Let us complete this analysis by a formula for the norms of the eigenvectors described in Proposition 11. In order to simplify the discussion we restrict ourselves to the domain $\mathbb{C}_0^\lambda$. Then instead of (32) one may write

$$\xi_k(z) = \left(\prod_{l=1}^{k} \frac{w_{l-1}}{z-\lambda_l}\right) \mathfrak{F}\left(\left\{\frac{{\gamma_l}^2}{\lambda_l - z}\right\}_{l=k+1}^{\infty}\right), \quad z \in \mathbb{C}_0^\lambda, \quad k \in \mathbb{Z}_+. \tag{41}$$

This is in fact nothing but the solution f_n from (21) restricted to nonnegative indices.

Proposition 18. *If $z \in \mathbb{C}_0^\lambda$ satisfies (28) then the functions $\xi_k(z)$, $k \in \mathbb{Z}_+$, defined in (41) fulfill*

$$\sum_{k=1}^{\infty} \xi_k(z)^2 = \xi_0'(z)\xi_1(z) - \xi_0(z)\xi_1'(z). \tag{42}$$

Particularly, if in addition $\mathcal{J}$ is real and $z \in \mathbb{R} \cap \mathbb{C}_0^\lambda$ is an eigenvalue of J then $\xi(z) = (\xi_k(z))_{k=1}^\infty$ is a corresponding eigenvector and

$$\|\xi(z)\|^2 = \xi_0'(z)\xi_1(z). \tag{43}$$

Proof. Put $\zeta_j^{(1)} = z - \lambda_j$, $\zeta_j^{(2)} = y - \lambda_j$, $j \in \mathbb{N}$, in Eq. (23), where $z, y \in \mathbb{C}_0^\lambda$. Then Proposition 6 is applicable to $f_j^{(1)} = \xi_j(z)$, $f_j^{(2)} = \xi_j(y)$, $j \in \mathbb{Z}_+$. Hence ($w_0 = 1$)

$$(z-y)\sum_{k=0}^{\infty} \xi_k(z)\xi_k(y) = \xi_1(z)\xi_0(y) - \xi_0(z)\xi_1(y).$$

Now the limit $y \to z$ can be treated in a routine way. □

Corollary 19. *Suppose $\mathcal{J}$ is real and let condition (28) be fulfilled for at least one $z_0 \in \mathbb{C}_0^\lambda$. Then the function $F_{\mathcal{J}}(z)$ has only simple real zeros on $\mathbb{C}_0^\lambda$.*

Proof. Suppose $z \in \mathbb{C}_0^\lambda$ is a zero of $F_{\mathcal{J}}(z)$, i.e. $F_{\mathcal{J}}(z) \equiv \xi_0(z) = 0$. Then z is a real eigenvalue of J where $J = J_{\max} = J_{\min}$ is self-adjoint, as we know from Corollary 17. Moreover, by Proposition 11, $\xi(z) \neq 0$ is a corresponding real eigenvector. Hence from (43) one infers that necessarily $\xi_0'(z) \neq 0$. □

4. Examples

4.1. Explicitly solvable examples of point spectra

In all examples presented below the Jacobi matrix $\mathcal{J}$ is real and symmetric. The set of accumulation points $\operatorname{der}(\lambda)$ is either empty or the one-point set $\{0\}$. Moreover, condition (28) is readily checked to be satisfied for any $z_0 \in \mathbb{C} \setminus \mathbb{R}$. Thus Corollary 17 applies to all these examples and may be used to determine the spectrum of the unique self-adjoint operator J whose matrix in the canonical basis equals $\mathcal{J}$ (recall also definition (31) of the zero set of the characteristic function). In addition, Proposition 11 and Eq. (32) (or (41)) provide us with explicit formulas for the corresponding eigenvectors.

Example 20. This is an example of an unbounded Jacobi operator. Let $\lambda_n = n\alpha$, where $\alpha \in \mathbb{R} \setminus \{0\}$, and $w_n = w > 0$ for all $n \in \mathbb{N}$. Thus

$$J = \begin{pmatrix} \alpha & w & & \\ w & 2\alpha & w & \\ & w & 3\alpha & w \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

One has $\operatorname{der}(\lambda) = \emptyset$ and $\operatorname{spec}(J) = \mathfrak{Z}(\mathcal{J})$. Using (15) one derives that

$$\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\alpha k - z}\right\}_{k=r+1}^{\infty}\right) = \mathfrak{F}\left(\left\{\frac{w}{\alpha k - z}\right\}_{k=r+1}^{\infty}\right) = \left(\frac{w}{\alpha}\right)^{-r+z/\alpha} \Gamma\left(1 + r - \frac{z}{\alpha}\right) J_{r-z/\alpha}\left(\frac{2w}{\alpha}\right)$$

for $r \in \mathbb{Z}_+$. It follows that

$$\operatorname{spec}(J) = \left\{ z \in \mathbb{R};\ J_{-z/\alpha}\left(\frac{2w}{\alpha}\right) = 0 \right\}. \tag{44}$$

For components of corresponding eigenvectors $v(z)$ one obtains

$$v_k(z) = (-1)^k J_{k-z/\alpha}\left(\frac{2w}{\alpha}\right), \quad k \in \mathbb{N}.$$

Let us remark that the characterization of the spectrum of J, as given in (44), was observed earlier by several authors, see [10, Section 3] and [13, Theorem 3.1]. We discuss in more detail solutions of the characteristic equation $J_{-z}(2w) = 0$ below in Section 4.3.

Further we describe four examples in which the Jacobi matrix always represents a compact operator on $\ell^2(\mathbb{N})$. In the first two of them we make use of the following construction. Let us fix positive constants c, α and β. For $n \in \mathbb{Z}_+$ we define the c-deformed number n as

$$[n]_c = \sum_{i=0}^{n-1} c^i.$$

Hence $[n]_c = (c^n - 1)/(c - 1)$ if $c \neq 1$ and $[n]_c = n$ for $c = 1$. Notice that

$$\frac{[n + m - 1]_c - [n - 1]_c}{[m]_c} = [n]_c - [n - 1]_c, \quad \forall n, m \in \mathbb{N}. \tag{45}$$

As for the Jacobi matrix $\mathcal{J}$, we put

$$\lambda_n = \frac{1}{\alpha + [n - 1]_c}, \quad w_n = \beta\sqrt{\lambda_n - \lambda_{n+1}}, \quad n = 1, 2, 3, \ldots. \tag{46}$$

Condition (28) is readily verified, for example for $z_0 < 0$.

Proposition 21. *Let $\mathcal{J}$ be defined by (46) for some $c \geqslant 1$ and $\alpha, \beta > 0$. Then for all $r \in \mathbb{Z}_+$,*

$$\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - z}\right\}_{n=r+1}^{\infty}\right) = \sum_{s=0}^{\infty} \frac{\beta^{2s}}{z^s} \prod_{i=1}^{s} \left([\,i\,]_c\left(1 - \frac{z}{\lambda_{r+i}}\right)\right)^{-1}. \tag{47}$$

Proof. We claim that

$$\sum_{k_1=r}^{\infty}\sum_{k_2=k_1+2}^{\infty}\cdots\sum_{k_s=k_{s-1}+2}^{\infty} \times \frac{w_{k_1}^2}{(\lambda_{k_1}-z)(\lambda_{k_1+1}-z)}\,\frac{w_{k_2}^2}{(\lambda_{k_2}-z)(\lambda_{k_2+1}-z)}\cdots\frac{w_{k_s}^2}{(\lambda_{k_s}-z)(\lambda_{k_s+1}-z)}$$

$$= \frac{(-1)^s}{z^s}\,\beta^{2s}\prod_{i=1}^{s}\left([\,i\,]_c\left(1-\frac{z}{\lambda_{r+i-1}}\right)\right)^{-1} \tag{48}$$

holds for every $r, s \in \mathbb{N}$. In fact, to show (48) one can proceed by mathematical induction in s. The case $s = 1$ as well as all induction steps are straightforward consequences of the equality

$$\frac{w_j^2}{(\lambda_j-z)(\lambda_{j+1}-z)}\prod_{i=1}^{s-1}\left([\,i\,]_c\left(1-\frac{z}{\lambda_{j+i+1}}\right)\right)^{-1}$$
$$= -\frac{\beta^2}{z}\left(\prod_{i=1}^{s}\left([\,i\,]_c\left(1-\frac{z}{\lambda_{j+i-1}}\right)\right)^{-1}-\prod_{i=1}^{s}\left([\,i\,]_c\left(1-\frac{z}{\lambda_{j+i}}\right)\right)^{-1}\right),$$

with $s = 1, 2, 3, \ldots$, which in turn can be verified with the aid of (45). Identity (47) follows from (48) and definition (1). □

Example 22. In (46), let us put $c = 1$ and $\alpha = 1$ while β is arbitrary positive. Then $\lambda_n = 1/n$, $w_n = \beta/\sqrt{n(n+1)}$, for all $n \in \mathbb{N}$, and so

$$J = \begin{pmatrix} 1 & \beta/\sqrt{2} & & & \\ \beta/\sqrt{2} & 1/2 & \beta/\sqrt{6} & & \\ & \beta/\sqrt{6} & 1/3 & \beta/\sqrt{12} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

One finds that

$$\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n-z}\right\}_{n=k+1}^{\infty}\right)=\sum_{s=0}^{\infty}\frac{\beta^{2s}}{s!\,z^s}\prod_{j=1}^{s}\frac{1}{1-(k+j)z} = {}_0F_1\left(k+1-\frac{1}{z};-\frac{\beta^2}{z^2}\right)$$
$$=\left(\frac{z}{\beta}\right)^{k-1/z}\Gamma\left(k+1-\frac{1}{z}\right)J_{k-1/z}\left(\frac{2\beta}{z}\right), \tag{49}$$

with $k \in \mathbb{Z}_+$, see [1, Eq. (9.1.69)]. Then

$$F_{\mathcal{J}}(z) = \Gamma\left(1-\frac{1}{z}\right)\left(\frac{z}{\beta}\right)^{-1/z}J_{-1/z}\left(\frac{2\beta}{z}\right)$$

and

$$\operatorname{spec}(J) = \left\{z \in \mathbb{R}\setminus\{0\};\ J_{-1/z}\left(\frac{2\beta}{z}\right)=0\right\}\cup\{0\}.$$

For components of corresponding eigenvectors $v(z)$ one has

$$v_k(z) = \sqrt{k}\,J_{k-1/z}\left(\frac{2\beta}{z}\right),\quad k \in \mathbb{N}.$$

Example 23. Now we suppose in (46) that $c > 1$ and put $\alpha = 1/(c-1)$. Then $\lambda_n = (c-1)c^{-n+1}$ and $w_n = \beta\,(c-1)c^{-1/2}c^{(-n+1)/2}$. In order to simplify the expressions let us divide all matrix elements by the term $c-1$. Furthermore, we also replace the parameter β by $\beta c^{1/2}$ and use the substitution $c = 1/q$, with $0 < q < 1$. Thus for the matrix simplified in this way we have $\lambda_n = q^{n-1}$ and $w_n = \beta q^{(n-1)/2}$. Hence

$$J = \begin{pmatrix} 1 & \beta & & & \\ \beta & q & \beta\sqrt{q} & & \\ & \beta\sqrt{q} & q^2 & \beta q & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Eq. (47) then becomes

$$\mathfrak{F}\left(\left\{\frac{\gamma_n^{\,2}}{\lambda_n - z}\right\}_{n=r+1}^{\infty}\right) = \mathfrak{F}\left(\left\{\frac{\beta q^{(2n-3)/4}}{q^{n-1} - z}\right\}_{n=r+1}^{\infty}\right) = \sum_{s=0}^{\infty}(-1)^s\,\frac{q^{s(s-1)}}{(q;q)_s(q^r/z;q)_s}\left(\frac{q^{r/2}\beta}{z}\right)^{2s}$$
$$= {}_0\phi_1\left(;\frac{q^r}{z};q,-\frac{q^r\beta^2}{z^2}\right),\ r \in \mathbb{Z}_+. \tag{50}$$

Thus we get

$$\operatorname{spec}(J) = \left\{z \in \mathbb{R}\setminus\{0\};\ \left(\frac{1}{z};q\right)_{\infty}\ {}_0\phi_1\left(;\frac{1}{z};q,-\frac{\beta^2}{z^2}\right) = 0\right\} \cup \{0\}.$$

The kth entry of an eigenvector $v(z)$ corresponding to a nonzero point of the spectrum z may be written in the form

$$v_k(z) = q^{(k-1)(k-2)/4}\left(\frac{\beta}{z}\right)^{k-1}\left(\frac{q^k}{z};q\right)_{\infty}\ {}_0\phi_1\left(;\frac{q^k}{z};q,-\frac{q^k\beta^2}{z^2}\right),\quad k \in \mathbb{N}.$$

Further we shortly discuss two examples of Jacobi matrices with zero diagonal and $w \in \ell^2(\mathbb{N})$. Such a Jacobi matrix represents a compact operator (even Hilbert–Schmidt). The characteristic function is an even function,

$$\mathfrak{F}\left(\left\{\frac{\gamma_n^{\,2}}{z}\right\}_{n=1}^{\infty}\right) = \sum_{m=0}^{\infty}\frac{(-1)^m}{z^{2m}}\sum_{k_1=1}^{\infty}\sum_{k_2=k_1+2}^{\infty}\cdots\sum_{k_m=k_{m-1}+2}^{\infty} w_{k_1}^2 w_{k_2}^2\cdots w_{k_m}^2.$$

Hence the spectrum of J is symmetric with respect to the origin.

Though 0 always belongs to the spectrum of a compact Jacobi operator, one may ask under which conditions 0 is even an eigenvalue (necessarily simple). An answer can be deduced directly from the eigenvalue equation (11). One immediately finds that any eigenvector x must satisfy $x_{2k} = 0$ and $x_{2k-1} = (-1)^{k+1}x_1/\gamma_{2k-1}$, $k \in \mathbb{N}$. Consequently, zero is a simple eigenvalue of J iff

$$\sum_{k=1}^{\infty}\frac{1}{\gamma_{2k-1}^{\,2}} = \sum_{k=1}^{\infty}\prod_{j=1}^{k-1}\left(\frac{w_{2j-1}}{w_{2j}}\right)^2 < \infty. \tag{51}$$

Example 24. Let $\lambda_n = 0$ and $w_n = 1/\sqrt{(n+\alpha)(n+\alpha+1)}$, $n \in \mathbb{N}$, where $\alpha > -1$ is fixed. According to (15) one has, for $k \in \mathbb{Z}_+$,

$$\mathfrak{F}\left(\left\{\frac{\gamma_n^{\,2}}{z}\right\}_{n=k+1}^{\infty}\right) = \mathfrak{F}\left(\left\{\frac{1}{z(\alpha+n)}\right\}_{n=k+1}^{\infty}\right) = \Gamma(\alpha+k+1)z^{\alpha+k}J_{\alpha+k}\left(\frac{2}{z}\right).$$

Hence

$$\operatorname{spec}(J) = \left\{ z \in \mathbb{R}\setminus\{0\};\ J_\alpha\left(\frac{2}{z}\right) = 0 \right\} \cup \{0\}.$$

The kth entry of an eigenvector $v(z)$ corresponding to a nonzero eigenvalue z may be written in the form

$$v_k(z) = \sqrt{\alpha + k}\, J_{\alpha+k}\left(\frac{2}{z}\right), \quad k \in \mathbb{N}.$$

It is well known that for $\alpha \in 1/2 + \mathbb{Z}$, the Bessel function $J_\alpha(z)$ can be expressed as a linear combination of sine and cosine functions, the simplest cases being

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\, \cos(z), \quad J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\, \sin(z).$$

Thus for $\alpha = \pm 1/2$ the spectrum of J is described fully explicitly. In other cases the eigenvalues of J close to zero can approximately be determined from the known asymptotic formulas for large zeros of Bessel functions, see [1, Eq. (9.5.12)].

Example 25. Suppose that $0 < q < 1$ and put $\lambda_n = 0$, $w_n = q^{n-1}$, $n \in \mathbb{N}$. With the aid of (17) one derives that

$$\mathfrak{F}\left(\left\{ \frac{{\gamma_n}^2}{z} \right\}_{n=k+1}^{\infty}\right) = {}_0\phi_1(;0;q^2, -q^{2k}z^{-2}), \quad k \in \mathbb{Z}_+.$$

It follows that

$$\operatorname{spec}(J) = \{z \in \mathbb{R} \setminus \{0\};\ {}_0\phi_1(;0;q^2, -z^{-2}) = 0\} \cup \{0\}.$$

The components of an eigenvector $v(z)$ corresponding to an eigenvalue $z \neq 0$ may be expressed as

$$v_k(z) = q^{(k-1)(k-2)/2} z^{-k+1}\, {}_0\phi_1(;0;q^2, -q^{2k}z^{-2}), \quad k \in \mathbb{N}.$$

In this example as well as in the previous one, 0 belongs to the continuous spectrum of J since the condition (51) is not fulfilled.

Example 26. Finally we give another example of an unbounded Jacobi operator. It is obtained by modifying Example 23 in which we replace decreasing geometric sequences on the diagonals by increasing ones. Thus we put $\lambda_n = q^{-n+1}$ and $w_n = \beta q^{-(n-1)/2}$ where again $0 < q < 1$, $\beta > 0$. From (50) one infers that

$$\mathfrak{F}\left(\left\{ \frac{{\gamma_n}^2}{\lambda_n - z} \right\}_{n=r+1}^{\infty}\right) = \mathfrak{F}\left(\left\{ \frac{\beta q^{(2n-1)/4}}{z\,(q^{n-1} - z^{-1})} \right\}_{n=r+1}^{\infty}\right) ={}_0\, \phi_1(;q^r z;q, -q^{r+1}\beta^2),\ r \in \mathbb{Z}_+.$$

Thus one has

$$\operatorname{spec}(J) = \left\{ z \in \mathbb{R};\ (z\,;q)_\infty\ {}_0\phi_1(;z;q,-q\beta^2) = 0 \right\}.$$

The kth entry of an eigenvector $v(z)$ corresponding to an eigenvalue z can be written in the form

$$v_k(z) = q^{k(k+1)/4}\,(-\beta)^{k-1}\,(q^k z\,;q)_\infty\ {}_0\phi_1(;q^k z;q, -q^{k+1}\beta^2), \quad k \in \mathbb{N}.$$

4.2. Applications of Proposition 18

Here we apply identity (42) to the six examples of Jacobi matrices described above (though not in the same order). Without going into details, the final form of the presented identities is achieved after some simple substitutions. On the other hand, no attempt is made here to optimize the range of involved parameters; it is basically the same as it was for the Jacobi matrix in question.

(1) In case of Example 20 one gets

$$\sum_{k=1}^{\infty} J_{\nu+k}(x)^2 = \frac{x}{2}\left(J_{\nu+1}(x)\,\frac{\partial}{\partial\nu}J_\nu(x) - J_\nu(x)\,\frac{\partial}{\partial\nu}J_{\nu+1}(x)\right),$$

where $x > 0$ and $\nu \in \mathbb{C}$. This is in fact a particular case of (18).

(2) In case of Example 22 one gets

$$\sum_{k=1}^{\infty} k J_{-\alpha z+k}(z)^2 = \frac{z^2}{2}\left(J_{-\alpha z}(z)\,\frac{\mathrm{d}}{\mathrm{d}z}J_{-\alpha z+1}(z) - J_{-\alpha z+1}(z)\,\frac{\mathrm{d}}{\mathrm{d}z}J_{-\alpha z}(z)\right),$$

where $\alpha > 0$ and $z \in \mathbb{C}$.

(3) In case of Example 24 one gets

$$\sum_{k=1}^{\infty} (\alpha + k)J_{\alpha+k}(z)^2 = \frac{z^2}{2}\left(J_\alpha(z)\,\frac{\mathrm{d}}{\mathrm{d}z}J_{\alpha+1}(z) - J_{\alpha+1}(z)\,\frac{\mathrm{d}}{\mathrm{d}z}J_\alpha(z)\right),$$

where $\alpha > -1$ and $z \in \mathbb{C}$.

(4) In case of Example 23 one gets

$$\begin{aligned}
&\sum_{k=1}^{\infty} q^{(k-1)(k-2)/2}\,(tz^2)^{k-1}\left((q^k z;q)_\infty\ {}_0\phi_1(;q^k z;q,-q^k tz^2)\right)^2\\
&\quad= (qz;q)_\infty^2\left({}_0\phi_1(;z;q,-tz^2)\,{}_0\phi_1(;qz;q,-qtz^2)\right.\\
&\qquad\quad + z\,(z-1)\Big({}_0\phi_1(;qz;q,-qtz^2)\,\frac{\mathrm{d}}{\mathrm{d}z}\,{}_0\phi_1(;z;q,-tz^2)\\
&\qquad\quad\left. - {}_0\phi_1(;z;q,-tz^2)\,\frac{\mathrm{d}}{\mathrm{d}z}\,{}_0\phi_1(;qz;q,-qtz^2)\Big)\right),
\end{aligned}$$

where $0 < q < 1$, $t > 0$ and $z \in \mathbb{C}$.

(5) In case of Example 26 one gets

$$\begin{aligned}
&\sum_{k=1}^{\infty} q^{k(k-1)/2} t^{k-1}\,(q^k z;q)_\infty^2\ {}_0\phi_1(;q^k z;q,-q^k t)^2\\
&\quad= (qz;q)_\infty^2\left({}_0\phi_1(;z;q,-t)\,{}_0\phi_1(;qz;q,-qt) + (z-1)\Big({}_0\phi_1(;qz;q,-qt)\,\frac{\mathrm{d}}{\mathrm{d}z}\,{}_0\phi_1(;z;q,-t)\right.\\
&\qquad\quad\left. - {}_0\phi_1(;z;q,-t)\,\frac{\mathrm{d}}{\mathrm{d}z}\,{}_0\phi_1(;qz;q,-qt)\Big)\right),
\end{aligned}$$

where $0 < q < 1$, $t > 0$ and $z \in \mathbb{C}$.

(6) In case of Example 25 one gets

$$\begin{aligned}
\sum_{k=1}^{\infty} q^{(k-1)(k-2)/2}\,z^{k-1}\,{}_0\phi_1(;0;q,-q^k z)^2 &= {}_0\phi_1(;0;q,-z)\,{}_0\phi_1(;0;q,-qz)\\
&\quad + 2z\left({}_0\phi_1(;0;q,-z)\,\frac{\mathrm{d}}{\mathrm{d}z}\,{}_0\phi_1(;0;q,-qz) - {}_0\phi_1(;0;q,-qz)\,\frac{\mathrm{d}}{\mathrm{d}z}\,{}_0\phi_1(;0;q,-z)\right),
\end{aligned}$$

where $0 < q < 1$ and $z \in \mathbb{C}$.

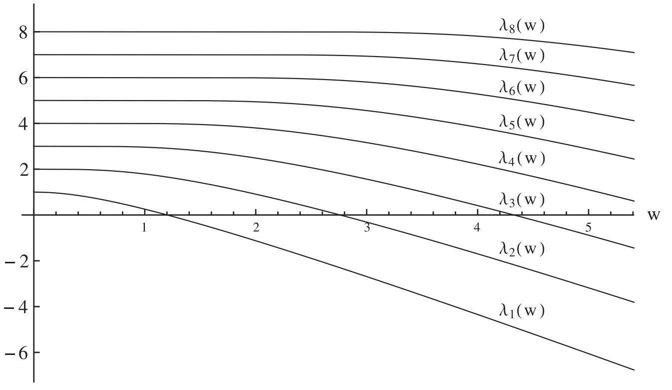

Fig. 1. Several first eigenvalues $\lambda_s(w)$ as functions of the parameter w for the Jacobi operator $J = J(w)$ from Example 20, with $\alpha = 1$.

4.3. A Jacobi matrix with a linear diagonal and constant parallels

Here we discuss in somewhat more detail Example 20 concerned with a Jacobi matrix having a linear diagonal and constant parallels. For simplicity and with no loss of generality we put $\alpha = 1$. Our goal is to study how the spectrum of the Jacobi operator J depends on the real parameter w. We treat J as a linear operator-valued function, $J = J(w)$. One may write $J(w) = L + wT$ where L is the diagonal operator with the diagonal sequence $\lambda_n = n$, $\forall n \in \mathbb{N}$, and T has all units on the parallels neighboring to the diagonal and all zeros elsewhere. Notice that $\|T\| \leqslant 2$.

We know that $J(w)$ has, for all $w \in \mathbb{R}$, a semibounded simple discrete spectrum. Let us enumerate the eigenvalues in ascending order as $\lambda_s(w)$, $s \in \mathbb{N}$. From the standard perturbation theory one infers that all functions $\lambda_s(w)$ are real analytic, with $\lambda_s(0) = s$. Moreover, the functions $\lambda_s(w)$ are also known to be even and so we restrict w to the positive real half-axis. In Example 20 we learned that for every $w > 0$ fixed, the roots of the equation $J_{-z}(2w) = 0$ are exactly $\lambda_s(w)$, $s \in \mathbb{N}$. Several first eigenvalues $\lambda_s(w)$ as functions of w are depicted in Fig. 1.

The problem of roots of a Bessel function depending on the order, with the argument being fixed, has a long history. Here we make use of some results derived in the classical paper [5]. Some numerical aspects of the problem are discussed in [10]. For comparatively recent results in this domain one may consult [13] and references therein.

In [5] it is shown that

$$\frac{d\lambda_s(w)}{dw} = -\left(2w\int_0^\infty K_0(4w\sinh(t))\exp\left(2\lambda_s(w)t\right)dt\right)^{-1}.$$

From this relation one immediately deduces a few basic qualitative properties of the spectrum of the Jacobi operator.

Proposition 27 (M.J. Coulomb)**.** *The spectrum* $\{\lambda_s(w);\ s \in \mathbb{N}\}$ *of the above introduced Jacobi operator, depending on the parameter* $w \geqslant 0$*, has the following properties.*

(i) For every $s \in \mathbb{N}$*, the function* $\lambda_s(w)$ *is strictly decreasing.*
(ii) If $r < s$ *then* $\lambda_r{}'(w) < \lambda_s{}'(w)$*.*
(iii) In particular, the distance between two neighboring eigenvalues $\lambda_{s+1}(w) - \lambda_s(w)$*,* $s \in \mathbb{N}$*, increases with increasing* w *and is always greater than or equal to* 1*, with the equality only for* $w = 0$*.*

Let us next check the asymptotic behavior of $\lambda_s(w)$ at infinity. The asymptotic expansion at infinity of the sth root $x = j_s(\nu)$ of the equation $J_\nu(x) = 0$ reads [1, Eq. (9.5.22)]

$$j_s(\nu) = \nu - 2^{-1/3}a_s\nu^{1/3} + O\left(\nu^{-1/3}\right) \text{ as } \nu \to +\infty,$$

where a_s is the sth negative zero of the Airy function $\mathrm{Ai}(x)$. From here one deduces that

$$\lambda_s(w) = -2w - a_s w^{1/3} + O\Big(w^{-1/3}\Big) \text{ as } w \to +\infty.$$

Concerning the asymptotic behavior of $\lambda_s(w)$ at $w = 0$, one may use the expression for the Bessel function as a power series and apply the substitution, $\lambda_s(w) = s - z(w)$, $s = 1, 2, 3, \ldots$. The solution $z = z(w)$, with $z(0) = 0$, is then defined implicitly near $w = 0$ by the equation

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+1-s+z(w))}\, w^{2m} = 0.$$

The computation is straightforward and based on the relation

$$\frac{1}{\Gamma(-m+z)} = (-1)^m m! \Big(z - \psi^{(0)}(m+1)\, z^2\Big) + O\Big(z^3\Big), \quad m = 0, 1, 2, 3, \ldots,$$

where $\psi^{(0)}$ is the polygamma function. This way one derives that, as $w \to 0$,

$$\begin{aligned} \lambda_1(w) &= 1 - w^2 + \frac{1}{2}w^4 + O\Big(w^6\Big), \\ \lambda_s(w) &= s - \frac{1}{(s-1)!s!}\, w^{2s} + \frac{2s}{(s-1)(s-1)!(s+1)!}\, w^{2s+2} + O\Big(w^{2s+4}\Big), \quad \text{for } s \geqslant 2. \end{aligned} \tag{52}$$

The same asymptotic formulas, as given in (52), can also be derived using the standard perturbation theory [12, Section II.2]. Alternatively, one may use equivalent formulas for coefficients of the perturbation series derived in [6,7] which are perhaps more convenient for this particular example.

The distance of $s \in \mathbb{N}$ to the rest of the spectrum of the diagonal operator L equals 1. The Kato–Rellich theorem tells us that there exists exactly one eigenvalue of $J(w)$ in the disk centered at s and of radius $1/2$ as long as $|w| < 1/4$. The explicit expression for the leading term in (52) suggests, however, that the eigenvalue $\lambda_s(w)$ may stay close to s on a much larger interval at least for high orders s. It turns out that actually $\lambda_s(w)$ is well approximated by this leading asymptotic term on an interval $[0, \beta_s)$, with $\beta_s \sim s/e$ for $s \gg 1$. A precise formulation is given in Proposition 30 below.

Denote by $y_k(\nu)$ the kth root of the Bessel function $Y_\nu(z)$, $k \in \mathbb{N}$. Let us put

$$\beta_s := \left(\frac{(s-1)!\, s!}{\pi}\right)^{1/(2s)}, \quad s \in \mathbb{N}. \tag{53}$$

In order to avoid confusion with the usual notation for Bessel functions, the nth truncation of $J(w)$ is now denoted by a bold letter as $\boldsymbol{J}_n(w)$.

Lemma 28. *The following estimate holds true:*

$$\beta_s < \frac{1}{2}\, y_1\Big(s - \frac{1}{2}\Big), \quad \forall s \in \mathbb{N}. \tag{54}$$

Proof. One knows that $\nu < y_1(\nu)$, $\forall \nu \geqslant 0$ [1, Eq. (9.5.2)], and in particular this is true for $\nu = s - 1/2$, $s \in \mathbb{N}$. On the other hand, the sequence

$$\phi_s = \frac{\pi}{(s-1)!\, s!} \left(s - \frac{1}{2}\right)^{2s} 2^{-2s}$$

is readily verified to be increasing, and $1 < \phi_4$. This shows (54) for all $s \geqslant 4$. The cases $s = 1, 2, 3$ can be checked numerically. □

Lemma 29. *Denote by $\chi_n(w;z)$ the characteristic polynomial of the nth truncation $\boldsymbol{J}_n(w)$ of the Jacobi matrix $J(w)$. If $0 \leqslant w \leqslant \beta_s$ for some $s \in \mathbb{N}$ then $z = \lambda_s(w)$ solves the equation*

$$\chi_{2s-1}(w;z) - \frac{wJ_{2s-z}(2w)}{J_{2s-1-z}(2w)}\,\chi_{2s-2}(w;z) = 0. \tag{55}$$

Proof. Let $\{e_k;\ k \in \mathbb{N}\}$ be the canonical basis in $\ell^2(\mathbb{N})$. Let us split the Hilbert space into the orthogonal sum

$$\ell^2(\mathbb{N}) = \operatorname{span}\{e_k;\ 1 \leqslant k \leqslant 2s-1\} \oplus \overline{\operatorname{span}\{e_k;\ 2s \leqslant k\}}.$$

Then $J(w)$ splits correspondingly into four matrix blocks,

$$J(w) = \begin{pmatrix} A(w) & B(w) \\ C(w) & D(w) \end{pmatrix}.$$

Here $A(w) = \boldsymbol{J}_{2s-1}(w)$, $D(w) = J(w) + (2s-1)I$, the block $B(w)$ has just one nonzero element in the lower left corner and $C(w)$ is transposed to $B(w)$.

By the min max principle, the minimal eigenvalue of $D(w)$ is greater than or equal to $2s - 2w$. Since $\lambda_s(w) \leqslant s$ one can estimate

$$\min \operatorname{spec}(D(w)) - \lambda_s(w) = \lambda_1(w) - \lambda_s(w) + 2s - 1 \geqslant s - 2w.$$

We claim that $0 \leqslant w \leqslant \beta_s$ implies $\min\operatorname{spec}(D(w)) - \lambda_s(w) > 0$. This is obvious for $s = 1$. For $s \geqslant 2$, it suffices to show that $\beta_s < s/2$. This can be readily done by induction in s. Hence, under this assumption, $D(w) - z$ is invertible for $z = \lambda_s(w)$.

Solving the eigenvalue equation $J(w)\boldsymbol{v} = z\boldsymbol{v}$ one can write the eigenvector as a sum $\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{y}$, in accordance with the above orthogonal decomposition. If $D(w) - z$ is invertible then the eigenvalue equation reduces to the finite-dimensional linear system

$$\left(A - z - B(D-z)^{-1}C\right)\boldsymbol{x} = 0. \tag{56}$$

One observes that $B(D-z)^{-1}C$ has all entries equal to zero except of the element in the lower right corner. Using (35) and (15) one finds that this nonzero entry equals

$$wJ_{2s-z}(2w)J_{2s-1-z}(2w)^{-1}.$$

Eq. (55) then immediately follows from (56). □

Proposition 30. *For $s \in \mathbb{N}$ and $0 \leqslant w \leqslant \beta_s$, with β_s given in (53), one has*

$$0 \leqslant s - \lambda_s(w) \leqslant \frac{1}{\pi}\arcsin\!\left(\frac{\pi w^{2s}}{(s-1)!s!}\right).$$

Proof. We start from Lemma 29 and Eq. (55). Let us recall from [14, Proposition 30] that

$$\det(\boldsymbol{J}_{2s-1}(w) - s - x) = (-1)^s x \sum_{k=0}^{s-1} \binom{2s-k-1}{k} w^{2k} \prod_{j=1}^{s-k-1}\left(j^2 - x^2\right).$$

Hence if $z \in \mathbb{R}$, $|z - s| \leqslant 1$, then

$$|\chi_{2s-1}(w;z)| \geqslant |z-s| \prod_{j=1}^{s-1}\left(j^2 - (z-s)^2\right). \tag{57}$$

Since $J_{-s+1/2}(x) = (-1)^s Y_{s-1/2}(x)$ it is true that for $2w = y_1(s-1/2)$ one has $\lambda_s(w) = s-1/2$. Because of monotonicity of $\lambda_s(w)$ one makes the following observation: if $2w \leqslant y_1(s-1/2)$ then $s \geqslant \lambda_s(w) \geqslant s-1/2$.

By Lemma 28, if $w \leqslant \beta_s$ then $2w \leqslant y_1(s-1/2)$, and so the estimate (57) applies for $z = \lambda_s(w)$. Using also Proposition 4 to express $\chi_{2s-2}(w;z)$ one derives from (55) that

$$|\lambda - s| \leqslant w \left|\frac{J_{2s-\lambda}(2w)}{J_{2s-1-\lambda}(2w)}\right| \left|\frac{s-\lambda}{2s-1-\lambda}\,\mathfrak{F}\!\left(\frac{w}{1-\lambda},\frac{w}{2-\lambda},\ldots,\frac{w}{2s-2-\lambda}\right)\right| \tag{58}$$

where as well as in the remainder of the proof we write for short λ instead of $\lambda_s(w)$.

Starting from the equation

$$\mathfrak{F}\!\left(\frac{w}{1-\lambda},\frac{w}{2-\lambda},\frac{w}{3-\lambda},\ldots\right) = 0, \quad \text{with } \lambda = \lambda_s(w),$$

and using (4), (15) one derives that, for all $k \in \mathbb{Z}_+$,

$$\left(\prod_{j=1}^{k}(j-\lambda)\right)\mathfrak{F}\!\left(\frac{w}{1-\lambda},\frac{w}{2-\lambda},\ldots,\frac{w}{k-\lambda}\right) = w^k\,\frac{J_{k+1-\lambda}(2w)}{J_{1-\lambda}(2w)}\,. \tag{59}$$

Combining (58) and (59) we get (knowing that $0 \leqslant s-\lambda \leqslant 1/2$ for $\lambda = \lambda_s(w)$)

$$s-\lambda \leqslant w^{2s-1}\left|\left(\prod_{j=1}^{s-1}(\lambda-j)\,\prod_{j=1}^{s-1}(j+s-\lambda)\right)^{-1}\frac{J_{2s-\lambda}(2w)}{J_{1-\lambda}(2w)}\right|.$$

But notice that, by expressing the sine function as an infinite product,

$$\prod_{j=1}^{s-1}(\lambda-j)\,\prod_{j=1}^{s-1}(j+s-\lambda) = ((s-1)!)^2\,\frac{\sin(\pi(s-\lambda))}{\pi(s-\lambda)}\left(\prod_{j=s}^{\infty}\left(1-\frac{(s-\lambda)^2}{j^2}\right)\right)^{-1}.$$

Hence

$$\sin(\pi(s-\lambda)) \leqslant \pi\,\frac{w^{2s-1}}{((s-1)!)^2}\left|\frac{J_{2s-\lambda}(2w)}{J_{1-\lambda}(2w)}\right|.$$

From (26) one gets, while taking into account that $J_{-\lambda}(2w) = 0$,

$$\sin(\pi\lambda) = \pi w J_{\lambda}(2w) J_{1-\lambda}(2w).$$

In addition, one knows that

$$|J_\nu(x)| \leqslant \frac{1}{\Gamma(\nu+1)}\left|\frac{x}{2}\right|^{\nu}$$

provided $\nu > -1/2$ and $x \in \mathbb{R}$ [1, Eq. (9.1.62)]. Hence

$$\sin(\pi(s-\lambda))^2 \leqslant \frac{\pi^2 w^{4s}}{((s-1)!)^2\,\Gamma(2s+1-\lambda)\Gamma(\lambda+1)}\,.$$

Writing $\lambda = s-\zeta$, with $0 \leqslant \zeta \leqslant 1/2$, one has

$$\frac{d}{d\zeta}\log\!\left(\frac{1}{\Gamma(s+\zeta+1)\Gamma(s-\zeta+1)}\right) = -\psi^{(0)}(s+\zeta+1) + \psi^{(0)}(s-\zeta+1) < 0.$$

Thus we arrive at the estimate

$$\sin(\pi(s-\lambda))^2 \leqslant \frac{\pi^2 w^{4s}}{((s-1)!)^2\,(s!)^2}\,.$$

To complete the proof it suffices to notice that the assumption $w \leqslant \beta_s$ means nothing but $w^{2s} \big/ ((s-1)!s!) \leqslant 1$, and it also implies that $0 \leqslant s - \lambda \leqslant 1/2$. $\square$

Acknowledgments

The authors wish to acknowledge gratefully partial support from the following grants: Grant No. 201/09/0811 of the Czech Science Foundation (P.Š.) and Grant No. LC06002 of the Ministry of Education of the Czech Republic (F.Š.).

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Linear Algebra and its Applications ••• (••••) •••–•••

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Special functions and spectrum of Jacobi matrices

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ARTICLE INFO

Article history:
Received 10 December 2012
Accepted 20 June 2013
Available online xxxx
Submitted by V. Mehrmann

MSC:
47B36
39A70
47A10
33D15

Keywords:
Infinite Jacobi matrix
Spectral problem
Special functions

ABSTRACT

Several examples of Jacobi matrices with an explicitly solvable spectral problem are worked out in detail. In all discussed cases the spectrum is discrete and coincides with the set of zeros of a special function. Moreover, the components of corresponding eigenvectors are expressible in terms of special functions as well. Our approach is based on a recently developed formalism providing us with explicit expressions for the characteristic function and eigenvectors of Jacobi matrices. This is done under an assumption of a simple convergence condition on matrix entries. Among the treated special functions there are regular Coulomb wave functions, confluent hypergeometric functions, q-Bessel functions and q-confluent hypergeometric functions. In addition, in the case of q-Bessel functions, we derive several useful identities.

1. Introduction

Special functions usually depend on a complex variable and an additional parameter called order. Typically, they obey a three-term recurrence relation with respect to the order. This is the basis of their relationship to Jacobi (tridiagonal) matrices. In more detail, the zeros of an appropriate special function are directly related to eigenvalues of a Jacobi matrix operator, and components of corresponding eigenvectors can be expressed in terms of special functions as well. One may also say that the characteristic function of the (infinite) matrix operator in question is written explicitly in terms

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0024-3795/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.laa.2013.06.024

Please cite this article in press as: F. Štampach, P. Šťovíček, Special functions and spectrum of Jacobi matrices, Linear Algebra Appl. (2013), http://dx.doi.org/10.1016/j.laa.2013.06.024

of special functions. Particularly, Gard and Zakrajšek reported in [3] a matrix equation approach for numerical computation of the zeros of Bessel functions; on this point see also [8]. In [7], Ikebe then showed that the same approach was applicable, too, for determining the zeros of regular Coulomb wave functions. In practical computations, an infinite tridiagonal matrix should be truncated which raises a question of error estimates. Such an analysis has been carried out in [9,11].

In [13], the authors initiated an approach to a class of Jacobi matrices with discrete spectra. The basic tool is a function $\mathfrak{F}$ depending on a countable number of variables. In more detail, we define $\mathfrak{F}: D \to \mathbb{C}$,

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1}, \tag{1}$$

where the set D is formed by complex sequences $x = \{x_k\}_{k=1}^{\infty}$ obeying

$$\sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty. \tag{2}$$

For a finite number of variables we identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$. By convention, we put $\mathfrak{F}(\emptyset) = 1$ where $\emptyset$ is the empty sequence. Notice that the domain D is not a linear space though $\ell^2(\mathbb{N}) \subset D$.

In the same paper, two examples are given of special functions expressed directly in terms of $\mathfrak{F}$. The first example is concerned with Bessel functions of the first kind. For $w, \nu \in \mathbb{C}$, $\nu \notin -\mathbb{N}$, one has

$$J_\nu(2w) = \frac{w^\nu}{\Gamma(\nu+1)} \mathfrak{F}\left(\left\{ \frac{w}{\nu+k} \right\}_{k=1}^{\infty} \right). \tag{3}$$

Secondly, the formula

$$\mathfrak{F}\left(\left\{t^{k-1} w\right\}_{k=1}^{\infty}\right) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{t^{m(2m-1)} w^{2m}}{(1-t^2)(1-t^4)\dots(1-t^{2m})} = {}_0\phi_1\left(; 0; t^2, -tw^2\right) \tag{4}$$

holds for $t, w \in \mathbb{C}$, $|t| < 1$. Here ${}_0\phi_1$ is the basic hypergeometric series (also called q-hypergeometric series) being defined by

$${}_0\phi_1(; b; q, z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{(q;q)_k (b;q)_k} z^k,$$

and

$$(a; q)_k = \prod_{j=0}^{k-1} \left(1 - aq^j\right), \quad k = 0, 1, 2, \dots,$$

is the q-Pochhammer symbol, see [4].

In [14], the approach is further developed and a construction in terms of $\mathfrak{F}$ of the characteristic function of certain Jacobi matrices is established. As an application, a series of examples of Jacobi matrices with explicitly expressible characteristic functions is described. The method works well for Jacobi matrices obeying a simple convergence condition imposed on the matrix entries which is in principle dictated by condition (2) characterizing the domain of $\mathfrak{F}$.

In the current paper, we present more interesting examples of Jacobi matrices whose spectrum coincides with the set of zeros of a particular special function. As a byproduct, we provide examples of sequences on which the function $\mathfrak{F}$ can be evaluated explicitly. The paper is organized as follows. In Section 2 we recall from [13,14] some basic facts needed in the current paper. Section 3 is concerned with regular Coulomb wave functions. Here we reconsider the example due to Ikebe while using our formalism. In Section 4 we deal with confluent hypergeometric functions. Here we go beyond the above mentioned convergence condition (see (14) below) which is violated in this example. Section 5 is concerned with q-Bessel functions. This example is particular in that respect that the constructed second order difference operator is bilateral, i.e. it acts in $\ell^2(\mathbb{Z})$ rather than in $\ell^2(\mathbb{N})$. We first derive several useful properties of q-Bessel functions and then we use this knowledge to solve the spectral problem for the bilateral difference operator fully explicitly. Finally, another interplay between special functions, namely q-confluent hypergeometric functions, and an appropriate Jacobi matrix is demonstrated in Section 6.

2. Preliminaries

Let us recall from [13,14] some basic facts concerning the function $\mathfrak{F}$ and its properties and possible applications. First of all, quite crucial property of $\mathfrak{F}$ is the recurrence rule

$$\mathfrak{F}\big(\{x_k\}_{k=1}^{\infty}\big) = \mathfrak{F}\big(\{x_k\}_{k=2}^{\infty}\big) - x_1 x_2 \mathfrak{F}\big(\{x_k\}_{k=3}^{\infty}\big). \tag{5}$$

In addition, $\mathfrak{F}(x_1, x_2, \ldots, x_{k-1}, x_k) = \mathfrak{F}(x_k, x_{k-1}, \ldots, x_2, x_1)$. Furthermore, for $x \in D$,

$$\lim_{n\to\infty} \mathfrak{F}\big(\{x_k\}_{k=n}^{\infty}\big) = 1 \quad \text{and} \quad \lim_{n\to\infty} \mathfrak{F}(x_1, x_2, \ldots, x_n) = \mathfrak{F}(x). \tag{6}$$

Let us note that the definition of $\mathfrak{F}$ naturally extends to more general ranges of indices. For any sequence $\{x_n\}_{n=N_1}^{N_2}$, $N_1, N_2 \in \mathbb{Z} \cup \{-\infty, +\infty\}$, $N_1 \leqslant N_2 + 1$ (if $N_1 = N_2 + 1 \in \mathbb{Z}$ then the sequence is considered as empty) such that $\sum_{k=N_1}^{N_2-1} |x_k x_{k+1}| < \infty$ one defines

$$\mathfrak{F}\big(\{x_k\}_{k=N_1}^{N_2}\big) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k \in \mathcal{I}(N_1, N_2, m)} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \ldots x_{k_m} x_{k_m+1}$$

where

$$\mathcal{I}(N_1, N_2, m) = \big\{k \in \mathbb{Z}^m;\ k_j + 2 \leqslant k_{j+1} \text{ for } 1 \leqslant j \leqslant m-1,\ N_1 \leqslant k_1,\ k_m < N_2\big\}.$$

With this definition, one has the generalized recurrence rule

$$\mathfrak{F}\big(\{x_k\}_{k=N_1}^{N_2}\big) = \mathfrak{F}\big(\{x_k\}_{k=N_1}^{n}\big)\mathfrak{F}\big(\{x_k\}_{k=n+1}^{N_2}\big) - x_n x_{n+1} \mathfrak{F}\big(\{x_k\}_{k=N_1}^{n-1}\big)\mathfrak{F}\big(\{x_k\}_{k=n+2}^{N_2}\big) \tag{7}$$

provided $n \in \mathbb{Z}$ satisfies $N_1 \leqslant n < N_2$.

Let us denote by J an infinite Jacobi matrix of the form

$$J = \begin{pmatrix} \lambda_1 & w_1 & & & \\ w_1 & \lambda_2 & w_2 & & \\ & w_2 & \lambda_3 & w_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{8}$$

where $\{w_n;\ n \in \mathbb{N}\} \subset \mathbb{C}\backslash\{0\}$ and $\{\lambda_n;\ n \in \mathbb{N}\} \subset \mathbb{C}$. In all examples treated in the current paper, the matrix J determines in a natural way a unique closed operator in $\ell^2(\mathbb{N})$ (in other words, $J_{\min} = J_{\max}$; see, for instance, [2]). If the matrix is real then the operator is self-adjoint. For the sake of simplicity of the notation the operator is again denoted by J. One notes, too, that all eigenvalues of J, if any, are simple since any solution $\{x_k\}$ of the formal eigenvalue equation

$$\lambda_1 x_1 + w_1 x_2 = z x_1, \qquad w_{k-1} x_{k-1} + \lambda_k x_k + w_k x_{k+1} = z x_k \quad \text{for } k \geqslant 2, \tag{9}$$

with $z \in \mathbb{C}$, is unambiguously determined by its first component x_1.

Let $\{\gamma_k\}$ be any sequence fulfilling $\gamma_k \gamma_{k+1} = w_k$, $k \in \mathbb{N}$. If J_n is the principal $n \times n$ submatrix of J then

$$\det(J_n - z I_n) = \left(\prod_{k=1}^{n} (\lambda_k - z)\right) \mathfrak{F}\left(\frac{\gamma_1^2}{\lambda_1 - z}, \frac{\gamma_2^2}{\lambda_2 - z}, \ldots, \frac{\gamma_n^2}{\lambda_n - z}\right). \tag{10}$$

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The function $\mathfrak{F}$ can also be applied to bilateral difference equations. Suppose that sequences $\{w_n\}_{n=-\infty}^{\infty}$ and $\{\zeta_n\}_{n=-\infty}^{\infty}$ are such that $w_n \neq 0$, $\zeta_n \neq 0$ for all n, and

$$\sum_{k=-\infty}^{\infty} \left| \frac{w_k^2}{\zeta_k \zeta_{k+1}} \right| < \infty.$$

Consider the difference equation

$$w_n x_{n+1} - \zeta_n x_n + w_{n-1} x_{n-1} = 0, \quad n \in \mathbb{Z}. \tag{11}$$

Define the sequence $\{\mathcal{P}_n\}_{n\in\mathbb{Z}}$ by $\mathcal{P}_0 = 1$ and $\mathcal{P}_{n+1} = (w_n/\zeta_{n+1})\mathcal{P}_n$ for all n. The sequence $\{\gamma_n\}_{n\in\mathbb{Z}}$ is again defined by the rule $\gamma_n\gamma_{n+1} = w_n$ for all $n \in \mathbb{Z}$, and any choice of $\gamma_1 \neq 0$. Then the sequences $\{f_n\}_{n\in\mathbb{Z}}$ and $\{g_n\}_{n\in\mathbb{Z}}$,

$$f_n = \mathcal{P}_n \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\zeta_k}\right\}_{k=n+1}^{\infty}\right), \qquad g_n = \frac{1}{w_{n-1}\mathcal{P}_{n-1}} \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\zeta_k}\right\}_{k=-\infty}^{n-1}\right), \tag{12}$$

represent two solutions of the bilateral difference equation (11). With the usual definition of the Wronskian, $\mathcal{W}(f,g) = w_n(f_n g_{n+1} - f_{n+1} g_n)$, one has

$$\mathcal{W}(f,g) = \mathfrak{F}\left(\left\{\gamma_n^2/\zeta_n\right\}_{n=-\infty}^{\infty}\right). \tag{13}$$

For $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ let us denote $\mathbb{C}_0^{\lambda} := \mathbb{C} \setminus \overline{\{\lambda_n;\ n \in \mathbb{N}\}}$, and let $\mathrm{der}(\lambda)$ stand for the set of all finite accumulation points of the sequence λ. Further, for $z \in \mathbb{C} \setminus \mathrm{der}(\lambda)$, let $r(z)$ be the number of members of the sequence λ coinciding with z (hence $r(z) = 0$ for $z \in \mathbb{C}_0^{\lambda}$). We assume everywhere that $\mathbb{C}_0^{\lambda} \neq \emptyset$.

Suppose

$$\sum_{n=1}^{\infty} \left| \frac{w_n^2}{(\lambda_n - z_0)(\lambda_{n+1} - z_0)} \right| < \infty \tag{14}$$

for at least one $z_0 \in \mathbb{C}_0^{\lambda}$. Then (14) is true for all $z_0 \in \mathbb{C}_0^{\lambda}$ [14]. In particular, the following definitions make good sense. For $k \in \mathbb{Z}_+$ ($\mathbb{Z}_+$ standing for nonnegative integers) and $z \in \mathbb{C} \setminus \mathrm{der}(\lambda)$ put

$$\xi_k(z) := \lim_{u \to z} (u - z)^{r(z)} \left(\prod_{l=1}^{k} \frac{w_{l-1}}{u - \lambda_l} \right) \mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l - u}\right\}_{l=k+1}^{\infty}\right). \tag{15}$$

Here one sets $w_0 := 1$. Particularly, for $z \in \mathbb{C}_0^{\lambda}$, one simply has

$$\xi_k(z) = \left(\prod_{l=1}^{k} \frac{w_{l-1}}{z - \lambda_l} \right) \mathfrak{F}\left(\left\{\frac{\gamma_l^2}{\lambda_l - z}\right\}_{l=k+1}^{\infty}\right) \tag{16}$$

(this is in fact nothing but the solution f_n from (12) restricted to nonnegative indices). All functions $\xi_k(z)$, $k \in \mathbb{Z}_+$, are holomorphic on $\mathbb{C}_0^{\lambda}$ and extend to meromorphic functions on $\mathbb{C} \setminus \mathrm{der}(\lambda)$, with poles at the points $z = \lambda_n$, $n \in \mathbb{N}$, and with orders of the poles not exceeding $r(z)$. This justifies definition (15).

The sequence $\{\xi_k(z)\}$ solves the second order difference equation

$$w_{k-1} x_{k-1} + (\lambda_k - z) x_k + w_k x_{k+1} = 0 \quad \text{for } k \geqslant 2. \tag{17}$$

In addition, $(\lambda_1 - z)\xi_1(z) + w_1 \xi_2(z) = 0$ provided $\xi_0(z) = 0$. Proceeding this way one can show [14, Section 3.3] that if $\xi_0(z)$ does not vanish identically on $\mathbb{C}_0^{\lambda}$ then

$$\mathrm{spec}(J) \setminus \mathrm{der}(\lambda) = \left\{ z \in \mathbb{C} \setminus \mathrm{der}(\lambda);\ \xi_0(z) = 0 \right\}. \tag{18}$$

Moreover, if $z \in \mathbb{C} \setminus \mathrm{der}(\lambda)$ is an eigenvalue of J then $\xi(z) := (\xi_1(z), \xi_2(z), \xi_3(z), \ldots)$ is a corresponding eigenvector. If J is real and $z \in \mathbb{R} \cap \mathbb{C}_0^{\lambda}$ is an eigenvalue then $\|\xi(z)\|^2 = \xi_0'(z)\xi_1(z)$. Finally, let us remark that the Weyl m-function can be expressed as $m(z) = -\xi_1(z)/\xi_0(z)$.

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Lemma 1. *For $p, r, \ell \in \mathbb{N}$, $1 < p \leqslant r+1 \leqslant \ell$, and any ℓ-tuple of complex numbers x_j, $1 \leqslant j \leqslant \ell$, it holds true that*

$$\mathfrak{F}\big(\{x_j\}_{j=1}^{r}\big)\mathfrak{F}\big(\{x_j\}_{j=p}^{\ell}\big) - \mathfrak{F}\big(\{x_j\}_{j=1}^{\ell}\big)\mathfrak{F}\big(\{x_j\}_{j=p}^{r}\big) = \left(\prod_{j=p-1}^{r} x_j x_{j+1}\right)\mathfrak{F}\big(\{x_j\}_{j=1}^{p-2}\big)\mathfrak{F}\big(\{x_j\}_{j=r+2}^{\ell}\big).$$

If $p, r \in \mathbb{N}$, $1 < p \leqslant r+1$, and a complex sequence $\{x_j\}_{j=1}^{\infty}$ fulfills (2) then

$$\mathfrak{F}\big(\{x_j\}_{j=1}^{r}\big)\mathfrak{F}\big(\{x_j\}_{j=p}^{\infty}\big) - \mathfrak{F}\big(\{x_j\}_{j=p}^{r}\big)\mathfrak{F}\big(\{x_j\}_{j=1}^{\infty}\big) = \left(\prod_{j=p-1}^{r} x_j x_{j+1}\right)\mathfrak{F}\big(\{x_j\}_{j=1}^{p-2}\big)\mathfrak{F}\big(\{x_j\}_{j=r+2}^{\infty}\big).$$

Proof. Suppose $\{z_j\}_{j=-\infty}^{\infty}$ is any nonvanishing bilateral complex sequence. In [14, Section 2] it is shown (under somewhat more general circumstances) that there exists an antisymmetric matrix $\mathfrak{J}(m,n)$, $m, n \in \mathbb{Z}$, such that

$$\mathfrak{J}(m,n) = \left(\prod_{j=m+1}^{n-1} \frac{1}{z_j}\right)\mathfrak{F}(z_{m+1}, z_{m+2}, \ldots, z_{n-1})$$

for $m < n$, and $\mathfrak{J}(m,k)\mathfrak{J}(n,\ell) - \mathfrak{J}(m,\ell)\mathfrak{J}(n,k) = \mathfrak{J}(m,n)\mathfrak{J}(k,\ell)$ for all $m, n, k, \ell \in \mathbb{Z}$. In particular, assuming that indices p, r, ℓ obey the restrictions from the lemma,

$$\mathfrak{J}(0, r+1)\mathfrak{J}(p-1, \ell+1) - \mathfrak{J}(0, \ell+1)\mathfrak{J}(p-1, r+1) = \mathfrak{J}(0, p-1)\mathfrak{J}(r+1, \ell+1).$$

After obvious cancellations in this equation one can drop the assumption on nonvanishing sequences. The lemma readily follows. □

Lemma 2. *Let $x = \{x_n\}_{n=1}^{\infty}$ be a nonvanishing complex sequence satisfying (2). Then*

$$F_n := \mathfrak{F}\big(\{x_k\}_{k=n}^{\infty}\big), \quad n \in \mathbb{N}, \tag{19}$$

is the unique solution of the second order difference equation

$$F_n - F_{n+1} + x_n x_{n+1} F_{n+2} = 0, \quad n \in \mathbb{N}, \tag{20}$$

satisfying the boundary condition $\lim_{n\to\infty} F_n = 1$.

Proof. The sequence $\{F_n\}$ defined in (19) fulfills all requirements, as stated in (5) and (6). It suffices to show that there exists another solution $\{G_n\}$ of (20) such that $\lim_{n\to\infty} G_n = \infty$. If $F_1 = \mathfrak{F}(x) \neq 0$ then $\{G_n\}$ can be defined by $G_1 = 0$ and

$$G_n = \left(\prod_{k=1}^{n-2} \frac{1}{x_k x_{k+1}}\right)\mathfrak{F}\big(\{x_k\}_{k=1}^{n-2}\big), \quad \text{for } n \geqslant 2. \tag{21}$$

If $F_1 = 0$ then necessarily $F_2 \neq 0$ since otherwise (20) would imply $F_n = 0$ for all n which is impossible. Hence in that case one can shift the index by 1, i.e. one can put $G_2 = 0$,

$$G_n = \left(\prod_{k=2}^{n-2} \frac{1}{x_k x_{k+1}}\right)\mathfrak{F}\big(\{x_k\}_{k=2}^{n-2}\big), \quad \text{for } n \geqslant 3$$

(and $G_1 = -x_1 x_2$). In any case, F_n is the minimal solution of (20), see [5]. □

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Remark 3. If $\mathfrak{F}(x) = 0$ then $\mathfrak{F}(x_1, x_2, \ldots, x_n)$ tends to 0 as $n \to \infty$ quite rapidly, more precisely,

$$\mathfrak{F}(x_1, x_2, \ldots, x_{n+1}) = o\left(\prod_{k=1}^{n} x_k x_{k+1}\right), \quad \text{as } n \to \infty. \tag{22}$$

In fact, if $\mathfrak{F}(x) = 0$ then $F_2 \neq 0$ and the solutions $\{F_n\}$ and $\{G_n\}$ defined in (19) and (21), respectively, are linearly dependent, $F_n = F_2 G_n$, $\forall n$. Sending n to infinity one gets

$$1 = F_2 \lim_{n\to\infty}\left(\prod_{k=1}^{n} \frac{1}{x_k x_{k+1}}\right)\mathfrak{F}\big(\{x_k\}_{k=1}^{n}\big) = \lim_{n\to\infty}\left(\prod_{k=1}^{n} \frac{1}{x_k x_{k+1}}\right)\mathfrak{F}\big(\{x_k\}_{k=1}^{n}\big)\mathfrak{F}\big(\{x_k\}_{k=2}^{n+1}\big).$$

Now, Lemma 1 provides us with the identity

$$\mathfrak{F}\big(\{x_k\}_{k=1}^{n}\big)\mathfrak{F}\big(\{x_k\}_{k=2}^{n+1}\big) - \mathfrak{F}\big(\{x_k\}_{k=1}^{n+1}\big)\mathfrak{F}\big(\{x_k\}_{k=2}^{n}\big) = \prod_{k=1}^{n} x_k x_{k+1},$$

and so one arrives at the equation

$$\lim_{n\to\infty}\left(\prod_{k=1}^{n} \frac{1}{x_k x_{k+1}}\right)\mathfrak{F}\big(\{x_k\}_{k=1}^{n+1}\big)\mathfrak{F}\big(\{x_k\}_{k=2}^{n}\big) = 0.$$

Since $\mathfrak{F}(\{x_k\}_{k=2}^{\infty}) \neq 0$ this shows (22).

3. Coulomb wave functions

For $x > 1$, $y \in \mathbb{R}$, put

$$\lambda(x, y) = \frac{y}{(x-1)x}, \qquad w(x, y) = \frac{1}{x}\sqrt{\frac{x^2 + y^2}{4x^2 - 1}},$$

and

$$\gamma(x, y) = \frac{\Gamma(\frac{1}{2}x)}{\sqrt{2x-1}\,\Gamma(\frac{1}{2}(x+1))}\left|\frac{\Gamma(\frac{1}{2}(x+iy+1))}{\Gamma(\frac{1}{2}(x+iy))}\right|.$$

Then $\gamma(x, y)\gamma(x+1, y) = w(x, y)$. For $\mu > 0$, $\nu \in \mathbb{R}$, consider the Jacobi matrix $J = J(\mu, \nu)$ of the form (8), with

$$\lambda_k = \lambda(\mu + k, \nu), \qquad w_k = w(\mu + k, \nu), \quad k = 1, 2, 3, \ldots. \tag{23}$$

Similarly, $\gamma_k = \gamma(\mu + k, \nu)$. Clearly, the matrix $J(\mu, \nu)$ represents a Hermitian Hilbert–Schmidt operator in $\ell^2(\mathbb{N})$. Moreover, the convergence condition (14) is satisfied for any $z_0 \in \mathbb{C}\backslash\{0\}$ such that $z_0 \neq \lambda_k$, $\forall k \in \mathbb{N}$.

Recall the definition of regular Coulomb wave functions [1, Eq. 14.1.3]

$$F_L(\eta, \rho) = 2^L e^{-\pi\eta/2}\frac{|\Gamma(L+1+i\eta)|}{\Gamma(2L+2)}\rho^{L+1}e^{-i\rho}\,{}_1F_1(L+1-i\eta; 2L+2; 2i\rho), \tag{24}$$

valid for $L \in \mathbb{Z}_+$, $\eta \in \mathbb{R}$, $\rho > 0$. Let us remark that, though not obvious from its form, the values of the regular Coulomb wave function in the indicated range are real. But nothing prevents us to extend, by analyticity, the Coulomb wave function to the values $L > -1$ and $\rho \in \mathbb{C}$ (assuming that a proper branch of ρ^{L+1} has been chosen).

As observed in [7], the eigenvalue equation for $J(\mu, \nu)$ may be written in the form $F_{\mu-1}(-\nu, z^{-1}) = 0$. Moreover, if $z \neq 0$ is an eigenvalue of $J(\mu, \nu)$ then the components $v_n(z)$, $n \in \mathbb{N}$, of a corresponding eigenvector $v(z)$ are proportional to $\sqrt{2\mu + 2n - 1}F_{\mu+n-1}(-\nu, z^{-1})$. Thus, using definition (24), one can write

$$\operatorname{spec}\bigl(J(\mu,\nu)\bigr)\backslash\{0\} = \bigl\{\zeta^{-1};\ e^{-i\zeta}\,{}_1F_1(\mu+i\nu;2\mu;2i\zeta)=0\bigr\} \tag{25}$$

and

$$v_n\bigl(\zeta^{-1}\bigr) = \sqrt{2\mu+2n-1}\,\frac{|\Gamma(\mu+n+i\nu)|}{\Gamma(2\mu+2n)}(2\zeta)^{n-1}e^{-i\zeta}\,{}_1F_1(\mu+n+i\nu;2\mu+2n;2i\zeta). \tag{26}$$

Here we wish to shortly reconsider this example while using our formalism.

Proposition 4. *Under the above assumptions (see (23)),*

$$\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k-\zeta^{-1}}\right\}_{k=1}^{\infty}\right) = \frac{\Gamma(\frac{1}{2}+\mu-\frac{1}{2}\sqrt{1+4\nu\zeta})\Gamma(\frac{1}{2}+\mu+\frac{1}{2}\sqrt{1+4\nu\zeta})}{\Gamma(\mu)\Gamma(\mu+1)}e^{-i\zeta}\,{}_1F_1(\mu+i\nu;2\mu;2i\zeta). \tag{27}$$

Proof. Observe that the convergence condition (14) is satisfied in this example. For $n\in\mathbb{N}$ put

$$f_{1,n} = \mathfrak{F}\left(\left\{\frac{{\gamma_k}^2}{\lambda_k-\zeta^{-1}}\right\}_{k=n}^{\infty}\right),$$

and let $f_{2,n}$ be equal to the RHS of (27) where we replace μ by $\mu+n-1$. According to (5), the sequence $\{f_{1,n}\}$ obeys the recurrence rule

$$f_{1,n} - f_{1,n+1} + X(\mu+n)f_{1,n+2} = 0, \quad n\in\mathbb{N}, \tag{28}$$

where

$$X(x) = \frac{w(x,\nu)^2}{(\lambda(x,\nu)-\zeta^{-1})(\lambda(x+1,\nu)-\zeta^{-1})} = \frac{(x^2-1)(x^2+\nu^2)\zeta^2}{(4x^2-1)((x-1)x-\nu\zeta)(x(x+1)-\nu\zeta)} \quad \text{for } x>1.$$

Next one can apply the identity

$${}_1F_1(a-1;b-2;z) - \frac{b^2-2b+(2a-b)z}{(b-2)b}\,{}_1F_1(a;b;z) - \frac{a(b-a)z^2}{(b^2-1)b^2}\,{}_1F_1(a+1;b+2;z) = 0, \tag{29}$$

as it follows from [1, §13.4], to verify that the sequence $\{f_{2,n}\}$ obeys (28) as well. Notice that, if rewritten in terms of Coulomb wave functions, (29) amounts to the recurrence rule [1, Eq. 14.2.3]

$$L\sqrt{(L+1)^2+\eta^2}\,u_{L+1} - (2L+1)\left(\eta+\frac{L(L+1)}{\rho}\right)u_L + (L+1)\sqrt{L^2+\eta^2}\,u_{L-1} = 0,$$

where $u_L = F_L(\eta,\rho)$.

To evaluate the limit of $f_{2,n}$, as $n\to\infty$, one may notice that

$$\lim_{n\to\infty}{}_1F_1(a+n;b+\kappa n;z) = e^{z/\kappa}$$

for $\kappa\neq 0$, and apply the Stirling formula. Alternatively, avoiding the Stirling formula, the limit is also obvious from the identity [6, Eq. 8.325(1)]

$$\prod_{k=0}^{\infty}\left(1+\frac{z}{(y+k)(y+k+1)}\right) = \frac{\Gamma(y)\Gamma(y+1)}{\Gamma(\frac{1}{2}+y-\frac{1}{2}\sqrt{1-4z})\Gamma(\frac{1}{2}+y+\frac{1}{2}\sqrt{1-4z})}. \tag{30}$$

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In any case, $\lim_{n\to\infty} f_{2,n} = 1$ and so, in virtue of Lemma 2, $f_{1,n} = f_{2,n}$, $\forall n$. In particular, for $n = 1$ one gets (27). □

Proof of formulas (25) and (26). As recalled in Section 2 (see (18)), $z = \zeta^{-1} \neq 0$ is an eigenvalue of $J(\mu,\nu)$ if and only if $\xi_0(z) = 0$ which means nothing but (25). In that case the components $\xi_n(z)$, $n \in \mathbb{N}$, of a corresponding eigenvector can be chosen as described in (15). Note that

$$\prod_{k=1}^{n-1} w_k = 2^{n-1}\sqrt{(2\mu+1)(2\mu+2n-1)}\left|\frac{\Gamma(\mu+n+i\nu)}{\Gamma(\mu+1+i\nu)}\right|\frac{\Gamma(2\mu+1)}{\Gamma(2\mu+2n)}$$

and that (30) means in fact the equality

$$\prod_{k=1}^{\infty}\frac{1}{1-\lambda(\mu+k,\nu)z} = \frac{\Gamma(\frac{1}{2}+\mu-\frac{1}{2}\sqrt{1+4\nu z})\Gamma(\frac{1}{2}+\mu+\frac{1}{2}\sqrt{1+4\nu z})}{\Gamma(\mu)\Gamma(\mu+1)}.$$

Using these equations and omitting a constant factor one finally arrives at formula (26). □

4. Confluent hypergeometric functions

First, let us show an identity.

Proposition 5. *The equation*

$$\begin{aligned}&\frac{\Gamma(x+\gamma+n)}{\Gamma(x+\gamma)}e^x\mathfrak{F}\left(\left\{\frac{\sqrt{2x}\,\Gamma(\frac{1}{2}(\gamma-\alpha+k+1))}{(x+\gamma+k-1)\Gamma(\frac{1}{2}(\gamma-\alpha+k))}\right\}_{k=1}^{n}\right)\\&\quad=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}{}_1F_1(\alpha;\gamma;x){}_1F_1(\alpha-\gamma-n;1-\gamma-n;x)\\&\qquad-\frac{\Gamma(\gamma-1)\Gamma(\gamma-\alpha+n+1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma+n+1)}x^{n+1}{}_1F_1(\alpha-\gamma+1;2-\gamma;x){}_1F_1(\alpha;\gamma+n+1;x), \qquad (31)\end{aligned}$$

is valid for $\alpha, \gamma, x \in \mathbb{C}$ and $n \in \mathbb{Z}_+$ (if considering the both sides as meromorphic functions).

Remark 6. For instance, as a particular case of (31) one gets, for $n = 0$,

$$\begin{aligned}&{}_1F_1(\alpha;\gamma;x){}_1F_1(\alpha-\gamma;1-\gamma;x)\\&\quad-\frac{(\gamma-\alpha)x}{\gamma(\gamma-1)}{}_1F_1(\alpha;\gamma+1;x){}_1F_1(\alpha-\gamma+1;2-\gamma;x) = e^x. \qquad (32)\end{aligned}$$

Proof. For α, γ and x fixed and $n \in \mathbb{Z}$, put

$$\varphi_n = \frac{1}{\Gamma(n+\gamma)}{}_1F_1(\alpha;n+\gamma;x), \qquad \psi_n = \frac{1}{\Gamma(n+\gamma-\alpha)}U(\alpha,n+\gamma,x).$$

Then $\{\varphi_n\}$ and $\{\psi_n\}$ obey the second order difference equation [1, Eqs. 13.4.2,13.4.16]

$$(n+\gamma-\alpha)xu_{n+1} - (n+\gamma+x-1)u_n + u_{n-1} = 0, \quad n \in \mathbb{Z}. \qquad (33)$$

Note also that

$$(\alpha-\gamma){}_1F_1(\alpha;\gamma+1;x)U(\alpha,\gamma,x) + \gamma\,{}_1F_1(\alpha;\gamma;x)U(\alpha,\gamma+1,x) = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha)}x^{-\gamma}e^x$$

(as it follows, for example, from Eqs. 13.4.12 and 13.4.25 combined with 13.1.22 in [1]). Whence

Please cite this article in press as: F. Štampach, P. Šťovíček, Special functions and spectrum of Jacobi matrices, Linear Algebra Appl. (2013), http://dx.doi.org/10.1016/j.laa.2013.06.024

$$\varphi_0\psi_1-\varphi_1\psi_0=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha+\gamma)}x^{-\gamma}e^x,$$

and so the solutions φ_n, ψ_n are linearly independent except of the cases $-\alpha\in\mathbb{Z}_+$ and $\alpha-\gamma\in\mathbb{N}$.

The difference equation (33) can be symmetrized using the substitution

$$u_n=\frac{x^{-n}}{\Gamma(\gamma-\alpha+n)}v_n.$$

Then $w_nv_{n+1}-\zeta_nv_n+w_{n-1}v_{n-1}=0$ where

$$w_n=\frac{x^{-n}}{\Gamma(\gamma-\alpha+n)},\qquad \zeta_n=\frac{(x+\gamma+n-1)x^{-n}}{\Gamma(\gamma-\alpha+n)}.$$

For a solution of the equation $\gamma_n\gamma_{n+1}=w_n$, $\forall n$, one can take

$$\gamma_n=2^{\frac{1}{4}}x^{-\frac{n}{2}+\frac{1}{4}}\sqrt{\frac{\Gamma(\frac{1}{2}(\gamma-\alpha+n+1))}{\Gamma(\gamma-\alpha+n)\Gamma(\frac{1}{2}(\gamma-\alpha+n))}}.$$

Referring to another solution, namely

$$v_n=\frac{1}{w_{n-1}\mathcal{P}_{n-1}}\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\zeta_k}\right\}_{k=1}^{n-1}\right),\quad \text{with } n\in\mathbb{N},$$

using otherwise the same notation as in (12), one concludes that there exist constants A and B such that

$$\frac{\Gamma(x+\gamma+n)}{\Gamma(\gamma-\alpha+n+1)}x^{-n}\mathfrak{F}\left(\left\{\frac{\sqrt{2x}\Gamma(\frac{1}{2}(\gamma-\alpha+k+1))}{(x+\gamma+k-1)\Gamma(\frac{1}{2}(\gamma-\alpha+k))}\right\}_{k=1}^{n}\right)=A\varphi_{n+1}+B\psi_{n+1}$$

for all $n\in\mathbb{Z}_+$. A and B can be determined from the values for $n=-1,0$ (putting $\mathfrak{F}(\{x_k\}_{k=1}^{-1})=0$, as dictated by the recurrence rule (7) provided the admissible values are extended to $N_1=N_2=1$, $n=0$). After some manipulations one gets

$$\begin{aligned}&\frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma-\alpha+n+1)}{}_1F_1(\alpha;\gamma;x)U(\alpha,\gamma+n+1,x)\\&\quad-\frac{\Gamma(\gamma)}{\Gamma(\gamma+n+1)}U(\alpha,\gamma,x){}_1F_1(\alpha;\gamma+n+1;x)\\&=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha)\Gamma(x+\gamma+n)}{\Gamma(\alpha)\Gamma(\gamma-\alpha+n+1)\Gamma(x+\gamma)}x^{-\gamma-n}e^x\mathfrak{F}\left(\left\{\frac{\sqrt{2x}\Gamma(\frac{1}{2}(\gamma-\alpha+k+1))}{(x+\gamma+k-1)\Gamma(\frac{1}{2}(\gamma-\alpha+k))}\right\}_{k=1}^{n}\right).\end{aligned}$$

Recall that

$$U(a,b,x)=\frac{\Gamma(1-b)}{\Gamma(a-b+1)}{}_1F_1(a;b;x)+\frac{\Gamma(b-1)}{\Gamma(a)}x^{1-b}{}_1F_1(a-b+1;2-b;x),\tag{34}$$

whence (31). □

Remark 7. Let us point out two particular cases of (31). Putting $\alpha=0$ one gets the identity

$$\mathfrak{F}\left(\left\{\frac{\sqrt{2x}\Gamma(\frac{1}{2}(\gamma+k+1))}{(x+\gamma+k-1)\Gamma(\frac{1}{2}(\gamma+k))}\right\}_{k=1}^{n}\right)=\frac{\Gamma(x+\gamma)}{\Gamma(x+\gamma+n)}\sum_{j=0}^{n}\frac{\Gamma(\gamma+n-j)}{\Gamma(\gamma)}x^j,\tag{35}$$

and for $\alpha=-1$ one obtains

$$\mathfrak{F}\left(\left\{\frac{\sqrt{2x}\,\Gamma(\frac{1}{2}(\gamma+k+2))}{(x+\gamma+k-1)\Gamma(\frac{1}{2}(\gamma+k+1))}\right\}_{k=1}^{n}\right)$$
$$=\frac{\Gamma(x+\gamma)}{\Gamma(x+\gamma+n)}\sum_{j=0}^{n}\frac{\Gamma(\gamma+n-j)}{\Gamma(\gamma+1)}\big(\gamma-j(n-j)\big)x^{j}. \tag{36}$$

Let us sketch a derivation of (36), Eq. (35) is simpler. Substitute -1 for α and $\gamma+n$ for γ in (32), and put

$$B_n=\frac{\Gamma(\gamma+n-1)}{\gamma-x+n}x^{-n}{}_1F_1(-\gamma-n;2-\gamma;x).$$

Then

$$B_{n+1}-B_n=\frac{\Gamma(\gamma+n+1)}{(\gamma-x+n)(\gamma-x+n+1)}x^{-n-1}e^{x}.$$

Whence

$$B_{n+1}=B_0+e^{x}\sum_{j=1}^{n+1}\frac{\Gamma(\gamma+j)}{(\gamma-x+j-1)(\gamma-x+j)}x^{-j}$$

which means nothing but

$$\Gamma(\gamma+n)(\gamma-x){}_1F_1(-\gamma-n-1;1-\gamma-n;x)$$
$$-\Gamma(\gamma-1)(\gamma-x+n+1)x^{n+1}{}_1F_1(-\gamma;2-\gamma;x)$$
$$=(\gamma-x+n+1)(\gamma-x)e^{x}\sum_{j=1}^{n+1}\frac{\Gamma(\gamma+j)}{(\gamma-x+j-1)(\gamma-x+j)}x^{n+1-j}. \tag{37}$$

Set $\alpha=-1$ in (31) and notice that ${}_1F_1(-1;b;x)=1-x/b$. After some simplifications, a combination of thus obtained identity with (37) gives (36).

As an application of (31) consider the Jacobi matrix operator $J(\alpha,\beta,\gamma)$ depending on parameters α, β, γ, with $\beta>0$, $\gamma>0$ and $\alpha+\beta>0$, as introduced in (8) where we put

$$\lambda_k=\gamma k,\qquad w_k=\sqrt{\alpha+\beta k},\quad k=1,2,3,\ldots. \tag{38}$$

For the sequence γ_k (fulfilling $\gamma_k\gamma_{k+1}=w_k$) one can take

$${\gamma_k}^2=\sqrt{2\beta}\,\Gamma\left(\frac{1}{2}\left(\frac{\alpha}{\beta}+k+1\right)\right)\Big/\Gamma\left(\frac{1}{2}\left(\frac{\alpha}{\beta}+k\right)\right).$$

Regarding the diagonal of $J(\alpha,\beta,\gamma)$ as an unperturbed part and the off-diagonal elements as a perturbation one immediately realizes that the matrix $J(\alpha.\beta,\gamma)$ determines a unique semibounded self-adjoint operator in $\ell^2(\mathbb{N})$. Moreover, the Weyl theorem about invariance of the essential spectrum tells us that its spectrum is discrete and simple. Our goal here is to show that one can explicitly construct a "characteristic" function of this operator in terms of confluent hypergeometric functions.

Proposition 8. *The spectrum of $J(\alpha,\beta,\gamma)$ defined in (8) and (38) coincides with the set of zeros of the function*

$$F_J(\alpha,\beta,\gamma;z)={}_1F_1\left(1-\frac{\alpha}{\beta}-\frac{\beta}{\gamma^2}-\frac{z}{\gamma};1-\frac{\beta}{\gamma^2}-\frac{z}{\gamma};\frac{\beta}{\gamma^2}\right)\Big/\Gamma\left(1-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}\right). \tag{39}$$

Moreover, if z is an eigenvalue then the components of a corresponding eigenvector v can be chosen as

$$v_k = (-1)^k \beta^{k/2} \gamma^{-k} \frac{\Gamma(\frac{\alpha}{\beta}+k)^{1/2}}{\Gamma(1-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}+k)} {}_1F_1\left(1-\frac{\alpha}{\beta}-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}; 1-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}+k; \frac{\beta}{\gamma^2}\right),$$
$$k \in \mathbb{N}. \tag{40}$$

Remark 9. (i) In principle it would be sufficient to consider the case $\gamma = 1$; observe that

$$F_J(\alpha, \beta, \gamma; z) = F_J\left(\frac{\alpha}{\gamma^2}, \frac{\beta}{\gamma^2}, 1; \frac{z}{\gamma}\right).$$

Thus for $\gamma = 1$ we get a simpler expression,

$$F_J(\alpha, \beta, 1; z) = {}_1F_1\left(1-\frac{\alpha}{\beta}-\beta-z; 1-\beta-z; \beta\right) / \Gamma(1-\beta-z).$$

(ii) Notice that the convergence condition (14) is violated in this example.

Before the proof we consider analogous results for finite matrices. Let $J_n(\alpha, \beta, \gamma)$ be the principal $n \times n$ submatrix of $J(\alpha, \beta, \gamma)$. The characteristic polynomial $F_{J_n}(z)$ of $J_n(\alpha, \beta, \gamma)$ can be expressed in terms of confluent hypergeometric functions, too. According to (10),

$$F_{J_n}(\alpha, \beta, \gamma; z) = \gamma^n \frac{\Gamma(1-\frac{z}{\gamma}+n)}{\Gamma(1-\frac{z}{\gamma})} \mathfrak{F}\left(\left\{ \frac{\sqrt{2\beta}}{\gamma} \frac{\Gamma(\frac{1}{2}(\frac{\alpha}{\beta}+k+1))}{(k-\frac{z}{\gamma})\Gamma(\frac{1}{2}(\frac{\alpha}{\beta}+k))} \right\}_{k=1}^{n}\right).$$

Applying (31) one arrives at the expression

$$\begin{aligned} &F_{J_n}(\alpha, \beta, \gamma; z) \\ &= \gamma^n e^{-\frac{\beta}{\gamma^2}} \left(\frac{\Gamma(n+1-\frac{\beta}{\gamma^2}-\frac{z}{\gamma})}{\Gamma(1-\frac{\beta}{\gamma^2}-\frac{z}{\gamma})} {}_1F_1\left(1-\frac{\alpha}{\beta}-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}; 1-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}; \frac{\beta}{\gamma^2}\right)\right. \\ &\quad \times {}_1F_1\left(-n-\frac{\alpha}{\beta}; -n+\frac{\beta}{\gamma^2}+\frac{z}{\gamma}; \frac{\beta}{\gamma^2}\right) \\ &\quad - \left(\frac{\beta}{\gamma^2}\right)^{n+1} \frac{\Gamma(n+1+\frac{\alpha}{\beta})\Gamma(-\frac{\beta}{\gamma^2}-\frac{z}{\gamma})}{\Gamma(\frac{\alpha}{\beta})\Gamma(n+2-\frac{\beta}{\gamma^2}-\frac{z}{\gamma})} {}_1F_1\left(1-\frac{\alpha}{\beta}; 1+\frac{\beta}{\gamma^2}+\frac{z}{\gamma}; \frac{\beta}{\gamma^2}\right) \\ &\quad \left. \times {}_1F_1\left(1-\frac{\alpha}{\beta}-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}; n+2-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}; \frac{\beta}{\gamma^2}\right)\right). \end{aligned}$$

Eigenvectors can be explicitly expressed as well. If z is an eigenvalue of $J_n(\alpha, \beta, \gamma)$ then formula (16) admits adaptation to this situation giving the expression for the components of a corresponding eigenvector,

$$\xi_k^{(n)} = (-1)^{k-1} \left(\prod_{j=1}^{k-1} w_j\right) \left(\prod_{j=k+1}^{n} (\lambda_j - z)\right) \mathfrak{F}\left(\left\{\frac{\gamma_j{}^2}{\lambda_j - z}\right\}_{j=k+1}^{n}\right), \quad k = 1, 2, \ldots, n.$$

Notice that $\xi_k^{(n)}$ makes sense also for $k = n+1$, and in that case its value is 0. Using (31) and omitting a redundant constant factor one arrives after some straightforward computation at the formula for an eigenvector $v^{(n)}$ of $J_n(\alpha, \beta, \gamma)$:

$$v_k^{(n)} = (-1)^k \beta^{k/2} \gamma^{-k} \frac{1}{\sqrt{\Gamma(\frac{\alpha}{\beta}+k)}} \left(\frac{\Gamma(\frac{\alpha}{\beta}+k)\Gamma(1-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}+n)}{\Gamma(1-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}+k)} \right.$$
$$\times {}_1F_1\left(1-\frac{\alpha}{\beta}-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}; 1-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}+k; \frac{\beta}{\gamma^2}\right) {}_1F_1\left(-\frac{\alpha}{\beta}-n; \frac{\beta}{\gamma^2}+\frac{z}{\gamma}-n; \frac{\beta}{\gamma^2}\right)$$
$$-\left(\frac{\beta}{\gamma^2}\right)^{n-k+1} \frac{\Gamma(1+\frac{\alpha}{\beta}+n)\Gamma(-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}+k)}{\Gamma(2-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}+n)}$$
$$\times {}_1F_1\left(1-\frac{\alpha}{\beta}-k; 1+\frac{\beta}{\gamma^2}+\frac{z}{\gamma}-k; \frac{\beta}{\gamma^2}\right)$$
$$\left. \times {}_1F_1\left(1-\frac{\alpha}{\beta}-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}; 2-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}+n; \frac{\beta}{\gamma^2}\right)\right), \quad 1 \leqslant k \leqslant n.$$

Remark 10. Formula (40) can be derived informally using a limit procedure. Suppose z is an eigenvalue of the infinite Jacobi matrix $J(\alpha, \beta, \gamma)$. For $k \in \mathbb{N}$ fixed, considering the asymptotic behavior of $v_k^{(n)}$ as $n \to \infty$ one expects that the leading term may give the component v_k of an eigenvector corresponding to the eigenvalue z. Omitting some constant factors one actually arrives in this way at (40). But having in hand the explicit expressions (39) and (40) it is straightforward to verify directly that the former one represents a characteristic function while the latter one describes an eigenvector.

Proof of Proposition 8. Observe first that for $k = 0$ the RHS of (40) is equal, up to a constant factor, to the announced characteristic function (39). If z solves the equation $v_0 = 0$ then one can make use of the identity [1, Eq. 13.4.2]

$$b(b-1)\,{}_1F_1(a; b-1; x) + b(1-b-x)\,{}_1F_1(a; b; x) + (b-a)x\,{}_1F_1(a; b+1; x) = 0$$

to verify that $v \in \ell^2(\mathbb{N})$ actually fulfills the eigenvalue equation (9). Note that the Stirling formula tells us that

$$v_k = \frac{(-1)^k}{(2\pi)^{1/4}} k^{-\frac{3}{4}+\frac{\alpha}{2\beta}+\frac{\beta}{\gamma^2}+\frac{z}{\gamma}} \left(\frac{\beta e}{\gamma^2 k}\right)^{k/2} \left(1 + O\left(\frac{1}{k}\right)\right) \quad \text{as } k \to \infty.$$

On the other hand, whatever the complex number z is, the sequence v_k, $k \in \mathbb{N}$, solves the second order difference equation (17), and in that case it is even true that

$$w_0 v_0 + (\lambda_1 - z)v_1 + w_1 v_2 = 0.$$

Let g_k, $k \in \mathbb{N}$, be any other independent solution of (17). Since the Wronskian

$$w_k(v_k g_{k+1} - v_{k+1} g_k) = \text{const} \neq 0$$

does not depend on k, and clearly $\lim_{k\to\infty} w_k v_k = \lim_{k\to\infty} w_k v_{k+1} = 0$, the sequence g_k cannot be bounded in any neighborhood of infinity. Hence, up to a multiplier, $\{v_k\}$ is the only square summable solution of (17). One concludes that z is an eigenvalue of $J(\alpha, \beta, \gamma)$ if and only if $w_0 v_0 = 0$ (which covers also the case $\alpha = 0$). □

Remark 11. A second independent solution of (17) can be found explicitly. For example, this is the sequence

$$g_k = (-1)^k \beta^{k/2} \gamma^{-k} \Gamma\left(\frac{\alpha}{\beta}+k\right)^{-1/2} U\left(1-\frac{\alpha}{\beta}-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}, 1-\frac{\beta}{\gamma^2}-\frac{z}{\gamma}+k, \frac{\beta}{\gamma^2}\right), \quad k \in \mathbb{N},$$

as it follows from the identity [1, Eq. 13.4.16]

$$(b-a-1)U(a, b-1, x) + (1-b-x)U(a, b, x) + xU(a, b+1, x) = 0.$$

But using once more relation (34) one may find as a more convenient the solution

$$g_k = \left(\frac{\beta}{\gamma^2}\right)^{\frac{\beta}{\gamma^2}+\frac{z}{\gamma}-\frac{k}{2}} \frac{1}{\sqrt{\Gamma(\frac{\alpha}{\beta}+k)\Gamma(1+\frac{\beta}{\gamma^2}+\frac{z}{\gamma}-k)}}$$
$$\times {}_1F_1\left(1-\frac{\alpha}{\beta}-k; 1+\frac{\beta}{\gamma^2}+\frac{z}{\gamma}-k; \frac{\beta}{\gamma^2}\right), \quad k \in \mathbb{N}.$$

Remark 12. Let us point out that for $\alpha = 0$ one gets a nontrivial example of an unbounded Jacobi matrix operator whose spectrum is known fully explicitly. In that case

$$\lambda_k = \gamma k, \qquad w_k = \sqrt{\beta k}, \quad k = 1, 2, 3, \ldots,$$

and

$$F_J(0, \beta, \gamma; z) = e^{\beta/\gamma^2} / \Gamma\left(1 - \frac{\beta}{\gamma^2} - \frac{z}{\gamma}\right).$$

Hence

$$\operatorname{spec} J(0, \beta, \gamma) = \left\{-\frac{\beta}{\gamma} + \gamma j;\ j = 1, 2, 3, \ldots\right\}.$$

Remark 13. Finally we remark that another particular case of interest is achieved in the formal limit $\beta \to 0$. Set $\alpha = w^2$ for some $w > 0$. Since [1, Eq. 13.3.2]

$$\lim_{a\to\infty} {}_1F_1\left(a; b; -\frac{z}{a}\right) = z^{(1-b)/2}\Gamma(b) J_{b-1}(2\sqrt{z})$$

one finds that

$$\lim_{\beta\to 0} F_J\left(w^2, \beta, \gamma; z\right) = \left(\frac{w}{\gamma}\right)^{z/\gamma} J_{-z/\gamma}\left(\frac{2w}{\gamma}\right).$$

It is known for quite a long time [3,8] that actually

$$\operatorname{spec} J\left(w^2, 0, \gamma\right) = \left\{z \in \mathbb{C};\ J_{-z/\gamma}\left(\frac{2w}{\gamma}\right) = 0\right\}.$$

5. Q-Bessel functions

5.1. Some properties of q-Bessel functions

Here we aim to explore a q-analogue to the following well-known property of Bessel functions. Consider the eigenvalue problem

$$w x_{k-1} - k x_k + w x_{k+1} = \nu x_k, \quad k \in \mathbb{Z},$$

for a second order difference operator acting in $\ell^2(\mathbb{Z})$ and depending on a parameter $w > 0$. If $\nu \notin \mathbb{Z}$ then one can take $\{J_{\nu+k}(2w)\}$ and $\{(-1)^k J_{-\nu-k}(2w)\}$ for two independent solutions of the formal eigenvalue equation while for $\nu \in \mathbb{Z}$ this may be the couple $\{J_{\nu+k}(2w)\}$ and $\{Y_{\nu+k}(2w)\}$. Taking into account the asymptotic behavior of Bessel functions for large orders (see [1, Eqs. 9.3.1, 9.3.2]) one finds that a square summable solution exists if and only if $\nu \in \mathbb{Z}$. Then $x_k = J_{\nu+k}(2w)$, $k \in \mathbb{Z}$, is such a solution and is unique up to a constant multiplier. Since

$$\sum_{k=-\infty}^{\infty} J_k(z)^2 = 1, \tag{41}$$

thus obtained eigenbasis $\boldsymbol{v}_\nu = \{v_{\nu,k}\}_{k=-\infty}^{\infty}$, $\nu \in \mathbb{Z}$, with $v_{\nu,k} = J_{\nu+k}(2w)$, is even orthonormal. One observes that the spectrum of the difference operator is stable and equals $\mathbb{Z}$ independently of the parameter w.

Hereafter we assume $0 < q < 1$. Recall the second definition of the q-Bessel function introduced by Jackson [10] (for some basic information and references one can also consult [4]),

$$J_\nu^{(2)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu {}_0\phi_1\left(;q^{\nu+1};q,-\frac{q^{\nu+1}x^2}{4}\right).$$

Here we prefer a slight modification of the second q-Bessel function, obtained just by some rescaling, and define

$$\begin{aligned}\mathrm{j}_\nu(x;q) &:= q^{\nu^2/4} J_\nu^{(2)}\big(q^{1/4}x;q\big)\\ &= q^{\nu(\nu+1)/4}\frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu {}_0\phi_1\left(;q^{\nu+1};q,-q^{\nu+3/2}\frac{x^2}{4}\right). \qquad (42)\end{aligned}$$

With our definition we have the following property.

Lemma 14. *For every $n \in \mathbb{N}$,*

$$\mathrm{j}_{-n}(x;q) = (-1)^n \mathrm{j}_n(x;q). \qquad (43)$$

Proof. One can readily verify that

$$\lim_{\nu\to -n}\big(1-q^{\nu+n}\big){}_0\phi_1\big(;q^{\nu+1};q,-q^{\nu+3/2}w^2\big) = -\frac{q^{n^2/2}w^{2n}}{(q;q)_{n-1}(q;q)_n}{}_0\phi_1\big(;q^{n+1};q,-q^{n+3/2}w^2\big)$$

and

$$\lim_{\nu\to -n}\frac{(q^{\nu+1};q)_\infty}{1-q^{\nu+n}} = (-1)^{n-1}q^{-n(n-1)/2}(q;q)_{n-1}(q;q)_\infty.$$

The lemma is an immediate consequence. □

Proposition 15. *For $0 < q < 1$, $w, \nu \in \mathbb{C}$, $q^{-\nu} \notin q^{\mathbb{Z}_+}$, one has*

$$\mathfrak{F}\left(\left\{\frac{w}{q^{-(\nu+k)/2}-q^{(\nu+k)/2}}\right\}_{k=0}^{\infty}\right) = {}_0\phi_1\big(;q^\nu;q,-q^{\nu+1/2}w^2\big). \qquad (44)$$

Remark 16. If rewritten in terms of q-Bessel functions, (44) becomes a q-analogue of (3). Explicitly,

$$\mathfrak{F}\left(\left\{\frac{w}{[\nu+k]_q}\right\}_{k=1}^{\infty}\right) = q^{\nu/4}\Gamma_q(\nu+1)w^{-\nu}J_\nu^{(2)}\big(2q^{-1/4}(1-q)w;q\big)$$

where [4]

$$[x]_q = \frac{q^{x/2}-q^{-x/2}}{q^{1/2}-q^{-1/2}}, \qquad \Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}.$$

Lemma 17. *For $\nu \in \mathbb{C}$, $q^{-\nu} \notin q^{\mathbb{Z}_+}$, and all $s \in \mathbb{N}$,*

$$\sum_{k=0}^{\infty}\frac{q^{sk}}{(q^{\nu+k};q)_{s+1}} = \frac{1}{(1-q^s)(q^\nu;q)_s}. \qquad (45)$$

Please cite this article in press as: F. Štampach, P. Šťovíček, Special functions and spectrum of Jacobi matrices, Linear Algebra Appl. (2013), http://dx.doi.org/10.1016/j.laa.2013.06.024

Proof. One can proceed by mathematical induction in s. The identity

$$\frac{q^{sk}}{(q^{\nu+k};q)_{s+1}}=\frac{q^{(s-1)k}}{q^{\nu}(1-q^{s})}\left(\frac{1}{(q^{\nu+k};q)_s}-\frac{1}{(q^{\nu+k+1};q)_s}\right)$$

can be used to verify both the case $s=1$ and the induction step $s\to s+1$. □

Proof of Proposition 15. One possibility how to prove (44) is based on Lemma 2. The proof presented below relies, however, on explicit evaluation of the involved sums. For $\nu\in\mathbb{C}$, $q^{\nu}\notin q^{\mathbb{Z}}$, $k\in\mathbb{Z}$, put

$$\rho_k=\frac{q^{(\nu+k)/2}}{1-q^{\nu+k}}.$$

Then (45) immediately implies that, for $n\in\mathbb{Z}$ and $s\in\mathbb{N}$,

$$\sum_{k=n}^{\infty}q^{(s-1)(\nu+k)/2}\rho_k\rho_{k+1}\dots\rho_{k+s}=\frac{q^{s(\nu+n+1)/2}}{1-q^{s}}\rho_n\rho_{n+1}\dots\rho_{n+s-1}.$$

This equation in turn can be used in the induction step on m to show that, for $m\in\mathbb{N}$, $n\in\mathbb{Z}$,

$$\sum_{k_1=n}^{\infty}\sum_{k_2=k_1+2}^{\infty}\dots\sum_{k_m=k_{m-1}+2}^{\infty}\rho_{k_1}\rho_{k_1+1}\rho_{k_2}\rho_{k_2+1}\dots\rho_{k_m}\rho_{k_m+1}$$
$$=\frac{q^{m(3m+1)/4}q^{m(\nu+n-1)/2}}{(q;q)_m}\rho_n\rho_{n+1}\dots\rho_{n+m-1}.$$

In particular, for $n=1$ one gets

$$\sum_{k_1=1}^{\infty}\sum_{k_2=k_1+2}^{\infty}\dots\sum_{k_m=k_{m-1}+2}^{\infty}\rho_{k_1}\rho_{k_1+1}\rho_{k_2}\rho_{k_2+1}\dots\rho_{k_m}\rho_{k_m+1}=\frac{q^{m(2m+1)/2+\nu m}}{(q;q)_m(q^{\nu+1};q)_m},\quad m\in\mathbb{N}.$$

Now, in order to evaluate $\mathfrak{F}(\{w\rho_k\}_{k=1}^{\infty})$, it suffices to apply the very definition (1). □

The q-hypergeometric function is readily seen to satisfy the recurrence rule

$${}_0\phi_1(;q^{\nu};q,z)-{}_0\phi_1(;q^{\nu+1};q,qz)-\frac{z}{(1-q^{\nu})(1-q^{\nu+1})}{}_0\phi_1(;q^{\nu+2};q,q^2z)=0.$$

Consequently,

$$w\mathrm{j}_{\nu}(2w;q)-\left(q^{-(\nu+1)/2}-q^{(\nu+1)/2}\right)\mathrm{j}_{\nu+1}(2w;q)+w\mathrm{j}_{\nu+2}(2w;q)=0.$$

This is in agreement with (12) if applied to the bilateral second order difference equation

$$wx_{n-1}-\left(q^{-(\nu+n)/2}-q^{(\nu+n)/2}\right)x_n+wx_{n+1}=0,\quad n\in\mathbb{Z}. \tag{46}$$

Suppose $q^{\nu}\notin q^{\mathbb{Z}}$. Then the two solutions described in (12) in this case give

$$f_n=q^{-\nu(\nu+1)/4}\frac{(q;q)_\infty}{(q^{\nu+1};q)_\infty}w^{-\nu}\mathrm{j}_{\nu+n}(2w;q), \tag{47}$$

$$g_n=(-1)^{n+1}q^{-\nu(\nu+1)/4}\frac{(q;q)_\infty}{(q^{-\nu};q)_\infty}w^{\nu}\mathrm{j}_{-\nu-n}(2w;q),\quad n\in\mathbb{Z}. \tag{48}$$

Let us show that they are generically independent. For the proof we need the identity [4, §1,3]

$$\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}}{(q;q)_k}z^k=(-z;q)_\infty.$$

Lemma 18. *For $\nu \in \mathbb{C}$, $q^\nu \notin q^{\mathbb{Z}}$, the Wronskian of the solutions of (46), $\{f_n\}$ and $\{g_n\}$ defined in (47) and (48), respectively, fulfills*

$$\mathcal{W}(f,g) = \mathfrak{F}\left(\left\{\frac{wq^{(\nu+k)/2}}{1-q^{\nu+k}}\right\}_{k=-\infty}^{\infty}\right) = \left(-q^{1/2}w^2; q\right)_\infty. \tag{49}$$

Proof. The first equality in (49) is nothing but (13). Further, in virtue of (44), the second member in (49) equals

$$\begin{aligned}\lim_{N\to\infty} {}_0\phi_1\left(; q^{\nu-N}; q, -q^{\nu-N+1/2}w^2\right) &= \lim_{M\to\infty}\sum_{k=0}^{\infty}\frac{q^{k(k-1)}}{(q;q)_k(q^{-M};q)_k}\left(-q^{-M}q^{1/2}w^2\right)^k \\ &= \sum_{k=0}^{\infty}\frac{q^{k^2/2}}{(q;q)_k}w^{2k} = \left(-q^{1/2}w^2; q\right)_\infty.\end{aligned}$$

The lemma follows. □

At the same time, $\mathcal{W}(f,g)$ equals

$$\frac{q^{-\nu(\nu+1)/2}(q;q)_\infty{}^2 w}{(q^{\nu+1};q)_\infty(q^{-\nu};q)_\infty}\left(\mathrm{j}_\nu(2w;q)\mathrm{j}_{-\nu-1}(2w;q) + \mathrm{j}_{\nu+1}(2w;q)\mathrm{j}_{-\nu}(2w;q)\right).$$

This implies the following result.

Proposition 19. *For $w \in \mathbb{C}$ one has*

$$\begin{aligned}&\mathrm{j}_\nu(2w;q)\mathrm{j}_{-\nu-1}(2w;q) + \mathrm{j}_{\nu+1}(2w;q)\mathrm{j}_{-\nu}(2w;q) \\ &\quad = \frac{q^{\nu(\nu+1)/2}(q^{\nu+1};q)_\infty(q^{-\nu};q)_\infty(-q^{1/2}w^2;q)_\infty}{(q;q)_\infty{}^2 w}\end{aligned} \tag{50}$$

and, rewriting (50) in terms of q-hypergeometric functions,

$$\begin{aligned}&{}_0\phi_1\left(; q^{\nu+1}; q, -q^{\nu+1}z\right){}_0\phi_1\left(; q^{-\nu}; q, -q^{-\nu}z\right) \\ &\quad - \frac{q^\nu z}{(1-q^\nu)(1-q^{\nu+1})}{}_0\phi_1\left(; q^{\nu+2}; q, -q^{\nu+2}z\right){}_0\phi_1\left(; q^{-\nu+1}; q, -q^{-\nu+1}z\right) \\ &\quad = (-z;q)_\infty.\end{aligned} \tag{51}$$

Remark 20. Let us examine the limit $q \to 1-$ applied to (50) while replacing w by $(1-q)w$. One knows that [4]

$$\lim_{q\to 1-}\mathrm{j}_\nu\left((1-q)z;q\right) = J_\nu(z), \qquad \lim_{q\to 1-}(1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty} = \Gamma(x).$$

Thus one finds that the limiting equation coincides with the well-known identity

$$J_\nu(2w)J_{-\nu-1}(2w) + J_{\nu+1}(2w)J_{-\nu}(2w) = \frac{1}{w\Gamma(\nu+1)\Gamma(-\nu)} = -\frac{\sin(\pi\nu)}{\pi w}.$$

It is desirable to have some basic information about the asymptotic behavior of q-Bessel functions for large orders. It is straightforward to see that

$$\mathrm{j}_\nu(x;q) = q^{\nu(\nu+1)/4}\frac{1}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu\left(1 + O\left(q^\nu\right)\right) \quad \text{as } \operatorname{Re}\nu \to +\infty. \tag{52}$$

The asymptotic behavior at $-\infty$ is described as follows.

Lemma 21. *For $\sigma, w \in \mathbb{C}$, $q^\sigma \notin q^{\mathbb{Z}}$, one has*

$$\begin{aligned}&\lim_{\substack{|\nu|\to\infty\\ \nu\in-\sigma-\mathbb{N}}} \sin(\pi\nu) q^{\nu(\nu+1)/4} w^{-\nu} \mathrm{j}_\nu(2w;q)\\ &\quad = -\sin(\pi\sigma) q^{-\sigma(1-\sigma)/2} \frac{(q^\sigma;q)_\infty (q^{1-\sigma};q)_\infty (-q^{1/2}w^2;q)_\infty}{(q;q)_\infty}. \end{aligned} \tag{53}$$

Proof. Put $\nu = -\sigma - n$ where $n \in \mathbb{N}$. Using (44) and (7) one can write

$$\begin{aligned}&{}_0\phi_1\big(;q^{-\sigma-n};q,-q^{-\sigma-n+1/2}w^2\big)\\ &\quad = \mathfrak{F}\left(\left\{\frac{w}{q^{(\sigma+k)/2}-q^{-(\sigma+k)/2}}\right\}_{k=0}^{n}\right)\mathfrak{F}\left(\left\{\frac{w}{q^{(\sigma-k)/2}-q^{-(\sigma-k)/2}}\right\}_{k=1}^{\infty}\right)\\ &\qquad + \frac{w^2}{(q^{\sigma/2}-q^{-\sigma/2})(q^{(1-\sigma)/2}-q^{-(1-\sigma)/2})}\\ &\qquad \times \mathfrak{F}\left(\left\{\frac{w}{q^{(\sigma+k)/2}-q^{-(\sigma+k)/2}}\right\}_{k=1}^{n}\right)\mathfrak{F}\left(\left\{\frac{w}{q^{(\sigma-k)/2}-q^{-(\sigma-k)/2}}\right\}_{k=2}^{\infty}\right).\end{aligned}$$

Applying the limit $n \to \infty$ one obtains

$$\begin{aligned}&\lim_{n\to\infty} {}_0\phi_1\big(;q^{-\sigma-n};q,-q^{-\sigma-n+1/2}w^2\big)\\ &\quad = {}_0\phi_1\big(;q^\sigma;q,-q^{\sigma+1/2}w^2\big){}_0\phi_1\big(;q^{1-\sigma};q,-q^{-\sigma+3/2}w^2\big)\\ &\qquad + \frac{w^2}{(q^{\sigma/2}-q^{-\sigma/2})(q^{(1-\sigma)/2}-q^{-(1-\sigma)/2})}\\ &\qquad \times {}_0\phi_1\big(;q^{1+\sigma};q,-q^{\sigma+3/2}w^2\big){}_0\phi_1\big(;q^{2-\sigma};q,-q^{-\sigma+5/2}w^2\big)\\ &\quad = \big(-q^{1/2}w^2;q\big)_\infty.\end{aligned}$$

To get the last equality we have used (51). Notice also that

$$\lim_{n\to\infty} (-1)^n q^{(-\sigma-n)(-\sigma-n+1)/2}\big(q^{-\sigma-n+1};q\big)_\infty = q^{\sigma(\sigma-1)/2}\big(q^\sigma;q\big)_\infty\big(q^{1-\sigma};q\big)_\infty.$$

The limit (53) then readily follows. □

Finally we establish an identity which can be viewed as a q-analogue to (41).

Proposition 22. *For $0 < q < 1$ and $w \in \mathbb{C}$ one has*

$$\sum_{k=-\infty}^{\infty} q^{-k/2}\mathrm{j}_k(2w;q)^2 = \mathrm{j}_0(2w;q)^2 + \sum_{k=1}^{\infty}\big(q^{k/2}+q^{-k/2}\big)\mathrm{j}_k(2w;q)^2 = \big(-q^{1/2}w^2;q\big)_\infty. \tag{54}$$

Equivalently, if rewritten in terms of q-Bessel functions,

$$J_0^{(2)}(2w;q)^2 + \sum_{k=1}^{\infty}\big(q^{k/2}+q^{-k/2}\big)q^{k^2/2}J_k^{(2)}(2w;q)^2 = \big(-w^2;q\big)_\infty.$$

Proof. In [12, (1.20)] it is shown that

$$\frac{J_\nu^{(2)}(2w;q)^2}{(-w^2;q)_\infty} = \left(\frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\right)^2 w^{2\nu}{}_3\phi_2\big(q^{\nu+\frac12},-q^{\nu+\frac12},-q^{\nu+1};q^{\nu+1},q^{2\nu+1};q,-w^2\big),$$

and this can be rewritten as

$$
{}_0\phi_1\big(;q^{\nu+1};q,-q^{\nu+1}x\big)^2=(-x;q)_\infty {}_3\phi_2\big(q^{\nu+\frac{1}{2}},-q^{\nu+\frac{1}{2}},-q^{\nu+1};q^{\nu+1},q^{2\nu+1};q,-x\big).
$$

Hence (54) is equivalent to

$$
\begin{aligned}
&{}_3\phi_2\big(q^{1/2},-q^{1/2},-q;q,q;q,-x\big)\\
&\quad+\sum_{k=1}^{\infty}\frac{(q^{-k/2}+q^{k/2})q^{k^2/2}}{(q;q)_k{}^2}\,{}_3\phi_2\big(q^{k+\frac{1}{2}},-q^{k+\frac{1}{2}},-q^{k+1};q^{k+1},q^{2k+1};q,-x\big)x^k=1.
\end{aligned}
$$

Looking at the power expansion in x one gets, equivalently, a countable system of equations, for $n=1,2,3,\ldots$,

$$
\begin{aligned}
&\frac{(q^{1/2};q)_n(-q^{1/2};q)_n(-q;q)_n}{(q;q)_n{}^3}\\
&\quad+\sum_{k=1}^{n}(-1)^k\frac{(q^{-k/2}+q^{k/2})q^{k^2/2}}{(q;q)_k{}^2}\,\frac{(q^{k+1/2};q)_{n-k}(-q^{k+1/2};q)_{n-k}(-q^{k+1};q)_{n-k}}{(q;q)_{n-k}(q^{k+1};q)_{n-k}(q^{2k+1};q)_{n-k}}=0.
\end{aligned}
$$

The equations can be brought to the form

$$
\frac{1}{(q;q)_n{}^2}+\sum_{k=1}^{n}(-1)^k\frac{q^{k(k-1)/2}(1+q^k)}{(q;q)_{n+k}(q;q)_{n-k}}=0
$$

or, more conveniently,

$$
\sum_{j=0}^{2n}(-1)^j\frac{q^{-j(2n-j+1)/2}}{(q;q)_{2n-j}(q;q)_j}=0.
$$

This is true indeed since, for any $m\in\mathbb{Z}_+$,

$$
\sum_{j=0}^{m}(-1)^j\frac{(q;q)_m}{(q;q)_{m-j}(q;q)_j}q^{-j(m-j)/2}x^j=\big(q^{-(m-1)/2}x;q\big)_m=\prod_{k=0}^{m-1}\big(1-q^{-(m-1)/2+k}x\big).
$$

This concludes the proof. □

5.2. A bilateral second order difference equation

We know that the sequence $u_n=\mathrm{j}_{\nu+n}(2w;q)$ obeys (46). Applying the substitution $q^{-\nu-1}=z$, $w=q^{\frac{\nu}{2}+\frac{1}{4}}\beta$, one finds that the sequence

$$
\begin{aligned}
v_n&=q^{-n/4}u_n=q^{-n/4}\mathrm{j}_{\nu+n}\big(2q^{(2\nu+1)/4}\beta;q\big) &(55)\\
&=q^{-(\nu^2+2\nu+2)/4}q^{(n-1)(n-2)/4}\frac{(q^nz^{-1};q)_\infty}{(q;q)_\infty}\left(\frac{\beta}{z}\right)^{\nu+n}{}_0\phi_1\big(;q^nz^{-1};q,-q^nz^{-2}\beta^2\big),
\end{aligned}
$$

fulfills

$$
q^{(n-1)/2}\beta v_n+\big(q^n-z\big)v_{n+1}+q^{n/2}\beta v_{n+2}=0,\quad n\in\mathbb{Z}. \tag{56}
$$

Remark 23. One can as well consider the unilateral second order difference equation

$$
(1-z)v_1+\beta v_2=0,\qquad q^{(n-1)/2}\beta v_n+\big(q^n-z\big)v_{n+1}+q^{n/2}\beta v_{n+2}=0,\quad n=1,2,3,\ldots.
$$

From (52) it can be seen that the sequence $\{v_n\}$ given in (55) is square summable over $\mathbb{N}$. Considering the Wronskian one also concludes that any other linearly independent solution of (56) cannot be bounded on any neighborhood of $+\infty$. Hence the sequence v_n, $n \in \mathbb{N}$, solves the eigenvalue problem in $\ell^2(\mathbb{N})$ iff $v_0 = 0$, i.e. iff $\mathrm{j}_\nu(2w; q) = 0$. In terms of the new parameters β, z this condition becomes the characteristic equation for an eigenvalue z,

$$\left(z^{-1}; q\right)_\infty {}_0\phi_1\left(; z^{-1}; q, -z^{-2}\beta^2\right) = 0.$$

This example has already been treated in [14, Sec. 4.1].

For the bilateral equation it may be more convenient shifting the index by 1 in (56). This is to say that we are going to solve the equation

$$q^{(n-1)/2}\beta v_{n-1} + \left(q^n - z\right)v_n + q^{n/2}\beta v_{n+1} = 0, \quad n \in \mathbb{Z}, \tag{57}$$

rather than (56). Denote again by $J = J(\beta, q)$, with $\beta \in \mathbb{R}$ and $0 < q < 1$, the corresponding matrix operator in $\ell^2(\mathbb{Z})$. One knows, however, that $J(-\beta, q)$ and $J(\beta, q)$ are unitarily equivalent and so, if convenient, one can consider just the values $\beta \geqslant 0$. In Eq. (57), z is playing the role of a spectral parameter. Using a notation analogous to (8) (now for the bilateral case), this means that

$$w_n = q^{n/2}\beta, \qquad \lambda_n = q^n, \quad \text{and} \quad \zeta_n := z - \lambda_n = z - q^n, \quad n \in \mathbb{Z}. \tag{58}$$

Notice that for a sequence $\{\gamma_n\}$ obeying $\gamma_n\gamma_{n+1} = w_n$, $\forall n \in \mathbb{Z}$, one can take

$$\gamma_{2k-1}{}^2 = q^{k-1}, \qquad \gamma_{2k}{}^2 = q^k\beta^2.$$

Since the sequence $\{w_n/(\lambda_n + 1)\}$ is summable over $\mathbb{Z}$, the Weyl theorem tells us that the essential spectrum of the self-adjoint operator $J(\beta, q)$ contains just one point, namely 0. Hence all nonzero spectral points are eigenvalues.

Proposition 24. *For $0 < q < 1$ and $\beta > 0$, the spectrum of the Jacobi matrix operator $J(\beta, q)$ in $\ell^2(\mathbb{Z})$, as introduced above (see (58)), is pure point, all eigenvalues are simple and*

$$\operatorname{spec}_p J(\beta, q) = \left(-\beta^2 q^{\mathbb{Z}_+}\right) \cup q^{\mathbb{Z}}.$$

Eigenvectors $\boldsymbol{v}_m^{(+)}$ corresponding to the eigenvalues q^m, $m \in \mathbb{Z}$, can be chosen as $\boldsymbol{v}_m^{(+)} = \{v_{m,k}^{(+)}\}_{k=-\infty}^{\infty}$, with

$$v_{m,k}^{(+)} = q^{(m-k)/4}\mathrm{j}_{-m+k}\left(2q^{-(2m+1)/4}\beta; q\right).$$

They are normalized as follows:

$$\left\|\boldsymbol{v}_m^{(+)}\right\|^2 = \sum_{k=-\infty}^{\infty} q^{-k/2}\mathrm{j}_k\left(2q^{-(2m+1)/4}\beta; q\right)^2 = \left(-q^{-m}\beta^2; q\right)_\infty, \quad \forall m \in \mathbb{Z}.$$

Eigenvector $\boldsymbol{v}_m^{(-)}$ corresponding to the eigenvalues $-\beta^2 q^m$, $m \in \mathbb{Z}_+$, can be chosen as $\boldsymbol{v}_m^{(-)} = \{v_{m,k}^{(-)}\}_{k=-\infty}^{\infty}$, with

$$\begin{aligned} v_{m,k}^{(-)} = {} & \frac{(-1)^k q^{k(k-4m-1)/4}}{(q;q)_\infty}\beta^{-k}\left(-q^{-m+k+1}\beta^{-2}; q\right)_\infty \\ & \times {}_0\phi_1\left(; -q^{-m+k+1}\beta^{-2}; q, -q^{-2m+k+1}\beta^{-2}\right). \end{aligned} \tag{59}$$

Remark 25. An expression for the norms of vectors $\boldsymbol{v}_m^{(-)}$ can be found, too,

$$\begin{aligned} \left\|\boldsymbol{v}_m^{(-)}\right\|^2 = {} & (-1)^m q^{-m(3m+1)/2}\frac{(-q\beta^{-2}; q)_\infty(-q^{-m}\beta^{-2}; q)_\infty(-q^{m+1}\beta^2; q)_\infty}{(-q\beta^2; q)_\infty(q^{m+1}; q)_\infty} \\ & \times \frac{{}_0\phi_1(; -q\beta^{-2}; q, -q^{-m+1}\beta^{-2})}{{}_0\phi_1(; -q\beta^2; q, -q^{-m+1}\beta^2)}, \quad \forall m \in \mathbb{Z}_+. \end{aligned}$$

But the formula is rather cumbersome and its derivation somewhat lengthy and this is why we did not include it in the proposition and omit its proof.

Proof. We use the substitution $z = q^{-\nu}$ where ν is in general complex. The RHS in (12) can be evaluated using (44) and (42). Applying some easy simplifications one gets two solutions of (57):

$$v_n = q^{-(\nu+n)/4} \mathrm{j}_{\nu+n}\big(2q^{(2\nu-1)/4}\beta; q\big), \qquad \tilde{v}_n = (-1)^n q^{-(\nu+n)/4} \mathrm{j}_{-\nu-n}\big(2q^{(2\nu-1)/4}\beta; q\big), \quad n \in \mathbb{Z}.$$

One can argue that in the bilateral case, too, all eigenvalues of $J(\beta, q)$ are simple. In fact, the solution $\{v_n\}$ asymptotically behaves as

$$v_n = \frac{1}{(q;q)_\infty} q^{\frac{1}{4}(n^2+(4\nu-1)n+(3\nu-1)\nu)} \beta^{\nu+n}\big(1 + O\big(q^n\big)\big) \quad \text{as } n \to +\infty.$$

For any other independent solution $\{y_n\}$ of (57), $q^{n/2}(y_n v_{n+1} - y_{n+1} v_n)$ is a nonzero constant. Obviously, such a sequence $\{y_n\}$ cannot be bounded on any neighborhood of $+\infty$. A similar argument applies to the solution $\{\tilde{v}_n\}$ for n large but negative. In particular, one concludes that $z = q^{-\nu}$ is an eigenvalue of $J(\beta, q)$ if and only if $\{v_n\}$ and $\{\tilde{v}_n\}$ are linearly dependent.

Using (50) one can derive a formula for the Wronskian,

$$\begin{aligned}
\mathcal{W}(v, \tilde{v}) &= q^{k/2}\beta(v_k \tilde{v}_{k+1} - v_{k+1}\tilde{v}_k) \\
&= (-1)^{k+1}\beta q^{-(2\nu+1)/4}\big(\mathrm{j}_{\nu+k}\big(2q^{(2\nu-1)/4}\beta; q\big)\mathrm{j}_{-\nu-k-1}\big(2q^{(2\nu-1)/4}\beta; q\big) \\
&\quad + \mathrm{j}_{\nu+k+1}\big(2q^{(2\nu-1)/4}\beta; q\big)\mathrm{j}_{-\nu-k}\big(2q^{(2\nu-1)/4}\beta; q\big)\big) \\
&= \frac{q^{\nu(\nu-3)/2}(q^\nu; q)_\infty (q^{1-\nu}; q)_\infty (-q^\nu \beta^2; q)_\infty}{(q;q)_\infty{}^2}.
\end{aligned}$$

Thus z is an eigenvalue if and only if either $(z^{-1}; q)_\infty (qz; q)_\infty = 0$ or $(-z^{-1}\beta^2; q)_\infty = 0$. In the former case $z \in q^{\mathbb{Z}}$, in the latter case $-z \in \beta^2 q^{\mathbb{Z}_+}$.

Thus in the case of positive eigenvalues one can put $\nu = -m$, with $m \in \mathbb{Z}$. With this choice, $\{v_k\}$ coincides with $\{v_{m,k}^{(+)}\}$. Notice that then the linear dependence of the sequences $\{v_k\}$ and $\{\tilde{v}_k\}$ is also obvious from (43). Normalization of the eigenvectors $\boldsymbol{v}_m^{(+)}$ is a consequence of (54).

As far as the negative spectrum is concerned, one can put, for example, $\tau = -(i\pi + \log\beta^2)/\log q$ and $\nu = \tau - m$, $m \in \mathbb{Z}_+$. Then the sequence

$$v_k = q^{-(\tau-m+k)/4} \mathrm{j}_{\tau-m+k}\big(-2iq^{-(2m+1)/4}; q\big), \quad k \in \mathbb{Z},$$

represents an eigenvector corresponding to the eigenvalue $-\beta^2 q^m$. But it is readily seen to be proportional to the RHS of (59) whose advantage is to be manifestly real.

Finally let is show that 0 can never be an eigenvalue of $J(\beta, q)$. We still assume $\beta > 0$. For $z = 0$, one can find two mutually complex conjugate solutions of (57) explicitly. Let us call them $v_{\pm,n}$, $n \in \mathbb{Z}$, where

$$v_{\pm,n} = i^{\pm n} q^{-n/4} {}_1\phi_1\left(0; -q^{1/2}; q^{1/2}, \pm\frac{iq^{(2n+3)/4}}{\beta}\right) = i^{\pm n} q^{-n/4} \sum_{k=0}^\infty \frac{q^{k(k+2)/4}}{(q;q)_k}\left(\mp\frac{iq^{n/2}}{\beta}\right)^k.$$

Clearly,

$$v_{\pm,n} = i^{\pm n} q^{-n/4}\big(1 + O\big(q^{n/2}\big)\big) \quad \text{as } n \to +\infty.$$

Using the asymptotic expansion one can evaluate the Wronskian getting

$$\mathcal{W}(v_+, v_-) = q^{n/2}\beta(v_{+,n}v_{-,n+1} - v_{+,n+1}v_{-,n}) = -2iq^{-1/4}\beta.$$

Hence the two solutions are linearly independent. It is also obvious from the asymptotic expansion that no nontrivial linear combination of these solutions can be square summable. Hence 0 cannot be an eigenvalue of $J(\beta, q)$ whatever β is, and this concludes the proof. □

So one observes that the positive part of the spectrum of $J(\beta, q)$ is stable and does not depend on the parameter β. This behavior is very similar to what one knows from the non-deformed case. On the other hand, there is an essentially new feature in the q-case when a negative part of the spectrum emerges for $\beta \neq 0$, and it is even infinite-dimensional though it shrinks to zero with the rate β^2 as β tends to 0.

6. Q-confluent hypergeometric functions

In this section we deal with the q-confluent hypergeometric function

$$ {}_1\phi_1\big(a; b; q, z\big) = \sum_{k=0}^{\infty} (-1)^k q^{k(k-1)/2} \frac{(a;q)_k}{(b;q)_k (q;q)_k} z^k. $$

It can readily be checked to obey the recurrence rules

$$ -\frac{q^{\alpha+\gamma}(1-q^{\gamma-\alpha+1})}{(1-q^{\gamma})(1-q^{\gamma+1})} z\, {}_1\phi_1\big(q^{\alpha}; q^{\gamma+2}; q, q^{\gamma+2}z\big) - \left(1 - \frac{q^{\gamma}z}{1-q^{\gamma}}\right) {}_1\phi_1\big(q^{\alpha}; q^{\gamma+1}; q, q^{\gamma+1}z\big) + {}_1\phi_1\big(q^{\alpha}; q^{\gamma}; q, q^{\gamma}z\big) = 0 \tag{60} $$

and

$$ {}_1\phi_1\big(q^{\alpha-\gamma+1}; q^{2-\gamma}; q, z\big) + \frac{q(q-q^{\gamma}-q^{1-\gamma}+1)}{q^{\gamma}-q^{\alpha}} {}_1\phi_1\big(q^{\alpha-\gamma-1}; q^{-\gamma}; q, z\big) - \left(\frac{q(q-q^{\gamma}-q^{1-\gamma}+1)}{q^{\gamma}-q^{\alpha}} + \frac{q-q^{\gamma}}{q^{\gamma}-q^{\alpha}} z\right) {}_1\phi_1\big(q^{\alpha-\gamma}; q^{1-\gamma}; q, z\big) = 0. $$

Put, for $n \in \mathbb{Z}$,

$$ \varphi_n = \big(q^{n+\gamma}; q\big)_{\infty}\, {}_1\phi_1\big(q^{\alpha}; q^{n+\gamma}; q, -q^{n+\gamma}z\big), \tag{61} $$

$$ \psi_n = q^{-\alpha(n+\gamma)-(n+\gamma-1)(n+\gamma-2)/2} \frac{(q^{n+\gamma-\alpha}; q)_{\infty}}{(q^{n+\gamma-1}; q)_{\infty}} z^{1-n-\gamma}\, {}_1\phi_1\big(q^{\alpha-n-\gamma+1}; q^{2-n-\gamma}; q, -qz\big). \tag{62} $$

Here $z, \alpha, \gamma \in \mathbb{C}$, $q^{\gamma} \notin q^{\mathbb{Z}}$. The recurrence rules imply that both $\{\varphi_n\}$ and $\{\psi_n\}$ solve the three-term difference equation

$$ q^{\alpha+\gamma+n-1}\big(1-q^{\gamma-\alpha+n}\big) z u_{n+1} - \big(1 - q^{\gamma+n-1} + q^{\gamma+n-1}z\big) u_n + u_{n-1} = 0, \quad n \in \mathbb{Z}. \tag{63} $$

Lemma 26. *The sequences $\{\varphi_n\}$ and $\{\psi_n\}$ defined in (61) and (62), respectively, fulfill*

$$ \varphi_0\psi_1 - \varphi_1\psi_0 = q^{-\alpha(\gamma+1)-\frac{1}{2}\gamma(\gamma-1)} \big(q^{\gamma-\alpha+1}; q\big)_{\infty} \big(-q^{\alpha}z; q\big)_{\infty} z^{-\gamma}. \tag{64} $$

Alternatively, (64) can be rewritten as

$$ {}_1\phi_1\big(q^{\alpha}; q^{\gamma}; q, q^{\gamma-\alpha}z\big)\, {}_1\phi_1\big(q^{\alpha-\gamma}; q^{1-\gamma}; q, q^{1-\alpha}z\big) + \frac{q^{\gamma-1}(1-q^{\gamma-\alpha})z}{(1-q^{\gamma-1})(1-q^{\gamma})} {}_1\phi_1\big(q^{\alpha}; q^{\gamma+1}; q, q^{\gamma-\alpha+1}z\big)\, {}_1\phi_1\big(q^{\alpha-\gamma+1}; q^{2-\gamma}; q, q^{1-\alpha}z\big) = (z; q)_{\infty}. $$

Proof. Checking the Wronskian of the solutions φ_n and ψ_n one finds that

$$q^{\frac{1}{2}n(n-1)+(\alpha+\gamma)n}\left(q^{\gamma-\alpha+1};q\right)_n z^n(\varphi_n\psi_{n+1}-\varphi_{n+1}\psi_n)=C \tag{65}$$

is a constant independent of n. In particular, $\varphi_0\psi_1-\varphi_1\psi_0=C$. It is straightforward to examine the asymptotic behavior for large n of the solutions in question getting $\varphi_n=1+O(q^n)$ and

$$\psi_n=q^{-\frac{1}{2}n(n-1)-(\alpha+\gamma-1)n-\frac{1}{2}(\gamma-1)(\gamma-2)-\alpha\gamma}z^{1-\gamma-n}\left(-q^\alpha z;q\right)_\infty\left(1+O\left(q^n\right)\right).$$

Sending n to infinity in (65) one finds that C equals the RHS of (64). □

Proposition 27. *For $\alpha,\gamma,z\in\mathbb{C}$,*

$$\mathfrak{F}\left(\left\{\frac{q^{\frac{1}{2}(\alpha+\gamma+k)-\frac{3}{4}}(q^{\gamma-\alpha+k};q^2)_\infty\sqrt{z}}{(q^{\gamma-\alpha+k+1};q^2)_\infty(1-(1-z)q^{\gamma+k-1})}\right\}_{k=1}^{\infty}\right)$$
$$=\frac{(q^\gamma;q)_\infty}{((1-z)q^\gamma;q)_\infty}\,{}_1\phi_1\left(q^\alpha;q^\gamma;q,-q^\gamma z\right). \tag{66}$$

Proof. The both sides of the identity are regarded as meromorphic functions in z. Setting $\operatorname{Im}\gamma$ to a constant, the both sides tend to 1 as $\operatorname{Re}\gamma$ tends to $+\infty$. In virtue of Lemma 2, it suffices to verify that the sequence

$$F_n=\frac{(q^{\gamma+n-1};q)_\infty}{((1-z)q^{\gamma+n-1};q)_\infty}\,{}_1\phi_1\left(q^\alpha;q^{\gamma+n-1};q,-q^{\gamma+n-1}z\right),\quad n\in\mathbb{N},$$

satisfies the three-term recurrence relation $F_n-F_{n+1}+s_nzF_{n+2}=0$, $n\in\mathbb{N}$, where

$$s_n=\frac{q^{\alpha+\gamma+n-1}(1-q^{\gamma-\alpha+n})}{(1-(1-z)q^{\gamma+n-1})(1-(1-z)q^{\gamma+n})}.$$

Since γ here is arbitrary one can consider just the equality for $n=1$. But then the three-term recurrence coincides with (60) (provided z is replaced by $-z$). □

Let us now focus on Eq. (63). One can extract from it a solvable eigenvalue problem for a Jacobi matrix obeying the convergence condition (14).

Proposition 28. *For $\sigma\in\mathbb{R}$ and $\gamma>-1$, let $J=J(\sigma,\gamma)$ be the Jacobi matrix operator in $\ell^2(\mathbb{N})$ defined by (8) and*

$$w_n=\frac{1}{2}\sinh(\sigma)q^{(n-\gamma-1)/2}\sqrt{1-q^{n+\gamma}},\qquad\lambda_n=q^{n-1}. \tag{67}$$

Then $z\neq0$ is an eigenvalue of $J(\sigma,\gamma)$ if and only if

$$\left(\cosh^2(\sigma/2)z^{-1};q\right)_\infty{}_1\phi_1\left(q^{-\gamma}\cosh^2(\sigma/2)z^{-1};\cosh^2(\sigma/2)z^{-1};q,-\sinh^2(\sigma/2)z^{-1}\right)=0.$$

Moreover, if $z\neq0$ solves this characteristic equation then the sequence $\{v_n\}_{n=1}^{\infty}$, with

$$v_n=q^{-\frac{1}{2}\gamma n+\frac{1}{4}n(n-3)}\frac{\sinh^n(\sigma)(2z)^{-n}}{\sqrt{(q^{\gamma+n};q)_\infty}}\left(q^n\cosh^2\left(\frac{\sigma}{2}\right)z^{-1};q\right)_\infty$$
$$\times{}_1\phi_1\left(q^{-\gamma}\cosh^2\left(\frac{\sigma}{2}\right)z^{-1};q^n\cosh^2\left(\frac{\sigma}{2}\right)z^{-1};q,-q^n\sinh^2\left(\frac{\sigma}{2}\right)z^{-1}\right), \tag{68}$$

is a corresponding eigenvector.

Remark 29. Notice that the matrix operator $J(\sigma,\gamma)$ is compact (even trace class).

Please cite this article in press as: F. Štampach, P. Šťovíček, Special functions and spectrum of Jacobi matrices, Linear Algebra Appl. (2013), http://dx.doi.org/10.1016/j.laa.2013.06.024

Proof. First, apply in (63) the substitution

$$\gamma = \tilde{\gamma} + \alpha, \qquad z = q^{\beta}, \qquad u_n = q^{-\alpha n}\tilde{u}_n,$$

and then forget about the tilde over γ and u. Next use the substitution

$$q^{\beta/2} = \tanh\left(\frac{\sigma}{2}\right), \qquad q^{\alpha} = q^{-\gamma}\cosh^2\left(\frac{\sigma}{2}\right)\tilde{z}^{-1}, \qquad u_n = \phi_n\tilde{u}_n,$$

where $\{\phi_n\}$ is a sequence obeying

$$\frac{\phi_n}{\phi_{n+1}} = q^{(\beta+\gamma+n-1)/2}\sqrt{1-q^{\gamma+n}}.$$

Up to a constant multiplier, ${\phi_n}^2 = q^{-\beta n-\gamma n-\frac{1}{2}n(n-3)}(q^{\gamma+n};q)_\infty$. We again forget about the tildes over z and u, and restrict the values of the index n to natural numbers. If $u_0 = 0$ then the transformed sequence $\{u_k\}_{k=1}^{\infty}$ solves the Jacobi eigenvalue problem (9) with w_n and λ_n given in (67).

Further apply the same sequence of transformations to the solution φ_n in (61). Let us call the resulting sequence $\{v_n\}$. A straightforward computation yields (68). Clearly, the sequence $\{v_k;\ k \geqslant 1\}$ is square summable. On general grounds, since $J(\sigma,\gamma)$ falls into the limit point case, any other linearly independent solution of the recurrence in question, (17), cannot be square summable. Hence the characteristic equation for this eigenvalue problem reads $v_0 = 0$. This shows the proposition. □

Remark 30. In the particular case $\gamma = 0$ the characteristic equation simplifies to the form

$$\left(\cosh^2(\sigma/2)z^{-1};q\right)_\infty\left(-\sinh^2(\sigma/2)z^{-1};q\right)_\infty = 0.$$

Hence in that case, apart of $z = 0$, one knows the point spectrum fully explicitly,

$$\operatorname{spec} J(\sigma,0)\setminus\{0\} = \left\{q^k\cosh^2(\sigma/2);\ k=0,1,2,\ldots\right\} \cup \left\{-q^k\sinh^2(\sigma/2);\ k=0,1,2,\ldots\right\}.$$

Remark 31. Of course, Proposition 28 can be as well derived using formulas (16), (18), while knowing that (14) is fulfilled. To evaluate $\xi_n(z)$ one can make use of (66). Applying the same series of substitutions as above to Eq. (66) one gets

$$\mathfrak{F}\left(\left\{\frac{q^{\frac{1}{2}(k-\gamma)-\frac{3}{4}}\sinh(\sigma)(q^{\gamma+k};q^2)_\infty}{2(q^{\gamma+k+1};q^2)_\infty(q^{k-1}-z)}\right\}_{k=1}^{\infty}\right)$$
$$= \frac{(\cosh^2(\sigma/2)z^{-1};q)_\infty}{(z^{-1};q)_\infty}\,{}_1\phi_1\left(q^{-\gamma}\cosh^2\left(\frac{\sigma}{2}\right)z^{-1};\cosh^2\left(\frac{\sigma}{2}\right)z^{-1};q,-\sinh^2\left(\frac{\sigma}{2}\right)z^{-1}\right).$$

Then a straightforward computation yields

$$\xi_n(z) = \frac{2q^{(\gamma+1)/2}\sqrt{(q^{\gamma+1};q)_\infty}}{\sinh(\sigma)(z^{-1};q)_\infty}v_n, \quad n = 0,1,2,\ldots,$$

with v_n being given in (68).

Acknowledgements

The authors wish to acknowledge gratefully partial support from grant No. GA13-11058S of the Czech Science Foundation.

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Please cite this article in press as: F. Štampach, P. Šťovíček, Special functions and spectrum of Jacobi matrices, Linear Algebra Appl. (2013), http://dx.doi.org/10.1016/j.laa.2013.06.024

Factorization of the characteristic function of a Jacobi matrix

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Abstract

In a recent paper we have introduced a class of infinite Jacobi matrices with discrete character of spectra. With each Jacobi matrix from this class we have associated an analytic function, called the characteristic function, whose zero set coincides with the point spectrum of the corresponding Jacobi operator. Here we show that the characteristic function admits Hadamard's factorization in two possible ways – either in the spectral parameter or in an auxiliary parameter which may be called the coupling constant. As an intermediate result we get an explicit expression for the power series expansion of the logarithm of the characteristic function.

Keywords: infinite Jacobi matrix, characteristic function, Hadamard's factorization
AMS classification: 47B36; 33C99; 11A55

1 Introduction

In a recent paper [8] we have introduced a class of infinite Jacobi matrices characterized by a simple convergence condition. Each Jacobi matrix from this class unambiguously determines a closed operator on $\ell^2(\mathbb{N})$ having a discrete spectrum. Moreover, with such a matrix one associates a complex function, called the characteristic function, which is analytic on the complex plane with the closure of the range of the diagonal sequence being excluded, and meromorphic on the complex plane with the set of accumulation points of the diagonal sequence of the matrix being excluded. It turns out that the zero set of the characteristic function actually coincides with the point spectrum of the corresponding Jacobi operator on the domain of definition (with some subtleties when handling the poles; see Theorem 1 below).

The aim of the current paper is to show that the characteristic function admits Hadamard's factorization in two possible ways. First, assuming that the Jacobi matrix is real and the corresponding operator self-adjoint, we derive a factorization in the spectral parameter. Further, for symmetric complex Jacobi matrices we assume the off-diagonal elements to depend linearly on an auxiliary parameter which we call, following physical terminology, the coupling constant. The second factorization formula then concerns this parameter.

Many formulas throughout the paper are expressed in terms of a function, called $\mathfrak{F}$, which is defined on a suitable subset of the linear space of all complex sequences $x = \{x_k\}_{k=1}^{\infty}$; see [7] for its original definition. This function was also heavily employed in [8]. So we start from recalling the definition and basic properties as well as relevant results concerning $\mathfrak{F}$. In addition to Hadamard's factorization we derive, as an intermediate step, a formula for $\log \mathfrak{F}(x)$.

Define $\mathfrak{F} : D \to \mathbb{C}$,

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1} x_{k_1+1} x_{k_2} x_{k_2+1} \dots x_{k_m} x_{k_m+1}, \quad (1)$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}. \quad (2)$$

For a finite number of complex variables we identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$. By convention, let $\mathfrak{F}(\emptyset) = 1$ where $\emptyset$ is the empty sequence.

Note that $\ell^2(\mathbb{N}) \subset D$. For $x \in D$, one has the estimates

$$|\mathfrak{F}(x)| \le \exp\left(\sum_{k=1}^{\infty} |x_k x_{k+1}| \right), \quad |\mathfrak{F}(x) - 1| \le \exp\left(\sum_{k=1}^{\infty} |x_k x_{k+1}| \right) - 1, \quad (3)$$

and it is true that

$$\mathfrak{F}(x) = \lim_{n\to\infty} \mathfrak{F}(x_1, x_2, \dots, x_n). \quad (4)$$

Furthermore, $\mathfrak{F}$ satisfies the relation

$$\mathfrak{F}(\{x_n\}_{n=1}^{\infty}) = \mathfrak{F}(x_1, \dots, x_k)\, \mathfrak{F}(\{x_{k+n}\}_{n=1}^{\infty}) - \mathfrak{F}(x_1, \dots, x_{k-1}) x_k x_{k+1} \mathfrak{F}(\{x_{k+n+1}\}_{n=1}^{\infty}), \quad (5)$$

for any $k \in \mathbb{N}$ and $x \in D$. Let us also point out a simple invariance property. For $x \in D$ and $s \in \mathbb{C}$, $s \ne 0$, it is true that $y \in D$ and

$$\mathfrak{F}(x) = \mathfrak{F}(y), \text{ where } y_{2k-1} = s x_{2k-1},\ y_{2k} = x_{2k}/s,\ k \in \mathbb{N}. \quad (6)$$

We shall deal with symmetric Jacobi matrices

$$J = \begin{pmatrix} \lambda_1 & w_1 & & \\ w_1 & \lambda_2 & w_2 & \\ & w_2 & \lambda_3 & w_3 \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad (7)$$

where $\lambda = \{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$ and $w = \{w_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$. Let us put

$$\gamma_{2k-1} = \prod_{j=1}^{k-1} \frac{w_{2j}}{w_{2j-1}}\,, \ \gamma_{2k} = w_1 \prod_{j=1}^{k-1} \frac{w_{2j+1}}{w_{2j}}\,, \ k = 1, 2, 3, \ldots. \tag{8}$$

Then $\gamma_k \gamma_{k+1} = w_k$.

If for $n \in \mathbb{N}$, J_n is the $n \times n$ Jacobi matrix: $(J_n)_{j,k} = J_{j,k}$ for $1 \leq j, k \leq n$, and I_n is the $n \times n$ unit matrix, then the formula

$$\det(J_n - zI_n) = \left(\prod_{k=1}^{n} (\lambda_k - z)\right) \mathfrak{F}\left(\frac{\gamma_1^2}{\lambda_1 - z}, \frac{\gamma_2^2}{\lambda_2 - z}, \ldots, \frac{\gamma_n^2}{\lambda_n - z}\right). \tag{9}$$

holds true for all $z \in \mathbb{C}$ (after obvious cancellations, the RHS is well defined even for $z = \lambda_k$; here and throughout we use the shorthands LHS and RHS for "left-hand side" and "right-hand side", respectively).

Let us denote

$$\mathbb{C}_0^{\lambda} := \mathbb{C} \setminus \overline{\{\lambda_n;\, n \in \mathbb{N}\}}\,.$$

Moreover, $\operatorname{der}(\lambda)$ designates the set of all accumulation points of the sequence λ. The following theorem is a compilation of several results from [8, Subsec. 3.3].

Theorem 1. *Let a Jacobi matrix J be real and suppose that*

$$\sum_{n=1}^{\infty} \left| \frac{w_n^2}{(\lambda_n - z)(\lambda_{n+1} - z)} \right| < \infty \tag{10}$$

for at least one $z \in \mathbb{C}_0^{\lambda}$. Then
(i) J represents a unique self-adjoint operator on $\ell^2(\mathbb{N})$,
(ii) $\operatorname{spec}(J) \cap (\mathbb{C} \setminus \operatorname{der}(\lambda))$ consists of simple real eigenvalues with no accumulation points in $\mathbb{C} \setminus \operatorname{der}(\lambda)$,
(iii) the series (10) converges locally uniformly on $\mathbb{C}_0^{\lambda}$ and

$$F_J(z) := \mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - z}\right\}_{n=1}^{\infty}\right) \tag{11}$$

is a well defined analytic function on $\mathbb{C}_0^{\lambda}$,
(iv) $F_J(z)$ is meromorphic on $\mathbb{C} \setminus \operatorname{der}(\lambda)$, the order of a pole at $z \in \mathbb{C} \setminus \operatorname{der}(\lambda)$ is less than or equal to the number $r(z)$ of occurrences of z in the sequence λ,
(v) $z \in \mathbb{C} \setminus \operatorname{der}(\lambda)$ belongs to $\operatorname{spec}(J)$ iff

$$\lim_{u \to z} (z - u)^{r(z)} F_J(u) = 0$$

and, in particular, $\operatorname{spec}(J) \cap \mathbb{C}_0^{\lambda} = \operatorname{spec}_p(J) \cap \mathbb{C}_0^{\lambda} = F_J^{-1}(\{0\})$.

We shall mostly focus on real Jacobi matrices, with an exception of Section 4. For our purposes the following particular case, a direct consequence of a more general result derived in [8, Subsec. 3.3], will be sufficient.

Theorem 2. *Suppose J is a complex Jacobi matrix of the form (7) obeying $\lambda_n = 0$, $\forall n$, and $\{w_n\} \in \ell^2(\mathbb{N})$. Then J represents a Hilbert-Schmidt operator, $F_J(z)$ is analytic on $\mathbb{C} \setminus \{0\}$ and*

$$\operatorname{spec}(J) \setminus \{0\} = \operatorname{spec}_p(J) \setminus \{0\} = F_J^{-1}(\{0\}).$$

2 The logarithm of $\mathfrak{F}(x)$

$\mathfrak{F}(x_1, \ldots, x_n)$ is a polynomial function in n complex variables, with $\mathfrak{F}(0) = 1$, and so $\log \mathfrak{F}(x_1, \ldots, x_n)$ is a well defined analytic function in some neighborhood of the origin. The goal of the current section is to derive a formula for the coefficients of the corresponding power series.

For a multiindex $m \in \mathbb{N}^\ell$ denote by $|m|$ its order and by $d(m)$ its length, i.e.

$$|m| = \sum_{j=1}^{\ell} m_j, \;\; d(m) = \ell.$$

For $N \in \mathbb{N}$ define

$$\mathcal{M}(N) = \left\{ m \in \bigcup_{\ell=1}^{N} \mathbb{N}^\ell;\, |m| = N \right\}. \tag{12}$$

One has $\mathcal{M}(1) = \{(1)\}$ and

$$\begin{aligned} \mathcal{M}(N) &= \left\{ \left(1, m_1, m_2, \ldots, m_{d(m)}\right);\, m \in \mathcal{M}(N-1) \right\} \\ &\quad \cup \left\{ \left(m_1 + 1, m_2, \ldots, m_{d(m)}\right);\, m \in \mathcal{M}(N-1) \right\}. \end{aligned}$$

Hence $|\mathcal{M}(N)| = 2^{N-1}$. Here and everywhere in what follows, if M is a finite set then $|M|$ stands for the number of elements of M. Furthermore, for an multiindex $m \in \mathbb{N}^\ell$ put

$$\beta(m) := \prod_{j=1}^{\ell-1} \binom{m_j + m_{j+1} - 1}{m_{j+1}}, \;\; \alpha(m) := \frac{\beta(m)}{m_1}\,. \tag{13}$$

Proposition 3. *In the ring of formal power series in the variables $t_1, \ldots, t_n$, one has*

$$\log \mathfrak{F}(t_1, \ldots, t_n) = -\sum_{\ell=1}^{n-1} \sum_{m \in \mathbb{N}^\ell} \alpha(m) \sum_{k=1}^{n-\ell} \prod_{j=1}^{\ell} \left(t_{k+j-1} t_{k+j}\right)^{m_j}. \tag{14}$$

For a complex sequence $x = \{x_k\}_{k=1}^\infty$ such that $\sum_{k=1}^\infty |x_k x_{k+1}| < \log 2$ one has

$$\log \mathfrak{F}(x) = -\sum_{\ell=1}^{\infty} \sum_{m \in \mathbb{N}^\ell} \alpha(m) \sum_{k=1}^{\infty} \prod_{j=1}^{\ell} \left(x_{k+j-1} x_{k+j}\right)^{m_j}. \tag{15}$$

The proof of Proposition 3 is in principle a matter of some combinatorics. Below we reveal the meaning of the combinatorial numbers $\beta(m)$ and $\alpha(m)$. Let us first introduce a few notions. For $n \in \mathbb{N}$, $n \geq 2$, we regard the set

$$\Lambda_n = \{1, 2, \ldots, n\}$$

as a finite one-dimensional lattice. A loop of length $2N$, $N \in \mathbb{N}$, in Λ_n is a mapping

$$\pi : \{1, 2, \ldots, 2N, 2N+1\} \to \Lambda_n$$

such that $\pi(1) = \pi(2N+1)$ and $|\pi(j+1) - \pi(j)| = 1$ for $1 \le j \le 2N$. The vortex $\pi(1)$ is called the base point of a loop.

For $m \in \mathbb{N}^\ell$ denote by $\Omega(m)$ the set of all loops of length $2|m|$ in $\Lambda_{\ell+1}$ which encounter each edge $(j, j+1)$ exactly $2m_j$ times (counting both directions), $1 \le j \le \ell$. Let $\Omega_1(m)$ designate the subset of $\Omega(m)$ formed by those loops which are based at the vortex 1. If $\pi \in \Omega_1(m)$ then the sequence $(\pi(1), \pi(2), \ldots, \pi(2N))$ contains the vortex 1 exactly m_1 times, the vortices j, $2 \le j \le \ell$, are contained $(m_{j-1} + m_j)$ times in the sequence, and the number of occurrences of the vortex $\ell + 1$ equals m_ℓ.

Lemma 4. *For every $\ell \in \mathbb{N}$ and $m \in \mathbb{N}^\ell$, $|\Omega_1(m)| = \beta(m)$ and $|\Omega(m)| = 2|m|\alpha(m)$.*

Proof. To show the first equality one can proceed by induction in ℓ. For $\ell = 1$ and any $m \in \mathbb{N}$ one clearly has $|\Omega_1(m)| = 1$. Suppose now that $\ell \ge 2$ and fix $m \in \mathbb{N}^\ell$. Denote

$$m' = (m_2, \ldots, m_\ell) \in \mathbb{N}^{\ell-1}.$$

For any $\pi' \in \Omega_1(m')$ put

$$\tilde{\pi} = (1, \pi'(1) + 1, \pi'(2) + 1, \ldots, \pi'(2N' + 1) + 1, 1)$$

where $N' = |m'| = |m| - m_1$. The vortex 2 occurs in $\tilde{\pi}$ exactly $(m_2 + 1)$ times. After any such an occurrence of 2 one may insert none or several copies of the two-letter chain $(1, 2)$. Do it so while requiring that the total number of inserted couples equals $m_1 - 1$. This way one generates all loops from $\Omega_1(m)$, and each exactly once. This implies the recurrence rule

$$|\Omega_1(m_1, m_2, \ldots, m_\ell)| = \binom{m_1 - 1 + m_2}{m_2} |\Omega_1(m_2, \ldots, m_\ell)|,$$

thus proving that $|\Omega_1(m)| = \beta(m)$.

Let us proceed to the second equality. Put $N = |m|$. Consider the cyclic group $G = \langle g \rangle$, $g^{2N} = 1$. G acts on $\Omega(m)$ according to the rule

$$g \cdot \pi = (\pi(2), \pi(3), \ldots, \pi(2N+1), \pi(2)), \ \forall \pi \in \Omega(m).$$

Clearly, $G \cdot \Omega_1(m) = \Omega(m)$. Let us write $\Omega(m)$ as a disjoint union of orbits,

$$\Omega(m) = \bigcup_{s=1}^{M} \mathcal{O}_s.$$

For each orbit choose $\pi_s \in \mathcal{O}_s \cap \Omega_1(m)$. Let $H_s \subset G$ be the stabilizer of π_s. Then

$$|\Omega(m)| = \sum_{s=1}^{M} \frac{2N}{|H_s|}.$$

Denote further by G_s^1 the subset of G formed by those elements a obeying $a{\cdot}\pi_s \in \Omega_1(m)$ (i.e. the vortex 1 is still the base point). Then $|G_s^1| = m_1$ and $\mathcal{O}_s \cap \Omega_1(m) = G_s^1 \cdot \pi_s$. Moreover, $G_s^1 \cdot H_s = G_s^1$, i.e. H_s acts freely from the right on G_s^1, with orbits of this action being in one-to-one correspondence with elements of $\mathcal{O}_s \cap \Omega_1(m)$. Hence $|\mathcal{O}_s \cap \Omega_1(m)| = |G_s^1|/|H_s|$ and

$$|\Omega_1(m)| = \sum_{s=1}^{M} |\mathcal{O}_s \cap \Omega_1(m)| = \sum_{s=1}^{M} \frac{m_1}{|H_s|}\,.$$

This shows that $|\Omega(m)| = (2N/m_1)|\Omega_1(m)|$. In view of the first equality of the proposition and (13), the proof is complete. □

For $m \in \mathbb{N}^\ell$ let

$$\binom{|m|}{m} := \frac{|m|!}{m_1!\,m_2!\,\ldots\,m_\ell!}\,.$$

Lemma 5. *For $N \in \mathbb{N}$,*

$$\sum_{m\in\mathcal{M}(N)} \alpha(m) = \frac{1}{2N}\binom{2N}{N}. \tag{16}$$

Proof. According to Lemma 4, the sum

$$2N \sum_{m\in\mathcal{M}(N)} \alpha(m) = \sum_{m\in\mathcal{M}(N)} |\Omega(m)|$$

equals the number of all classes of loops of length $2N$ in the one-dimensional lattice $\mathbb{Z}$ provided loops differing by translations are identified. These classes are generated by making $2N$ choices, in all possible ways, each time choosing either the sign plus or minus (moving to the right or to the left on the lattice) while the total number of occurrences of each sign being equal to N. □

Lemma 6. *For every $\ell \in \mathbb{N}$ and $m \in \mathbb{N}^\ell$,*

$$\alpha(m) \le \frac{1}{|m|}\binom{|m|}{m},$$

and equality holds if and only if $\ell = 1$ or 2 .

Proof. Put $\gamma(m) = \alpha(m)/\binom{|m|}{m}$. To show that $\gamma(m) \le 1/|m|$ one can proceed by induction in ℓ. It is immediate to check that equality holds for $\ell = 1$ and 2 . For $\ell \ge 3$ and $m_1 > 1$ one readily verifies that

$$\gamma(m_1, m_2, m_3, \ldots, m_\ell) < \gamma(m_1 - 1, m_2 + 1, m_3, \ldots, m_\ell).$$

Furthermore, if $\ell \ge 3$, $m_1 = 1$ and the inequality is known to be valid for $\ell - 1$, one has

$$\gamma(m_1, m_2, m_3, \ldots, m_\ell) = \frac{m_2\,\gamma(m_2, m_3, \ldots, m_\ell)}{1 + m_2 + m_3 + \ldots + m_\ell} < \frac{1}{|m|}\,.$$

The lemma follows. □

Proof of Proposition 3. In the course of the proof and otherwise as well we repeatedly use the trivial observation that

$$\cup_{\ell=1}^{\infty}\mathbb{N}^{\ell}=\cup_{N=1}^{\infty}\mathcal{M}(N)\ \text{ and }\ \cup_{\ell=1}^{n-1}\mathbb{N}^{\ell}=\cup_{N=1}^{\infty}\{m\in\mathcal{M}(N);\ d(m)<n\}.$$

The coefficients of the power series expansion at the origin of the function $\log\mathfrak{F}\,(t_1,\ldots,t_n)$ can be calculated in the ring of formal power series. As observed in [8], one has

$$\mathfrak{F}(t_1,\ldots,t_n)=\det(I+T)$$

where

$$T=\begin{pmatrix} 0 & t_1 & & & & \\ t_2 & 0 & t_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & t_{n-1} & 0 & t_{n-1} \\ & & & & t_n & 0 \end{pmatrix}. \tag{17}$$

Since $\det\exp(A)=\exp(\operatorname{Tr}A)$ and so $\log\det(I+T)=\operatorname{Tr}\log(I+T)$, and noticing that $\operatorname{Tr}T^{2k+1}=0$, one gets

$$\log\mathfrak{F}\,(t_1,\ldots,t_n)=\operatorname{Tr}\log(I+T)=-\sum_{N=1}^{\infty}\frac{1}{2N}\,\operatorname{Tr}T^{2N}.$$

From (17) one deduces that

$$\operatorname{Tr}T^{2N}=\sum_{\pi\in\mathcal{L}(N)}\prod_{j=1}^{2N}t_{\pi(j)} \tag{18}$$

where $\mathcal{L}(N)$ stands for the set of all loops of length $2N$ in Λ_n (see the notation preceding Lemma 4). Further one observes that for each $\pi\in\mathcal{L}(N)$ there exist unambiguously defined $m\in\mathcal{M}(N)$, with $d(m)<n$, an integer k, $1\le k\le n-d(m)$, and $\tilde{\pi}\in\Omega(m)$ such that $\pi(j)=k+\tilde{\pi}(j)-1$ for $1\le j\le 2N+1$ (then $k=\min\{\pi(j);\ 1\le j\le 2N\}$). Hence the RHS of (18) equals

$$\sum_{\substack{m\in\mathcal{M}(N)\\ d(m)<n}}\ \sum_{k=1}^{n-d(m)}|\Omega(m)|\prod_{j=1}^{d(m)}\left(t_{k+j-1}t_{k+j}\right)^{m_j}.$$

To verify (14) it suffices to apply Lemma 4.

Suppose now x is a complex sequence. If $\sum_k|x_kx_{k+1}|<\log 2$ one has, by (3), $|\mathfrak{F}(x)-1|<1$ and so $\log\mathfrak{F}(x)$ is well defined. Moreover, according to (4),

$$\log\mathfrak{F}(x)=\lim_{n\to\infty}\log\mathfrak{F}(x_1,\ldots,x_n).$$

If $\sum_k |x_k x_{k+1}| < 1$ then the RHS of (14) admits the limit procedure, too, as demonstrated by the simple estimate (replacing t_js by x_js)

$$\begin{aligned} |\text{the RHS of (14)}| &\leq \sum_{N=1}^{\infty} \left[\max_{m \in \mathcal{M}(N)} \frac{\alpha(m)}{\binom{N}{m}} \right] \sum_{m \in \mathcal{M}(N)} \binom{N}{m} \sum_{k=1}^{\infty} \prod_{j=1}^{d(m)} |x_{k+j-1} x_{k+j}|^{m_j} \\ &\leq \sum_{N=1}^{\infty} \frac{1}{N} \left(\sum_{k=1}^{\infty} |x_k x_{k+1}| \right)^N = -\log\left(1 - \sum_{k=1}^{\infty} |x_k x_{k+1}| \right). \end{aligned}$$

Here we have used Lemma 6. □

3 Factorization in the spectral parameter

In this section, we introduce a regularized characteristic function of a Jacobi matrix and show that it can be expressed as a Hadamard infinite product.

Let $\lambda = \{\lambda_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$ be real sequences such that $\lim_{n\to\infty} \lambda_n = +\infty$ and $w_n \neq 0$, $\forall n$. In addition, without loss of generality, $\{\lambda_n\}_{n=1}^{\infty}$ is assumed to be positive. Moreover, suppose that

$$\sum_{n=1}^{\infty} \frac{w_n^2}{\lambda_n \lambda_{n+1}} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty. \tag{19}$$

Under these assumptions, by Theorem 1, J defined in (7) may be regarded as a self-adjoint operator on $\ell^2(\mathbb{N})$. Moreover, $\operatorname{der}(\lambda)$ is clearly empty and the characteristic function $F_J(z)$ is meromorphic on $\mathbb{C}$ with possible poles lying in the range of λ. To remove the poles let us define the function

$$\Phi_\lambda(z) := \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n} \right) e^{z/\lambda_n}.$$

Since $\sum_n \lambda_n^{-2} < \infty$, Φ_λ is a well defined entire function. Further, Φ_λ has zeros at the points $z = \lambda_n$, with multiplicity being equal to the number of repetitions of λ_n in the sequence λ, and no zeros otherwise; see, for instance, [5, Chp. 15].

Finally we define (see (11))

$$H_J(z) := \Phi_\lambda(z) F_J(z),$$

and call $H_J(z)$ the regularized characteristic function of the Jacobi operator J. Note that for $\varepsilon \geq 0$, $F_{J+\varepsilon I}(z) = F_J(z - \varepsilon)$ and so

$$H_{J+\varepsilon I}(z) = H_J(z - \varepsilon) \Phi_\lambda(-\varepsilon)^{-1} \exp\left(-z \sum_{n=1}^{\infty} \frac{\varepsilon}{\lambda_n(\lambda_n + \varepsilon)} \right). \tag{20}$$

According to Theorem 1, the spectrum of J is discrete, simple and real, and

$$\operatorname{spec}(J) = \operatorname{spec}_p(J) = H_J^{-1}(\{0\}).$$

As is well known, the determinant of an operator $I+A$ on a Hilbert space can be defined provided A belongs to the trace class. The definition, in a modified form, can be extended to other Schatten classes $\mathscr{I}_p$ as well, in particular to Hilbert-Schmidt operators; see [6] for a detailed survey of the theory. Let us denote, as usual, the trace class and the Hilbert-Schmidt class by $\mathscr{I}_1$ and $\mathscr{I}_2$, respectively. If $A \in \mathscr{I}_2$ then

$$(I+A)\exp(-A) - I \in \mathscr{I}_1,$$

and one defines

$$\det_2(I+A) := \det\left((I+A)\exp(-A)\right).$$

We shall need the following formulas [6, Chp. 9]. For $A, B \in \mathscr{I}_2$ one has

$$\det_2(I+A+B+AB) = \det_2(I+A)\det_2(I+B)\,\exp\left(-\operatorname{Tr}(AB)\right). \tag{21}$$

A factorization formula holds for $A \in \mathscr{I}_2$ and $z \in \mathbb{C}$,

$$\det_2(I+zA) = \prod_{n=1}^{N(A)} (1+z\mu_n(A))\exp\left(-z\mu_n(A)\right), \tag{22}$$

where $\mu_n(A)$ are all (nonzero) eigenvalues of A counted up to their algebraic multiplicity [6, Thm. 9.2]. In particular, $I+zA$ is invertible iff $\det_2(I+zA) \neq 0$. Moreover, the Plemejl-Smithies formula tells us that for $A \in \mathscr{I}_2$,

$$\det_2(I+zA) = \sum_{m=0}^{\infty} a_m(A)\,\frac{z^m}{m!}, \tag{23}$$

where

$$a_m(A) = \det\begin{pmatrix} 0 & m-1 & 0 & \dots & 0 & 0 \\ \operatorname{Tr} A^2 & 0 & m-2 & \dots & 0 & 0 \\ \operatorname{Tr} A^3 & \operatorname{Tr} A^2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \operatorname{Tr} A^{m-1} & \operatorname{Tr} A^{m-2} & \operatorname{Tr} A^{m-3} & \dots & 0 & 1 \\ \operatorname{Tr} A^m & \operatorname{Tr} A^{m-1} & \operatorname{Tr} A^{m-2} & \dots & \operatorname{Tr} A^2 & 0 \end{pmatrix} \tag{24}$$

for $m \geq 1$, and $a_0(A) = 1$ [6, Thm. 5.4]. Finally, there exists a constant C_2 such that for all $A, B \in \mathscr{I}_2$,

$$|\det_2(I+A) - \det_2(I+B)| \leq \|A-B\|_2 \exp\big(C_2(\|A\|_2 + \|B\|_2 + 1)^2\big), \tag{25}$$

where $\|\cdot\|_2$ stands for the Hilbert-Schmidt norm.

We write the Jacobi matrix in the form

$$J = L + W + W^*$$

where L is a diagonal matrix while W is lower triangular. By assumption (19), the operators L^{-1} and

$$K := L^{-1/2}(W+W^*)L^{-1/2} \tag{26}$$

are Hilbert-Schmidt. Hence for every $z \in \mathbb{C}$, the operator $L^{-1/2}(W + W^* - z)L^{-1/2}$, with its matrix being equal to

$$\begin{pmatrix} -z/\lambda_1 & w_1/\sqrt{\lambda_1\lambda_2} & & & \\ w_1/\sqrt{\lambda_1\lambda_2} & -z/\lambda_2 & w_2/\sqrt{\lambda_2\lambda_3} & & \\ & w_2/\sqrt{\lambda_2\lambda_3} & -z/\lambda_3 & w_3/\sqrt{\lambda_3\lambda_4} & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

belongs to the Hilbert-Schmidt class.

Lemma 7. *For every $z \in \mathbb{C}$,*

$$H_J(z) = \det\nolimits_2\left(I + L^{-1/2}(W + W^* - z)L^{-1/2}\right).$$

In particular,

$$H_J(0) = F_J(0) = \det\nolimits_2(I + K).$$

Proof. We first verify the formula for the truncated finite rank operator $J_N = P_N J P_N$, where P_N is the orthogonal projection onto the subspace spanned by the first N vectors of the canonical basis in $\ell^2(\mathbb{N})$. Using formula (9) one derives

$$\begin{aligned} &\det\left[(I + P_N L^{-1/2}(W + W^* - z)L^{-1/2}P_N)\exp\left(-P_N L^{-1/2}(W + W^* - z)L^{-1/2}P_N\right)\right] \\ &= \det(P_N L^{-1}P_N)\det(J_N - zI_N)\,\exp\left(z\operatorname{Tr}(P_N L^{-1}P_N)\right) \\ &= \left(\prod_{n=1}^{N}\left(1 - \frac{z}{\lambda_n}\right)e^{z/\lambda_n}\right)\mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - z}\right\}_{n=1}^{N}\right). \end{aligned}$$

Sending N to infinity it is clear, by (4) and (19), that the RHS tends to $H_J(z)$. Moreover, one knows that $\det_2(I + A)$ is continuous in A in the Hilbert-Schmidt norm, as it follows from (25). Thus to complete the proof it suffices to observe that if $A \in \mathscr{I}_2$ then $\|P_N A P_N - A\|_2 \to 0$ as $N \to \infty$. □

We intend to apply to $H_J(z)$ the Hadamard factorization theorem; see, for example, [1, Thm. XI.3.4]. For simplicity we assume that $F_J(0) \neq 0$ and so J is invertible. Otherwise one could replace J by $J + \varepsilon I$ for some $\varepsilon > 0$ and make use of (20).

As already mentioned, the operator K defined in (26) is Hilbert-Schmidt. At the same time, this is a Jacobi matrix operator with zero diagonal admitting application of Theorem 1. One readily finds that

$$F_K(z) = \mathfrak{F}\left(\left\{-\frac{\gamma_n^2}{z\lambda_n}\right\}_{n=1}^{\infty}\right).$$

Hence $F_K(-1) = F_J(0)$, and J is invertible if and only if the same is true for $(I + K)$. In that case, again by Theorem 1, 0 belongs to the resolvent set of J, and

$$J^{-1} = L^{-1/2}(I + K)^{-1}L^{-1/2}. \tag{27}$$

Lemma 8. *If J is invertible then J^{-1} is a Hilbert-Schmidt operator and*

$$\det_2\left(I - z(I+K)^{-1}L^{-1}\right) = \det_2\left(I - zJ^{-1}\right) \tag{28}$$

for all $z \in \mathbb{C}$.

Proof. By assumption (19), $L^{-1/2}$ belongs to the Schatten class $\mathscr{I}_4$. Since the Schatten classes are norm ideals and fulfill $\mathscr{I}_p\mathscr{I}_q \subset \mathscr{I}_r$ whenever $r^{-1} = p^{-1} + q^{-1}$ [6, Thm. 2.8], one deduces from (27) that $J^{-1} \in \mathscr{I}_2$.

Furthermore, one knows that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ provided $A \in \mathscr{I}_p$, $B \in \mathscr{I}_q$ and $p^{-1} + q^{-1} = 1$ [6, Cor. 3.8]. Hence

$$\mathrm{Tr}\left((I+K)^{-1}L^{-1}\right)^k = \mathrm{Tr}\left(L^{-1/2}(I+K)^{-1}L^{-1/2}\right)^k = \mathrm{Tr}(J^{-k}),\ \forall k \in \mathbb{N}, k \geq 2.$$

It follows that the coefficients a_m defined in (24) fulfill

$$a_m((I+K)^{-1}L^{-1}) = a_m(J^{-1}) \ \text{ for } m = 0, 1, 2, \ldots.$$

The Plemejl-Smithies formula (23) then implies (28). □

Theorem 9. *Using notation introduced in (7), suppose a real Jacobi matrix J obeys (19) and is invertible. Denote by $\lambda_n(J)$, $n \in \mathbb{N}$, the eigenvalues of J (all of them are real and simple). Then $L^{-1} - J^{-1} \in \mathscr{I}_1$,*

$$\sum_{n=1}^{\infty} \lambda_n(J)^{-2} < \infty, \tag{29}$$

and for the regularized characteristic function of J one has

$$H_J(z) = F_J(0)\, e^{bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n(J)}\right) e^{z/\lambda_n(J)} \tag{30}$$

where

$$b = \mathrm{Tr}\left(L^{-1} - J^{-1}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_n(J)}\right).$$

Proof. Recall (27). By a simple algebra, and since $L^{-1/2} \in \mathscr{I}_4$, $K \in \mathscr{I}_2$, one has

$$L^{-1} - J^{-1} = L^{-1/2}K(I+K)^{-1}L^{-1/2} \in \mathscr{I}_1. \tag{31}$$

By Lemma 8, the operator J^{-1} is Hermitian and Hilbert-Schmidt. This implies (29). Furthermore, by Lemma 7, formula (21) and Lemma 8,

$$\begin{aligned} H_J(z) &= \det_2(I + K - zL^{-1}) \\ &= \det_2(I+K)\det_2\left(I - z(I+K)^{-1}L^{-1}\right)\exp\left[z\,\mathrm{Tr}\left(K(I+K)^{-1}L^{-1}\right)\right] \\ &= F_J(0)\, e^{bz}\, \det_2\left(I - zJ^{-1}\right). \end{aligned}$$

Here we have used (31) implying

$$\operatorname{Tr}\left(K(I+K)^{-1}L^{-1}\right) = \operatorname{Tr}\left(L^{-1/2}K(I+K)^{-1}L^{-1/2}\right) = \operatorname{Tr}\left(L^{-1}-J^{-1}\right) = b.$$

Finally, by formula (22),

$$\det_2\left(I - zJ^{-1}\right) = \prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n(J)}\right)e^{z/\lambda_n(J)}.$$

This completes the proof. □

Corollary 10. *For each $\epsilon > 0$ there is $R_\epsilon > 0$ such that for $|z| > R_\epsilon$,*

$$|H_J(z)| < \exp\left(\epsilon|z|^2\right). \tag{32}$$

Proof. Theorem 9, and particularly the product formula (30) implies that $H_J(z)$ is an entire function of genus one. In that case the growth property (32) is known to be valid; see, for example, Theorem XI.2.6 in [1]. □

Example 11. Put $\lambda_n = n$ and $w_n = w \neq 0$, $\forall n \in \mathbb{N}$. As shown in [7], the Bessel functions of the first kind can be expressed as

$$J_\nu(2w) = \frac{w^\nu}{\Gamma(\nu+1)}\,\mathfrak{F}\left(\left\{\frac{w}{\nu+k}\right\}_{k=1}^{\infty}\right), \tag{33}$$

as long as $w, \nu \in \mathbb{C}$, $\nu \notin -\mathbb{N}$. Using (33) and the well known formula for the gamma function,

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n},$$

where γ is the Euler constant, one gets

$$H_J(z) = e^{\gamma z}w^z J_{-z}(2w).$$

Let us apply Theorem 9 to this Jacobi matrix. As a result one reveals the infinite product formula for a Bessel function considered as a function of its order. Assuming $J_0(2w) \neq 0$, the formula reads

$$\frac{w^z J_{-z}(2w)}{J_0(2w)} = e^{c(w)z}\prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n(J)}\right)e^{z/\lambda_n(J)}$$

where

$$c(w) = \frac{1}{J_0(2w)}\sum_{k=0}^{\infty}(-1)^k\psi(k+1)\,\frac{w^{2k}}{(k!)^2}$$

and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the polygamma function (recall that $\psi(1) = -\gamma$ and so $c(0) = -\gamma$). To derive the expression for $c(w)$ it suffices to compare the coefficients at z on both sides.

4 Factorization in the coupling constant

Let $x = \{x_n\}_{n=1}^{\infty}$ be a sequence of nonzero complex numbers belonging to the domain D defined in (2). Our goal in this section is to prove a factorization formula for the entire function

$$f(w) := \mathfrak{F}(wx),\ \ w \in \mathbb{C}.$$

Let us remark that $f(w)$ is even.

To this end, let us put $v_k = \sqrt{x_k}$, $\forall k$, (any branch of the square root is suitable) and introduce the auxiliary Jacobi matrix

$$A = \begin{pmatrix} 0 & a_1 & 0 & 0 & \cdots \\ a_1 & 0 & a_2 & 0 & \cdots \\ 0 & a_2 & 0 & a_3 & \cdots \\ 0 & 0 & a_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ with } a_k = v_k v_{k+1}\,,\ \ k \in \mathbb{N}. \tag{34}$$

Then A represents a Hilbert-Schmidt operator on $\ell^2(\mathbb{N})$ with the Hilbert-Schmidt norm

$$\|A\|_2^2 = 2\sum_{k=1}^{\infty} |a_k|^2 = 2\sum_{k=1}^{\infty} |x_k x_{k+1}|.$$

The relevance of A to our problem comes from the equality

$$F_A(z) = \mathfrak{F}\Big(\Big\{\frac{x_k}{z}\Big\}_{k=1}^{\infty}\Big) = f\big(z^{-1}\big)\,,$$

which can be verified with the aid of (6). Hence $F_A(z)$ is analytic on $\mathbb{C} \setminus \{0\}$. By Theorem 2, the set of nonzero eigenvalues of A coincides with the zero set of $F_A(z)$. It even turns out that the algebraic multiplicity of a nonzero eigenvalue ζ of A equals the multiplicity of ζ as a root of the function $F_A(z)$, as stated in the following supplement to Theorem 2.

Proposition 12. *Under the same assumptions as in Theorem 2, the algebraic multiplicity of any nonzero eigenvalue ζ of J is equal to the multiplicity of the root ζ^{-1} of the entire function $\varphi(z) = F_J(z^{-1}) = \mathfrak{F}(\{z\gamma_n^2\}_{n=1}^{\infty})$.*

Proof. Recall that $\gamma_n\gamma_{n+1} = w_n$ and so, by the assumptions of Theorem 2, $\{\gamma_n^2\} \in D$. Denote again by P_N, $N \in \mathbb{N}$, the orthogonal projection onto the subspace spanned by the first N vectors of the canonical basis in $\ell^2(\mathbb{N})$. From formula (9) we deduce that

$$\mathfrak{F}\left(\{z\gamma_n^2\}_{n=1}^{N}\right) = \det(I - zJ_N) = \det\big((I - zJ_N)e^{zJ_N}\big)\,,$$

where $J_N = P_N J P_N$. Since $P_N J P_N$ tends to J in the Hilbert-Schmidt norm, as $N \to \infty$, and by continuity of the generalized determinant as a functional on the space of Hilbert-Schmidt operators (see (25)) one immediately gets

$$\varphi(z) = \mathfrak{F}\big(\{z\gamma_n^2\}_{n=1}^{\infty}\big) = \det\big((I - zJ)e^{zJ}\big) = \det{}_2(I - zJ). \tag{35}$$

From (22) it follows that $\varphi(z) = (1 - \zeta z)^m\,\tilde{\varphi}(z)$ where m is the algebraic multiplicity of ζ, $\tilde{\varphi}(z)$ is an entire function and $\tilde{\varphi}(\zeta^{-1}) \neq 0$. □

The zero set of $f(w)$ is at most countable and symmetric with respect to the origin. One can split $\mathbb{C}$ into two half-planes so that the border line passes through the origin and contains no nonzero root of f. Fix one of the half-planes and enumerate all nonzero roots in it as $\{\zeta_k\}_{k=1}^{N(f)}$, with each root being repeated in the sequence according to its multiplicity. The number $N(f)$ may be either a non-negative integer or infinity. Then

$$\operatorname{spec}_p(A) \setminus \{0\} = \left\{\pm\zeta_k^{-1};\ k \in \mathbb{N}, k \leq N(f)\right\}.$$

Since A^2 is a trace class operator one has, by Proposition 12 and Lidskii's theorem,

$$\sum_{k=1}^{N(f)} \frac{1}{\zeta_k^2} = \frac{1}{2} \operatorname{Tr} A^2 = \sum_{k=1}^{\infty} x_k x_{k+1}. \tag{36}$$

Moreover, the sum on the LHS converges absolutely, as it follows from Weyl's inequality [6, Thm. 1.15].

Theorem 13. *Suppose $x = \{x_k\}_{k=1}^{\infty}$ is a sequence of nonzero complex numbers such that*

$$\sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty.$$

Then zeros of the entire even function $f(w) = \mathfrak{F}(wx)$ can be arranged into sequences

$$\{\zeta_k\}_{k=1}^{N(f)} \cup \{-\zeta_k\}_{k=1}^{N(f)},$$

with each zero being repeated according to its multiplicity, and

$$f(w) = \prod_{k=1}^{N(f)} \left(1 - \frac{w^2}{\zeta_k^2}\right). \tag{37}$$

Proof. Equality (37) can be deduced from Hadamard's factorization theorem; see, for example, [1, Chp. XI]. In fact, the absolute convergence of the series $\sum \zeta_k^{-2}$ in (36) means that the rank of f is at most 1. Furthermore, (3) implies that

$$|f(w)| \leq \exp\left(|w|^2 \sum_{k=1}^{\infty} |x_k x_{k+1}|\right),$$

and so the order of f is less than or equal to 2. Hadamard's factorization theorem tells us that the genus of f is at most 2. Taking into account that f is even and $f(0) = 1$, this means nothing but

$$f(w) = \exp(cw^2) \prod_{k=1}^{N(f)} \left(1 - \frac{w^2}{\zeta_k^2}\right)$$

for some $c \in \mathbb{C}$. Equating the coefficients at w^2 one gets

$$-\sum_{k=1}^{\infty} x_k x_{k+1} = c - \sum_{k=1}^{N(f)} \frac{1}{\zeta_k^2}\,.$$

According to (36), $c = 0$. ☐

Corollary 14. *For any $n \in \mathbb{N}$ (and recalling (12), (13)),*

$$\sum_{k=1}^{N(f)} \frac{1}{\zeta_k^{2n}} = n \sum_{m \in \mathcal{M}(n)} \alpha(m) \sum_{k=1}^{\infty} \prod_{j=1}^{d(m)} \left(x_{k+j-1} x_{k+j}\right)^{m_j}. \tag{38}$$

Proof. Using Proposition 3, one can expand $\log f(w)$ into a power series at $w = 0$. Applying log to (37) and equating the coefficients at w^{2n} gives (38). ☐

If the sequence $\{x_k\}$ in Theorem 13 is positive one has some additional information about the zeros of $f(w)$. In that case the v_ks in (34) can be chosen positive, and so A is a self-adjoint Hilbert-Schmidt operator. The zero set of f is countable and all roots are real, simple and have no finite accumulation points. Enumerating positive zeros in ascending order as ζ_k, $k \in \mathbb{N}$, factorization (37) and identities (38) hold true. Since the first positive root ζ_1 is strictly smaller than all other positive roots, one has

$$\zeta_1 = \lim_{N\to\infty} \left(\sum_{m \in \mathcal{M}(N)} \alpha(m) \sum_{k=1}^{\infty} \prod_{j=1}^{d(m)} \left(x_{k+j-1} x_{k+j}\right)^{m_j} \right)^{-1/(2N)}.$$

Remark 15. Still assuming the sequence $\{x_k\}$ to be positive let $g(z)$ be an entire function defined by

$$g(z) = 1 + \sum_{n=1}^{\infty} g_n z^n = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\zeta_k^2}\right),$$

i.e. $g(w^2) = f(w)$. In some particular cases the coefficients g_n may be known explicitly and then the spectral zeta function can be evaluated recursively. Put

$$\sigma(2n) = \sum_{k=1}^{\infty} \frac{1}{\zeta_k^{2n}}\,,\ n \in \mathbb{N}.$$

Taking the logarithmic derivative of $g(z)$ and equating coefficients at the same powers of z leads to the recurrence rule

$$\sigma(2) = -g_1,\ \sigma(2n) = -n g_n - \sum_{k=1}^{n-1} g_{n-k}\, \sigma(2k) \ \text{ for } n > 1. \tag{39}$$

Example 16. Put $x_k = (\nu + k)^{-1}$, with $\nu > -1$. Recalling (33) and letting $z = w/2$, one obtains the following factorization of the Bessel function [10],

$$\left(\frac{z}{2}\right)^{-\nu} \Gamma(\nu+1) J_\nu(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,k}^2}\right),$$

as a particular case of Theorem 13. Corollary 14 implies a formula for the so called Rayleigh function [4]

$$\sigma_\nu(s) = \sum_{k=1}^{\infty} \frac{1}{j_{\nu,k}^{\,s}}\,, \ \operatorname{Re} s > 1,$$

namely

$$\sigma_\nu(2N) = 2^{-2N} N \sum_{k=1}^{\infty} \sum_{m \in \mathcal{M}(N)} \alpha(m) \prod_{j=1}^{d(m)} \left(\frac{1}{(j+k+\nu-1)(j+k+\nu)}\right)^{m_j}, \ N \in \mathbb{N}.$$

Example 17. This examples is perhaps less commonly known and concerns the Ramanujan function, also interpreted as the q-Airy function by some authors [3, 9], and defined by

$$A_q(z) := {}_0\phi_1(\,;0;q,-qz) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} (-z)^n, \tag{40}$$

where ${}_0\phi_1(\,;b;q,z)$ is the basic hypergeometric series (q-hypergeometric series) and $(a;q)_k$ is the q-Pochhammer symbol (see, for instance, [2]). In (40) we suppose $0 < q < 1$ and $z \in \mathbb{C}$. In [7] we have shown that

$$A_q(w^2) = q\,\mathfrak{F}\Big(\big\{wq^{(2k-1)/4}\big\}_{k=1}^{\infty}\Big). \tag{41}$$

Denote by $0 < \zeta_1(q) < \zeta_2(q) < \zeta_3(q) < \ldots$ the positive zeros of $w \mapsto A_q(w^2)$ and put $\iota_k(q) = \zeta_k(q)^2$, $k \in \mathbb{N}$. Then Theorem 13 tells us that the zeros of $A_q(z)$ are exactly $0 < \iota_1(q) < \iota_2(q) < \iota_3(q) < \ldots$, all of them are simple and

$$A_q(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\iota_k(q)}\right).$$

One has $\big\{\iota_k(q)^{-1/2};\ k \in \mathbb{N}\big\} = \operatorname{spec}(\boldsymbol{A}(q)) \backslash \{0\}$ where $\boldsymbol{A}(q)$ is a Hilbert-Schmidt matrix operator in $\ell^2(\mathbb{N})$ whose matrix is of the form (34), with $a_k = q^{k/2}$. Corollary 14 yields a formula for the spectral zeta function $D_N(q)$ associated with $A_q(z)$, namely

$$D_N(q) := \sum_{k=1}^{\infty} \frac{1}{\iota_k(q)^N} = \frac{Nq^N}{1-q^N} \sum_{m \in \mathcal{M}(N)} \alpha(m)\, q^{\epsilon_1(m)}, \ N \in \mathbb{N},$$

where$\forall m \in \mathbb{N}^\ell$, $\epsilon_1(m) = \sum_{j=1}^{\ell} (j-1)\, m_j$. In accordance with (39), from the power series expansion of $A_q(z)$ we get the recurrence

$$D_n(q) = (-1)^{n+1} \frac{nq^{n^2}}{(q;q)_n} - \sum_{k=1}^{n-1} (-1)^k \frac{q^{k^2}}{(q;q)_k} D_{n-k}(q), \ n = 1, 2, 3, \ldots.$$

Consider now a real Jacobi matrix J of the form (7) such that the diagonal sequence $\{\lambda_n\}$ is semibounded. Suppose further that the off-diagonal elements w_n depend on a real parameter w as $w_n = w\omega_n$, $n \in \mathbb{N}$, with $\{\omega_n\}$ being a fixed sequence of positive numbers. Following physical terminology one may call w the coupling constant. Denote $\lambda_{\text{inf}} = \inf \lambda_n$. Assume that

$$\sum_{n=1}^{\infty} \frac{\omega_n^2}{(\lambda_n - z)(\lambda_{n+1} - z)} < \infty$$

for some and hence any $z < \lambda_{\text{inf}}$. For $z < \lambda_{\text{inf}}$, Theorem 13 can be applied to the sequence

$$x_n(z) = \frac{\kappa_n^2}{\lambda_n - z}\,,\ n \in \mathbb{N},$$

where $\{\kappa_n\}$ is defined recursively by $\kappa_1 = 1$, $\kappa_n\kappa_{n+1} = \omega_n$; comparing to (8) one has $\kappa_{2k-1} = \gamma_{2k-1}$, $\kappa_{2k} = \gamma_{2k}/w$. Let

$$F_J(z; w) = \mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - z}\right\}_{n=1}^{\infty}\right) = \mathfrak{F}(\{w\, x_n(z)\}_{n=1}^{\infty})$$

be the characteristic function of $J = J(w)$. We conclude that for every $z < \lambda_{\text{inf}}$ fixed, the equation $F_J(z; w) = 0$ in the variable w has a countably many positive simple roots $\zeta_k(z)$, $k \in \mathbb{N}$, enumerated in ascending order, and

$$F_J(z; w) = \prod_{k=1}^{\infty} \left(1 - \frac{w^2}{\zeta_k(z)^2}\right).$$

Acknowledgments

The authors wish to acknowledge gratefully partial support from the following grants: Grant No. 201/09/0811 of the Czech Science Foundation (P.Š.) and Grant No. LC06002 of the Ministry of Education of the Czech Republic (F.Š.).

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Part IV

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Journal of Mathematical Analysis and Applications

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Orthogonal polynomials associated with Coulomb wave functions

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ARTICLE INFO

Article history:
Received 8 November 2013
Available online xxxx
Submitted by M.J. Schlosser

Keywords:
Orthogonal polynomials
Measure of orthogonality
Lommel polynomials
Spectral zeta function
Coulomb wave function

ABSTRACT

A new class of orthogonal polynomials associated with Coulomb wave functions is introduced. These polynomials play a role analogous to that the Lommel polynomials have in the theory of Bessel functions. The orthogonality measure for this new class is described in detail. In addition, the orthogonality measure problem is discussed on a more general level. Apart from this, various identities derived for the new orthogonal polynomials may be viewed as generalizations of certain formulas known from the theory of Bessel functions. A key role in these derivations is played by a Jacobi (tridiagonal) matrix J_L whose eigenvalues coincide with the reciprocal values of the zeros of the regular Coulomb wave function $F_L(\eta,\rho)$. The spectral zeta function corresponding to the regular Coulomb wave function or, more precisely, to the respective tridiagonal matrix is studied as well.

1. Introduction

In [9], Ikebe showed the zeros of the regular Coulomb wave function $F_L(\eta,\rho)$ and its derivative $\partial_\rho F_L(\eta,\rho)$ (regarded as functions of ρ) to be related to eigenvalues of certain compact Jacobi matrices (see [1, Chp. 14] and references therein for basic information about Coulomb wave functions). He applied an approach originally suggested for Bessel functions by Grad and Zakrajšek [8]. In more detail, reciprocal values of the nonzero roots of $F_L(\eta,\rho)$ coincide with the nonzero eigenvalues of the Jacobi matrix

$$J_L = \begin{pmatrix} \lambda_{L+1} & w_{L+1} & & & \\ w_{L+1} & \lambda_{L+2} & w_{L+2} & & \\ & w_{L+2} & \lambda_{L+3} & w_{L+3} & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{1}$$

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http://dx.doi.org/10.1016/j.jmaa.2014.04.049
0022-247X/

Please cite this article in press as: F. Štampach, P. Šťovíček, Orthogonal polynomials associated with Coulomb wave functions, J. Math. Anal. Appl. (2014), http://dx.doi.org/10.1016/j.jmaa.2014.04.049

where

$$w_n = \frac{\sqrt{(n+1)^2+\eta^2}}{(n+1)\sqrt{(2n+1)(2n+3)}} \quad \text{and} \quad \lambda_n = -\frac{\eta}{n(n+1)} \tag{2}$$

for $n = L, L+1, L+2, \ldots$. Similarly, reciprocal values of the nonzero roots of $\partial_\rho F_L(\eta,\rho)$ coincide with the nonzero eigenvalues of the Jacobi matrix

$$\tilde{J}_L = \begin{pmatrix} \tilde{\lambda}_L & \tilde{w}_L & & & \\ \tilde{w}_L & \lambda_{L+1} & w_{L+1} & & \\ & w_{L+1} & \lambda_{L+2} & w_{L+2} & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{3}$$

where

$$\tilde{w}_L = \sqrt{\frac{2L+1}{L+1}}\, w_L \quad \text{and} \quad \tilde{\lambda}_L = -\frac{\eta}{(L+1)^2}. \tag{4}$$

The parameters have been chosen so that $L \in \mathbb{Z}_+$ (non-negative integers) and $\eta \in \mathbb{R}$. This is, however, unnecessarily restrictive and one may wish to extend the set of admissible values of L. Note also that J_L and $\tilde{J}_L$ are both compact, even Hilbert–Schmidt operators on $\ell^2(\mathbb{N})$.

Ikebe uses this observation for evaluating the zeros of $F_L(\eta,\rho)$ and $\partial_\rho F_L(\eta,\rho)$ approximately by computing eigenvalues of the respective finite truncated Jacobi matrices. In this paper, we are going to work with Jacobi matrices J_L and $\tilde{J}_L$ as well but with a fully different goal. We aim to establish a new class of orthogonal polynomials (shortly OPs) associated with Coulomb wave functions and to analyze their properties. In doing so, we make a thorough use of the formalism which has been introduced in [19] and further developed in [20]. The studied polynomials represent a two-parameter family which is a generalization of the well known Lommel polynomials associated with Bessel functions. Let us also note that another generalization of Lommel polynomials, although going in a completely different direction, has been described by Ismail in [10], see also [13,15].

When looking into a new class of OPs, our primary intention was to obtain the corresponding orthogonality relation. Before approaching this task we discuss the problem of finding a measure of orthogonality for a sequence of OPs on a more general level. In particular, we address the situation when a sequence of OPs is determined by a three-term recurrence whose coefficients satisfy a certain convergence condition. Apart from solving the orthogonality measure problem, various identities are derived for the newly identified class of OPs which may be viewed as generalizations of a number of formulas well known from the theory of Bessel functions. Finally, the last section is devoted to the study of spectral zeta functions corresponding to the regular Coulomb wave functions or, more precisely, to the respective tridiagonal matrices. In particular, we derive recursive formulas for the values of zeta functions. This result can be used to localize the smallest (in modulus) zero of $F_L(\eta,\rho)$, and hence the spectral radius of the Jacobi matrix J_L.

2. Preliminaries and selected useful identities

2.1. The function $\mathfrak{F}$

In order to keep the paper self-contained we first briefly summarize such information concerning the formalism originally introduced in [19] and [20] which will be needed in the course of this paper. Our approach is based on employing the function $\mathfrak{F}$ defined on the space of complex sequences. By definition, $\mathfrak{F} : D \to \mathbb{C}$,

$$\mathfrak{F}(x) = 1 + \sum_{m=1}^{\infty}(-1)^m \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+2}^{\infty} \dots \sum_{k_m=k_{m-1}+2}^{\infty} x_{k_1}x_{k_1+1}x_{k_2}x_{k_2+1}\dots x_{k_m}x_{k_m+1},$$

where

$$D = \left\{ \{x_k\}_{k=1}^{\infty} \subset \mathbb{C};\ \sum_{k=1}^{\infty} |x_k x_{k+1}| < \infty \right\}. \tag{5}$$

For $x \in D$ one has the estimate

$$\left|\mathfrak{F}(x)\right| \le \exp\left(\sum_{k=1}^{\infty} |x_k x_{k+1}| \right). \tag{6}$$

We identify $\mathfrak{F}(x_1, x_2, \dots, x_n)$ with $\mathfrak{F}(x)$ where $x = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$, and put $\mathfrak{F}(\emptyset) = 1$ where $\emptyset$ stands for an empty sequence.

Further we list from [19,20] several useful properties of $\mathfrak{F}$. First,

$$\mathfrak{F}(x) = \mathfrak{F}(x_1, \dots, x_k)\mathfrak{F}\left(T^k x\right) - \mathfrak{F}(x_1, \dots, x_{k-1})x_k x_{k+1}\mathfrak{F}\left(T^{k+1}x\right), \quad k = 1, 2, \dots,$$

where $x \in D$ and T denotes the shift operator from the left, i.e. $(Tx)_k = x_{k+1}$. In particular, for $k = 1$ one gets the rule

$$\mathfrak{F}(x) = \mathfrak{F}(Tx) - x_1 x_2 \mathfrak{F}\left(T^2 x\right). \tag{7}$$

Second, for $x \in D$ one has

$$\lim_{n\to\infty} \mathfrak{F}\left(T^n x\right) = 1, \qquad \lim_{n\to\infty} \mathfrak{F}(x_1, x_2, \dots, x_n) = \mathfrak{F}(x). \tag{8}$$

Third, one has (see [20, Subsection 2.3])

$$\begin{aligned} &\mathfrak{F}(x_1, x_2, \dots, x_d)\mathfrak{F}(x_2, x_3, \dots, x_{d+s}) - \mathfrak{F}(x_1, x_2, \dots, x_{d+s})\mathfrak{F}(x_2, x_3, \dots, x_d) \\ &= \left(\prod_{j=1}^{d} x_j x_{j+1} \right) \mathfrak{F}(x_{d+2}, x_{d+3}, \dots, x_{d+s}) \end{aligned} \tag{9}$$

where $d, s \in \mathbb{Z}_+$. By sending $s \to \infty$ in (9) one arrives at the equality

$$\mathfrak{F}(x_1, \dots, x_d)\mathfrak{F}(Tx) - \mathfrak{F}(x_2, \dots, x_d)\mathfrak{F}(x) = \left(\prod_{k=1}^{d} x_k x_{k+1} \right) \mathfrak{F}\left(T^{d+1}x\right) \tag{10}$$

which is true for any $d \in \mathbb{Z}_+$ and $x \in D$.

2.2. The characteristic function and Weyl m-function

Let us consider a semi-infinite symmetric Jacobi matrix J of the form

$$J = \begin{pmatrix} \lambda_0 & w_0 & & & \\ w_0 & \lambda_1 & w_1 & & \\ & w_1 & \lambda_2 & w_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{11}$$

where $w = \{w_n\}_{n=0}^{\infty} \subset (0, +\infty)$ and $\lambda = \{\lambda_n\}_{n=0}^{\infty} \subset \mathbb{R}$. In the present paper, such a matrix J is always supposed to represent a unique self-adjoint operator on $\ell^2(\mathbb{Z}_+)$, i.e. there exists exactly one such self-adjoint operator so that the canonical basis is contained in its domain and its matrix in the canonical basis coincides with J. For example, this hypothesis is evidently fulfilled if the sequence $\{w_n\}$ is bounded. With a certain degree of notation abuse we use the same symbol, J, to denote this unique self-adjoint operator.

In [20], we have introduced the characteristic function $\mathcal{F}_J$ for a Jacobi matrix J provided its elements satisfy the condition

$$\sum_{n=0}^{\infty} \frac{w_n^2}{|(\lambda_n - z)(\lambda_{n+1} - z)|} < \infty \tag{12}$$

for some (and hence any) $z \in \mathbb{C} \setminus \operatorname{der}(\lambda)$ where $\operatorname{der}(\lambda)$ denotes the set of all finite cluster points of the diagonal sequence λ, i.e. the set of limit values of all possible convergent subsequences of λ. By Corollary 17 in [20], condition (12) also guarantees that the matrix J represents a unique self-adjoint operator on $\ell^2(\mathbb{Z}_+)$. The definition of the characteristic function reads

$$\mathcal{F}_J(z) := \mathfrak{F}\left(\left\{\frac{\gamma_n^2}{\lambda_n - z}\right\}_{n=0}^{\infty}\right) \tag{13}$$

where $\{\gamma_n\}_{n=0}^{\infty}$ is determined by the off-diagonal sequence w recursively as follows: $\gamma_0 = 1$ and $\gamma_{k+1} = w_k/\gamma_k$, for $k \in \mathbb{Z}_+$. The zeros of the characteristic function have actually been shown in [20] to coincide with the eigenvalues of J. More precisely, under assumption (12) it holds true that

$$\operatorname{spec}(J) \setminus \operatorname{der}(\lambda) = \operatorname{spec}_p(J) \setminus \operatorname{der}(\lambda) = \mathfrak{Z}(J) \tag{14}$$

where

$$\mathfrak{Z}(J) := \left\{z \in \mathbb{C} \setminus \operatorname{der}(\lambda);\ \lim_{u \to z} (u - z)^{r(z)} \mathcal{F}_J(u) = 0\right\} \tag{15}$$

and $r(z) := \sum_{k=0}^{\infty} \delta_{z,\lambda_k} \in \mathbb{Z}_+$ is the number of occurrences of an element z in the sequence λ. Moreover, the eigenvalues of J have no accumulation points in $\mathbb{C} \setminus \operatorname{der}(\lambda)$ and all of them are simple.

Finally, denoting by $\{e_n;\ n \in \mathbb{Z}_+\}$ the canonical basis in $\ell^2(\mathbb{Z}_+)$, let us recall that the Weyl m-function $m(z) := \langle e_0, (J - z)^{-1} e_0 \rangle$ can be expressed in terms of $\mathfrak{F}$,

$$m(z) = \frac{1}{\lambda_0 - z} \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=1}^{\infty}\right) \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - z}\right\}_{k=0}^{\infty}\right)^{-1} \tag{16}$$

for $z \notin \operatorname{spec}(J) \cup \operatorname{der}(\lambda)$. From its definition it is clear that $m(z)$ is meromorphic on $\mathbb{C} \setminus \operatorname{der}(\lambda)$ having only simple real poles, and the set of these poles coincides with $\mathfrak{Z}(J)$.

3. Some general results on orthogonal polynomials

The theory of OPs has been developed into considerable depths. Let us just mention the fundamental monographs [2,3]. If convenient, a sequence of OPs, $\{P_n\}_{n=0}^{\infty}$, where $\deg P_n = n$, may be supposed to be already normalized. Then one way of defining such a sequence is by requiring the orthogonality relation

$$\int_{\mathbb{R}} P_m(x) P_n(x)\, \mathrm{d}\mu(x) = \delta_{mn}, \quad m, n \in \mathbb{Z}_+, \tag{17}$$

with respect to a positive Borel measure μ on $\mathbb{R}$ such that

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$$\int_{\mathbb{R}} x^{2n}\, \mathrm{d}\mu(x) < \infty, \quad \forall n \in \mathbb{Z}_+.$$

Without loss of generality, one may assume that μ is a probability measure, i.e. $\mu(\mathbb{R}) = 1$, and $P_0(x) = 1$. As usual, μ is unambiguously determined by the distribution function $x \mapsto \mu((-\infty, x])$. In particular, the distribution function is supposed to be continuous from the right. With some abuse of notation, the distribution function will again be denoted by the symbol μ. The set of monomials, $\{x^n;\ n \in \mathbb{Z}_+\}$, is required to be linearly independent in $L^2(\mathbb{R}, \mathrm{d}\mu)$ and so the function μ should have an infinite number of points of increase.

It is well known that a sequence of OPs, if normalized, satisfies a three-term recurrence relation,

$$xP_n(x) = w_{n-1}P_{n-1}(x) + \lambda_n P_n(x) + w_n P_{n+1}(x), \quad n \in \mathbb{N}, \tag{18}$$

with the initial conditions $P_0(x) = 1$ and $P_1(x) = (x - \lambda_0)/w_0$, where $\{\lambda_n\}_{n=0}^{\infty}$ is a real sequence and $\{w_n\}_{n=0}^{\infty}$ is a positive sequence [2,3]. However, due to Favard's theorem, the opposite statement is also true. For any sequence of real polynomials, $\{P_n\}_{n=0}^{\infty}$, with $\deg P_n = n$, satisfying the recurrence (18) with the above-given initial conditions, there exists a unique positive functional on the space of real polynomials which makes this sequence orthonormal. Moreover, if the matrix J given in (11) represents a unique self-adjoint operator on $\ell^2(\mathbb{Z}_+)$ then this functional is induced by a unique positive Borel measure μ on $\mathbb{R}$. This means that (17) is fulfilled. In other words, in that case the Hamburger moment problem is determinate; see, for instance, §4.1.1 and Corollary 2.2.4 in [2] or Theorem 3.4.5 in [14].

Using (7) one easily verifies that the solution of (18) with the given initial conditions is related to $\mathfrak{F}$ through the identity

$$P_n(x) = \left(\prod_{k=0}^{n-1} \frac{x - \lambda_k}{w_k}\right) \mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k - x}\right\}_{k=0}^{n-1}\right), \quad n \in \mathbb{Z}_+. \tag{19}$$

A second linearly independent solution of (18) can be written down in the form

$$Q_n(x) = \frac{1}{w_0}\left(\prod_{k=1}^{n-1} \frac{x - \lambda_k}{w_k}\right) \mathfrak{F}\left(\left\{\frac{\gamma_{k+1}^2}{\lambda_{k+1} - x}\right\}_{k=0}^{n-2}\right), \quad n \in \mathbb{N}.$$

The latter solution satisfies the initial conditions $Q_0(x) = 0$ and $Q_1(x) = 1/w_0$.

Being given a sequence of OPs, with $\{P_n\}_{n=0}^{\infty}$, defined via the recurrence rule (18), i.e. via formula (19), the crucial question is what does a measure of orthogonality looks like. Relying on the function $\mathfrak{F}$ we provide a partial description of the measure μ. Doing so we confine ourselves to such Jacobi matrices for which the set of cluster points of the diagonal sequence λ is discrete. This assumption is not too restrictive, though, since it turns out that $\mathrm{der}(\lambda)$ is a one-point set or even an empty set in many practical applications of interest.

Theorem 1. *Let J be a Jacobi matrix introduced in (11) and $\mathrm{der}(\lambda)$ be composed of isolated points only. Suppose there exists $z_0 \in \mathbb{C}$ such that (12) is fulfilled for $z = z_0$. Then the orthogonality relation for the sequence of OPs determined in (18) reads*

$$\int_{\mathbb{R}} P_m(x)P_n(x)\, d\nu(x) + \sum_{x \in \mathcal{D}} \frac{P_m(x)P_n(x)}{\|P(x)\|^2} = \delta_{mn}, \quad m, n \in \mathbb{Z}_+, \tag{20}$$

where $\mathcal{D} = \mathrm{spec}_p(J) \cap \mathrm{der}(\lambda)$ and $\|P(x)\|$ stands for the ℓ^2-norm of the vector $P(x) = (P_0(x), P_1(x), \ldots)$. The measure $d\nu$ is positive, purely discrete and supported on the set $\mathfrak{Z}(J)$. The magnitude of jumps of the step function $\nu(x)$ at those points $x \in \mathfrak{Z}(J)$ which do not belong to the range of λ equals

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$$\nu(x)-\nu(x-0)=\frac{1}{x-\lambda_0}\mathfrak{F}\left(\left\{\frac{\gamma_{k+1}^2}{\lambda_{k+1}-x}\right\}_{k=0}^{\infty}\right)\left[\frac{d}{dx}\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k-x}\right\}_{k=0}^{\infty}\right)\right]^{-1}. \tag{21}$$

Remark. In Theorem 1 we avoided considering the points from $\mathfrak{Z}(J)$ which belong to the range of λ. We remark, however, that such points, if any, can be addressed as well, similarly to (21), though in a somewhat more complicated way. But we omit further details for the sake of simplicity.

Proof. Let E_J stand for the projection-valued spectral measure of the self-adjoint operator J. As it is well known, the measure of orthogonality μ is related to E_J by the identity

$$\mu(M)=\langle e_0, E_J(M)e_0\rangle \tag{22}$$

holding for any Borel set $M\subset\mathbb{R}$. Here again, e_0 denotes the first vector of the canonical basis in $\ell^2(\mathbb{Z}_+)$. Moreover, $\operatorname{supp}(\mu)=\operatorname{spec}(J)$. In fact, let us recall that (22) follows from the observation that $e_n=P_n(J)e_0$ for all $n\in\mathbb{Z}_+$ and from the Spectral Theorem since

$$\delta_{mn}=\langle e_m,e_n\rangle=\langle e_0,P_m(J)P_n(J)e_0\rangle=\int_{\mathbb{R}}P_m(x)P_n(x)\,\mathrm{d}\mu(x).$$

The set $\operatorname{der}(\lambda)$ is closed and, by hypothesis, discrete – and therefore at most countable. We know, referring to (14), that the part of the spectrum of J lying in $\mathbb{C}\setminus\operatorname{der}(\lambda)$ is discrete, too. Consequently, $\operatorname{spec}(J)$ is countable and therefore the continuous part of the spectral measure E_J necessarily vanishes, i.e. J has a pure point spectrum. In that case, of course, in order to determine the spectral measure E_J it suffices to determine the projections $E_J(\{x\})$ for all $x\in\operatorname{spec}_p(J)$. Since the vector $P(z)$ is a formal solution of $(J-z)P(z)=0$, unique up to a constant multiplier, one has the well known criterion $x\in\operatorname{spec}_p(J)$ iff $\|P(x)\|<\infty$. Moreover, $P_0(x)=1$ and so

$$\langle e_0,E_J(\{x\})e_0\rangle=\frac{|\langle P(x),e_0\rangle|^2}{\|P(x)\|^2}=\frac{1}{\|P(x)\|^2}.$$

The point spectrum of J can be split into two disjoint sets, $\operatorname{spec}_p(J)=\mathfrak{Z}(J)\cup\mathcal{D}$. The Hilbert space and the spectral measure decompose correspondingly. Put

$$J'=JE_J(\mathfrak{Z}(J))\quad\text{and}\quad \nu(x)=\langle e_0,E_{J'}((-\infty,x])e_0\rangle\quad\text{for } x\in\mathbb{R}.$$

Then the measure $\mathrm{d}\nu$ is supported on $\mathfrak{Z}(J)$ and

$$\int_{\mathbb{R}}f(x)\,\mathrm{d}\mu(x)=\int_{\mathbb{R}}f(x)\,\mathrm{d}\nu(x)+\sum_{x\in\mathcal{D}}\frac{f(x)}{\|P(x)\|^2}$$

for all $f\in C(\mathbb{R})$. As pointed out in (14), any $x\in\mathfrak{Z}(J)$ is a simple isolated eigenvalue of J. Then $E_J(\{x\})$ can be written down as a Riezs spectral projection. When choosing a sufficiently small $\epsilon>0$, one has

$$\langle e_0,E_J(\{x\})e_0\rangle=-\frac{1}{2\pi i}\oint_{|x-z|=\epsilon}m(z)\,\mathrm{d}z=-\operatorname{Res}(m,x).$$

If, in addition, x does not belong to the range of λ then, in view of (16) and (13), (15) (with $r(x)=0$), we may evaluate

$$\operatorname{Res}(m,x)=\frac{1}{\lambda_0-x}\mathfrak{F}\left(\left\{\frac{\gamma_{k+1}^2}{\lambda_{k+1}-x}\right\}_{k=0}^{\infty}\right)\left[\frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=x}\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k-z}\right\}_{k=0}^{\infty}\right)\right]^{-1}.$$

This concludes the proof. □

Remark 2. Of course, the sum on the LHS of (20) is void if $\operatorname{der}(\lambda)=\emptyset$. The sum also simplifies in such a case when J is a compact operator satisfying (12). One can readily see that this happens iff $\lambda_n\to 0$ and $w\in\ell^2(\mathbb{Z}_+)$. Then Theorem 1 is applicable and the orthogonality relation (20) takes the form

$$\int_{\mathbb{R}} P_n(x)P_m(x)\,\mathrm{d}\nu(x)+\Lambda_0P_n(0)P_m(0)=\delta_{mn}.$$

If J is invertible then Λ_0 vanishes. Yet, in general, Λ_0 may be strictly positive – as demonstrated, for instance, by the example of q-Lommel polynomials, see [16, Theorem 4.2].

For the intended applications of Theorem 1, the following case is of importance. Let $\lambda\in\ell^1(\mathbb{Z}_+)$ be real and $w\in\ell^2(\mathbb{Z}_+)$ positive. Then J is compact and (12) holds for any $z\neq 0$ not belonging to the range of λ. Moreover, the characteristic function of J can be regularized with the aid of the entire function

$$\phi_\lambda(z):=\prod_{n=0}^{\infty}(1-z\lambda_n).$$

Let us define

$$\mathcal{G}_J(z):=\begin{cases}\phi_\lambda(z)\mathcal{F}_J(z^{-1}) & \text{if } z\neq 0,\\ 1 & \text{if } z=0.\end{cases}\tag{23}$$

The function $\mathcal{G}_J$ is entire and, referring to (14), one has

$$\operatorname{spec}(J)=\{0\}\cup\big\{z^{-1};\ \mathcal{G}_J(z)=0\big\}.\tag{24}$$

Let us also note that

$$\mathcal{G}_J\big(z^{-1}\big)=\lim_{n\to\infty}z^{-n}p_n(z)$$

where

$$p_n(x):=\left(\prod_{k=0}^{n-1}(x-\lambda_k)\right)\mathfrak{F}\left(\left\{\frac{\gamma_k^2}{\lambda_k-x}\right\}_{k=0}^{n-1}\right),\quad n\in\mathbb{Z}_+,$$

are the monic polynomials corresponding to the orthogonal polynomials $P_n(x)$ given in (19).

Since

$$m(z)=\int_{\mathbb{R}}\frac{\mathrm{d}\mu(x)}{x-z}$$

where $\mathrm{d}\mu$ is the measure from (22), formula (16) implies that the identity

$$\int_{\mathbb{R}}\frac{\mathrm{d}\mu(x)}{1-xz}=\frac{\mathcal{G}_{J^{(1)}}(z)}{\mathcal{G}_J(z)}\tag{25}$$

holds for any $z \notin \mathcal{G}_J^{-1}(\{0\})$. Here $J^{(1)}$ denotes the Jacobi operator determined by the diagonal sequence $\{\lambda_{n+1}\}_{n=0}^{\infty}$ and the weight sequence $\{w_{n+1}\}_{n=0}^{\infty}$; i.e. $J^{(1)}$ is obtained from J by deleting the first row and the first column.

Let us denote by $\{\mu_n\}_{n=1}^{\infty}$ the set of nonzero eigenvalues of the compact operator J. Remember that all eigenvalues of J are necessarily simple and particularly the multiplicity of 0 as an eigenvalue of J does not exceed 1. Since $\mathrm{d}\mu$ is supported by $\mathrm{spec}(J)$, formula (25) yields the Mittag–Leffler expansion

$$\Lambda_0 + \sum_{k=1}^{\infty} \frac{\Lambda_k}{1-\mu_k z} = \frac{\mathcal{G}_{J^{(1)}}(z)}{\mathcal{G}_J(z)} \tag{26}$$

where Λ_k denotes the jump of the piece-wise constant function $\mu(x)$ at $x = \mu_k$, and similarly for Λ_0 and $x = 0$. From (26) one deduces that

$$\Lambda_k = \lim_{z\to\mu_k^{-1}} (1-\mu_k z)\frac{\mathcal{G}_{J^{(1)}}(z)}{\mathcal{G}_J(z)} = -\mu_k \frac{\mathcal{G}_{J^{(1)}}(\mu_k^{-1})}{\mathcal{G}_J'(\mu_k^{-1})}$$

for $k \in \mathbb{N}$. This can be viewed as a regularized version of the identity (21) in this particular case. We have shown the following proposition:

Theorem 3. *Let λ be a real sequence from $\ell^1(\mathbb{Z}_+)$ and w be a positive sequence from $\ell^2(\mathbb{Z}_+)$. Then the measure of orthogonality $d\mu$ for the corresponding sequence of OPs defined in (18) fulfills*

$$\mathrm{supp}(d\mu) \setminus \{0\} = \{z^{-1};\ \mathcal{G}_J(z) = 0\}$$

where the RHS is a bounded discrete subset of $\mathbb{R}$ with 0 as the only accumulation point. Moreover, for $x \in \mathrm{supp}(d\mu) \setminus \{0\}$ one has

$$\mu(x) - \mu(x-0) = -x\frac{\mathcal{G}_{J^{(1)}}(x^{-1})}{\mathcal{G}_J'(x^{-1})}. \tag{27}$$

Let us denote $\xi_{-1}(z) := \mathcal{G}_J(z)$ and

$$\xi_k(z) := \left(\prod_{l=0}^{k-1} w_l\right) z^{k+1}\mathcal{G}_{J^{(k+1)}}(z), \quad k \in \mathbb{Z}_+, \tag{28}$$

where

$$J^{(k)} = \begin{pmatrix} \lambda_k & w_k & & & \\ w_k & \lambda_{k+1} & w_{k+1} & & \\ & w_{k+1} & \lambda_{k+2} & w_{k+2} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Lemma 4. *Let $\lambda \in \ell^1(\mathbb{Z}_+)$, $w \in \ell^2(\mathbb{Z}_+)$ and $z \neq 0$. Then the vector*

$$\xi(z) = \big(\xi_0(z), \xi_1(z), \xi_2(z), \ldots\big) \tag{29}$$

belongs to $\ell^2(\mathbb{Z}_+)$, and one has

$$\frac{1}{z^2}\sum_{k=0}^{\infty} \xi_k(z)^2 = \xi_{-1}(z)\xi_0'(z) - \xi_{-1}'(z)\xi_0(z). \tag{30}$$

Moreover, the vector $\xi(z)$ is nonzero and so

$$\xi_{-1}(z)\xi_0'(z) - \xi_{-1}'(z)\xi_0(z) > 0, \quad \forall z \in \mathbb{R} \setminus \{0\}, \tag{31}$$

provided that the sequences λ and w are both real.

Proof. First, choose $N \in \mathbb{Z}_+$ so that $z^{-1} \neq \lambda_k$ for all $k > N$. This is clearly possible since $\lambda_n \to 0$ as $n \to \infty$. Then we have, referring to (6) and (23),

$$\left|\mathcal{G}_{J^{(k)}}(z)\right| \leq \exp\left(\sum_{j=N+1}^{\infty} |z||\lambda_j| + \sum_{j=N+1}^{\infty} \frac{|z|^2\,|w_j|^2}{|(1-z\lambda_j)(1-z\lambda_{j+1})|}\right) \quad \text{for } k > N.$$

Observing that $w_n \to 0$ as $n \to \infty$, one concludes that there exists a constant $C > 0$ such that

$$|z|^{k+1} \prod_{l=0}^{k-1} w_l \leq C 2^{-k} \quad \text{for } k > N.$$

These estimates obviously imply the square summability of the vector $\xi(z)$.

Second, with the aid of (7), one verifies that for all $z \neq 0$ and $k \in \mathbb{Z}_+$,

$$w_{k-1}\xi_{k-1}(z) + \left(\lambda_k - z^{-1}\right)\xi_k(z) + w_k\xi_{k+1}(z) = 0$$

where we put $w_{-1} := 1$. From here one deduces that the equation

$$\left(z^{-1} - x^{-1}\right)\xi_k(z)\xi_k(x) = W_k(x,z) - W_{k-1}(x,z), \tag{32}$$

with

$$W_k(x,z) = w_k\left(\xi_{k+1}(z)\xi_k(x) - \xi_{k+1}(x)\xi_k(z)\right),$$

holds for all $k \in \mathbb{Z}_+$. Now one can derive (30) from (32) in a routine way.

Finally, it can be stated that the first equality in (8) implies the limit

$$\lim_{k\to\infty} \mathcal{G}_{J^{(k)}}(z) = 1.$$

Referring to (28) this means $\xi_k(z) \neq 0$ for all sufficiently large k. □

Proposition 5. *Let $\lambda \in \ell^1(\mathbb{Z}_+)$ be real, $w \in \ell^2(\mathbb{Z}_+)$ be positive and $z \neq 0$. If z^{-1} is an eigenvalue of the Jacobi operator J given in (11), then the vector (29) is a corresponding eigenvector.*

Proof. As mentioned in (24), z^{-1} is an eigenvalue of J iff $\mathcal{G}_J(z) \equiv \xi_{-1}(z) = 0$. Following from that, one readily verifies, with the aid of (7), that $\xi(z)$ is a formal solution of the eigenvalue equation $(J - z^{-1})\xi(z) = 0$. According to Lemma 4, $\xi(z) \neq 0$. Furthermore, it is true that $\xi_0(z) \neq 0$. Indeed, if $\xi_{-1}(z) = \xi_0(z) = 0$, then, by recurrence, $\xi_k(z) = 0$ for all $k \in \mathbb{Z}_+$, which is a contradiction. Moreover, Lemma 4 also tells us that $\xi(z) \in \ell^2(\mathbb{Z}_+)$. □

Theorem 6. *Let $\lambda \in \ell^1(\mathbb{Z}_+)$ be real and $w \in \ell^2(\mathbb{Z}_+)$ be positive. Then the zeros of the function $\mathcal{G}_J$ are all real and simple, and form a countable subset of $\mathbb{R} \setminus \{0\}$ with no finite accumulation points. Furthermore, the functions $\mathcal{G}_J$ and $\mathcal{G}_{J^{(1)}}$ have no common zeros, and the zeros of the same sign of $\mathcal{G}_J$ and $\mathcal{G}_{J^{(1)}}$ mutually separate each other, i.e. between any two consecutive zeros of $\mathcal{G}_J$ which have the same sign there is a zero of $\mathcal{G}_{J^{(1)}}$ and vice versa.*

Please cite this article in press as: F. Štampach, P. Šťovíček, Orthogonal polynomials associated with Coulomb wave functions, J. Math. Anal. Appl. (2014), http://dx.doi.org/10.1016/j.jmaa.2014.04.049

Proof. The first part of the proposition follows from (24). In fact, all zeros of $\mathcal{G}_J$ are with certainty real since J is a Hermitian operator in $\ell^2(\mathbb{Z}_+)$. Moreover, J is compact and all its eigenvalues are simple. Therefore the set of reciprocal values of nonzero eigenvalues of J is countable and has no finite accumulation points.

Thus we know that the zeros of $\mathcal{G}_J$ and $\mathcal{G}_{J^{(1)}}$ are all located in $\mathbb{R}\setminus\{0\}$ and $\xi_{-1}(z)=\mathcal{G}_J(z)$, $\xi_0(z)=z\mathcal{G}_{J^{(1)}}(z)$. Hence, as far as the zeros are concerned and we are considering an interval separated from the origin, we can speak about ξ_{-1} and ξ_0 instead of $\mathcal{G}_J$ and $\mathcal{G}_{J^{(1)}}$, respectively. The remainder of the proposition can be deduced from (31) in a usual way. Suppose a zero of ξ_{-1}, called z, is not simple. Then $\xi_{-1}(z)=\xi'_{-1}(z)=0$, which leads to a contradiction with (31). From (31) it arises that ξ_{-1} and ξ_0 have no common zeros in $\mathbb{R}\setminus\{0\}$. Furthermore, suppose z_1 and z_2 are two consecutive zeros of ξ_{-1} of the same sign. Since these zeros are simple, the numbers $\xi'_{-1}(z_1)$ and $\xi'_{-1}(z_2)$ differ in their sign. From (31) one deduces that $\xi_0(z_1)$ and $\xi_0(z_2)$ must differ in sign as well. Consequently, there is at least one zero of ξ_0 lying between z_1 and z_2. An entirely analogous argument applies if the roles of ξ_{-1} and ξ_0 are interchanged. □

4. Lommel polynomials

4.1. Basic properties and the orthogonality relation

In this section we deal with the Lommel polynomials since they represent one of the simplest and most interesting examples that enable to demonstrate the general results derived in Section 3. This is done having in mind the main goal of this paper, namely formulating a generalization of the Lommel polynomials established in the next section. Let us note that although the Lommel polynomials can be expressed in terms of hypergeometric series, they do not fit into Askey's scheme of hypergeometric orthogonal polynomials [12].

Let us recall that Lommel polynomials were introduced within the theory of Bessel function (see, for instance, [22, §9.6–9.73] or [7, Chp. VII]). They can be written explicitly in the form

$$R_{n,\nu}(x)=\sum_{k=0}^{[n/2]}(-1)^k\binom{n-k}{k}\frac{\Gamma(\nu+n-k)}{\Gamma(\nu+k)}\left(\frac{2}{x}\right)^{n-2k} \tag{33}$$

where $n\in\mathbb{Z}_+$, $\nu\in\mathbb{C}$, $-\nu\notin\mathbb{Z}_+$ and $x\in\mathbb{C}\setminus\{0\}$. Here we stick to the traditional terminology though, obviously, $R_{n,\nu}(x)$ is a polynomial in the variable x^{-1} rather than in x. Proceeding by induction in $n\in\mathbb{Z}_+$ one easily verifies the identity

$$R_{n,\nu}(x)=\left(\frac{2}{x}\right)^n\frac{\Gamma(\nu+n)}{\Gamma(\nu)}\mathfrak{F}\left(\left\{\frac{x}{2(\nu+k)}\right\}_{k=0}^{n-1}\right). \tag{34}$$

As it is well known, Lommel polynomials are directly related to Bessel functions,

$$\begin{aligned}R_{n,\nu}(x)&=\frac{\pi x}{2}\big(Y_{-1+\nu}(x)J_{n+\nu}(x)-J_{-1+\nu}(x)Y_{n+\nu}(x)\big)\\&=\frac{\pi x}{2\sin(\pi\nu)}\big(J_{1-\nu}(x)J_{n+\nu}(x)+(-1)^nJ_{-1+\nu}(x)J_{-n-\nu}(x)\big).\end{aligned}$$

From this and relation (37) below it follows that Lommel polynomials obey the recurrence

$$R_{n+1,\nu}(x)=\frac{2(n+\nu)}{x}R_{n,\nu}(x)-R_{n-1,\nu}(x),\quad n\in\mathbb{Z}_+, \tag{35}$$

with the initial conditions $R_{-1,\nu}(x)=0$, $R_{0,\nu}(x)=1$.

Please cite this article in press as: F. Štampach, P. Šťovíček, Orthogonal polynomials associated with Coulomb wave functions, J. Math. Anal. Appl. (2014), http://dx.doi.org/10.1016/j.jmaa.2014.04.049

The original meaning of the Lommel polynomials is revealed by the formula

$$J_{\nu+n}(x) = R_{n,\nu}(x)J_\nu(x) - R_{n-1,\nu+1}(x)J_{\nu-1}(x) \quad \text{for } n \in \mathbb{Z}_+. \tag{36}$$

As first observed by Lommel in 1871, (36) can be obtained by iterating the basic recurrence relation for Bessel functions, namely

$$J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x) - J_{\nu-1}(x). \tag{37}$$

Let us remark that (36) immediately follows from (10), (34) and the formula

$$\mathfrak{F}\left(\left\{\frac{\rho}{\nu+k}\right\}_{k=1}^{\infty}\right) = \Gamma(\nu+1)\rho^{-\nu}J_\nu(2\rho) \tag{38}$$

which has been observed in [19] and holds for any ν such that $-\nu \notin \mathbb{N}$ and $\rho \in \mathbb{C}$.

The orthogonality relation for Lommel polynomials is well known and is expressed in terms of the zeros of the Bessel function of order $\nu - 1$ as explained, for instance, in [4,6], see also [3, Chp. VI §6] and [10]. This relation can also be rederived as a corollary of Theorem 3. For $\nu > -1$ and $n \in \mathbb{Z}_+$, let us set temporarily

$$\lambda_n = 0 \quad \text{and} \quad w_n = 1/\sqrt{(\nu+n+1)(\nu+n+2)}.$$

Then the corresponding Jacobi operator J is compact, self-adjoint and 0 is not an eigenvalue. In fact, the invertibility of J can be verified straightforwardly by solving the formal eigenvalue equation for 0. Referring to (38), the regularized characteristic function of J equals

$$\mathcal{G}_J(z) = \mathcal{F}_J\left(z^{-1}\right) = \Gamma(\nu+1)z^{-\nu}J_\nu(2z).$$

Consequently, the support of the measure of orthogonality turns out to coincide with the zero set of $J_\nu(z)$. Remember that $x^{-\nu}J_\nu(x)$ is an even function. Let $j_{k,\nu}$ stand for the k-th positive zero of $J_\nu(x)$ and put $j_{-k,\nu} = -j_{k,\nu}$ for $k \in \mathbb{N}$. Theorem 3 then tells us that the orthogonality relation takes the form

$$-2(\nu+1)\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{J_{\nu+1}(j_{k,\nu})}{j_{k,\nu}^2 J_\nu'(j_{k,\nu})} P_m\left(\frac{2}{j_{k,\nu}}\right)P_n\left(\frac{2}{j_{k,\nu}}\right) = \delta_{mn}$$

where $J_\nu'(x)$ denotes the partial derivative of $J_\nu(x)$ with respect to x.

Furthermore, (19) and (34) imply

$$R_{n,\nu+1}(x) = \sqrt{\frac{\nu+1}{\nu+n+1}}P_n\left(\frac{2}{x}\right). \tag{39}$$

Using the identity

$$\partial_x J_\nu(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x),$$

the orthogonality relation simplifies to the well known formula

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} j_{k,\nu}^{-2}R_{n,\nu+1}(j_{k,\nu})R_{m,\nu+1}(j_{k,\nu}) = \frac{1}{2(n+\nu+1)}\delta_{mn}, \tag{40}$$

valid for $\nu > -1$ and $m, n \in \mathbb{Z}_+$.

4.2. Lommel polynomials in the variable ν

Lommel polynomials can also be addressed as polynomials in the parameter ν. Such polynomials are also orthogonal with the measure of orthogonality supported on the zero set of a Bessel function of the first kind regarded as a function of the order.

Let us consider a sequence of polynomials in the variable ν and depending on a parameter $u \neq 0$, $\{Q_n(u;\nu)\}_{n=0}^{\infty}$, determined by the recurrence

$$uQ_{n-1}(u;\nu) - nQ_n(u;\nu) + uQ_{n+1}(u;\nu) = \nu Q_n(u;\nu), \quad n \in \mathbb{Z}_+,$$

with the initial conditions $Q_{-1}(u;\nu) = 0$, $Q_0(u;\nu) = 1$. According to (19),

$$Q_n(u,\nu) = u^{-n}\frac{\Gamma(\nu+n)}{\Gamma(\nu)}\mathfrak{F}\left(\left\{\frac{u}{\nu+k}\right\}_{k=0}^{n-1}\right) \quad \text{for } n \in \mathbb{Z}_+.$$

Comparing the last formula with (34) one observes that

$$Q_n(u,\nu) = R_{n,\nu}(2u), \quad \forall n \in \mathbb{Z}_+.$$

The Bessel function $J_\nu(x)$ regarded as a function of ν has infinitely many simple real zeros which are all isolated provided that $x > 0$, see [20, Subsec. 4.3]. Below we denote the zeros of $J_{\nu-1}(2u)$ by $\theta_n = \theta_n(u)$, $n \in \mathbb{N}$, and restrict ourselves to the case $u > 0$ since $\theta_n(-u) = \theta_n(u)$.

The Jacobi matrix J corresponding to this case, i.e. J with the diagonal $\lambda_n = -n$ and the weights $w_n = u$, $n \in \mathbb{Z}_+$, is an unbounded self-adjoint operator with a discrete spectrum (see [20]). Hence the orthogonality measure for $\{Q_n(u;\nu)\}$ has the form stated in Remark 2. Thus, using (21) and (38), one arrives at the orthogonality relation

$$\sum_{k=1}^{\infty}\frac{J_{\theta_k}(2u)}{u(\partial_z|_{z=\theta_k}J_{z-1}(2u))}R_{n,\theta_k}(2u)R_{m,\theta_k}(2u) = \delta_{mn}, \quad m,n \in \mathbb{Z}_+.$$

Let us remark that initially it was Dickinson who formulated the problem of constructing the measure of orthogonality for the Lommel polynomials in the variable ν in 1958 [5]. Ten years later, Maki described such a construction in [17].

5. A new class of orthogonal polynomials

5.1. Characteristic functions of the Jacobi matrices J_L and $\tilde{J}_L$

In this section, we work with matrices J_L and $\tilde{J}_L$ defined in (1), (2) and (3), (4), respectively. In order to have a positive weight sequence w and a Hermitian matrix, we assume, in the case of J_L, that $-1 \neq L > -3/2$ if $\eta \in \mathbb{R} \setminus \{0\}$, and $L > -3/2$ if $\eta = 0$. Similarly, in the case of $\tilde{J}_L$ we assume that $L > -1/2$ and $\eta \in \mathbb{R}$.

Let us recall that the regular and irregular Coulomb wave functions, $F_L(\eta,\rho)$ and $G_L(\eta,\rho)$, are two linearly independent solutions of the second-order differential equation

$$\frac{d^2u}{d\rho^2} + \left(1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right)u = 0; \tag{41}$$

see, for instance, [1, Chp. 14]. One has the Wronskian formula (see [1, Eq. 14.2.5])

$$F_{L-1}(\eta,\rho)G_L(\eta,\rho) - F_L(\eta,\rho)G_{L-1}(\eta,\rho) = \frac{L}{\sqrt{L^2+\eta^2}}. \tag{42}$$

Furthermore, the function $F_L(\eta,\rho)$ admits the decomposition [1, Eqs. 14.1.3 and 14.1.7]

$$F_L(\eta,\rho) = C_L(\eta)\rho^{L+1}\phi_L(\eta,\rho) \tag{43}$$

where

$$C_L(\eta) := \sqrt{\frac{2\pi\eta}{e^{2\pi\eta}-1}}\,\frac{\sqrt{(1+\eta^2)(4+\eta^2)\dots(L^2+\eta^2)}}{(2L+1)!!L!}$$

and

$$\phi_L(\eta,\rho) := e^{-i\rho}\,{}_1F_1(L+1-i\eta, 2L+2, 2i\rho). \tag{44}$$

For L not an integer, $C_L(\eta)$ is to be understood as

$$C_L(\eta) = \frac{2^L e^{-\pi\eta/2}|\Gamma(L+1+i\eta)|}{\Gamma(2L+2)}. \tag{45}$$

In [21], a formula for the characteristic function of the matrix J_L has been derived. If expressed in terms of $\mathcal{G}_{J_L}$, as defined in (23), the formula simply reads

$$\mathcal{G}_{J_L}(\rho) = \left(\prod_{k=L+1}^{\infty}(1-\lambda_k\rho)\right)\mathfrak{F}\left(\left\{\frac{\gamma_k^2\rho}{1-\lambda_k\rho}\right\}_{k=L+1}^{\infty}\right) = \phi_L(\eta,\rho). \tag{46}$$

For the particular values of parameters, $L=\nu-1/2$ and $\eta=0$, one gets

$$\mathfrak{F}\left(\left\{\frac{\rho}{2(\nu+k)}\right\}_{k=1}^{\infty}\right) = \phi_{\nu-1/2}(0,\rho). \tag{47}$$

It is also known that, see (44) and Eqs. 14.6.6 and 13.6.1 in [1],

$$F_{\nu-1/2}(0,\rho) = \sqrt{\frac{\pi\rho}{2}}J_\nu(\rho), \tag{48}$$

$$\phi_{\nu-1/2}(0,\rho) = e^{-i\rho}\,{}_1F_1(\nu+1/2, 2\nu+1, 2i\rho) = \Gamma(\nu+1)\left(\frac{2}{\rho}\right)^{\nu}J_\nu(\rho). \tag{49}$$

Let us note that (47) with (49) jointly imply (38).

Using recurrence (7) for $\mathfrak{F}$, one can also obtain the characteristic function for $\tilde{J}_L$,

$$\mathcal{F}_{\tilde{J}_L}(\rho^{-1}) = \mathcal{F}_{J_L}(\rho^{-1}) - \frac{\tilde{w}_L^2}{(\rho^{-1}-\tilde{\lambda}_L)(\rho^{-1}-\lambda_{L+1})}\mathcal{F}_{J_{L+1}}(\rho^{-1}). \tag{50}$$

Be reminded that $\phi_L(\eta,\rho)$ obeys the equations

$$\partial_\rho\phi_{L+1}(\eta,\rho) = \frac{2L+3}{\rho}\phi_L(\eta,\rho) - \left(\frac{2L+3}{\rho} + \frac{\eta}{L+1}\right)\phi_{L+1}(\eta,\rho), \tag{51}$$

$$\partial_\rho\phi_L(\eta,\rho) = \frac{\eta}{L+1}\phi_L(\eta,\rho) - \frac{\rho}{2L+3}\left(1+\frac{\eta^2}{(L+1)^2}\right)\phi_{L+1}(\eta,\rho), \tag{52}$$

as it follows from [1, Eqs. 14.2.1 and 14.2.2]. A straightforward computation based on (46), (50) and (52) yields

$$\left(1+\frac{\eta\rho}{(L+1)^2}\right)\left(\prod_{n=L+1}^{\infty}\left(1+\frac{\eta\rho}{n(n+1)}\right)\right)\mathcal{F}_{\tilde{J}_L}\left(\rho^{-1}\right)=\phi_L(\eta,\rho)+\frac{\rho}{L+1}\partial_\rho\phi_L(\eta,\rho).$$

In view of (43), this can be rewritten as

$$\mathcal{G}_{\tilde{J}_L}(\rho)=\phi_L(\eta,\rho)+\frac{\rho}{L+1}\partial_\rho\phi_L(\eta,\rho)=\frac{1}{(L+1)C_L(\eta)}\rho^{-L}\partial_\rho F_L(\eta,\rho).$$

5.2. Orthogonal polynomials associated with $F_L(\eta,\rho)$

Following the general scheme outlined in Section 3 (see (18)), we denote by $\{P_n^{(L)}(\eta;z)\}_{n=0}^{\infty}$ the sequence of OPs given by the three-term recurrence

$$zP_n^{(L)}(\eta;z)=w_{L+n}P_{n-1}^{(L)}(\eta;z)+\lambda_{L+n+1}P_n^{(L)}(\eta;z)+w_{L+n+1}P_{n+1}^{(L)}(\eta;z),\quad n\in\mathbb{Z}_+,\tag{53}$$

with $P_{-1}^{(L)}(\eta;z)=0$ and $P_0^{(L)}(\eta;z)=1$. Again, we restrict ourselves to the range of parameters $-1\neq L>-3/2$ if $\eta\in\mathbb{R}\setminus\{0\}$, and $L>-3/2$ if $\eta=0$. Likewise to Lommel polynomials, these polynomials are not included in Askey's scheme [12]. Further let us denote

$$R_n^{(L)}(\eta;\rho):=P_n^{(L)}\left(\eta;\rho^{-1}\right)\tag{54}$$

for $\rho\neq 0$, $n\in\mathbb{Z}_+$. According to (19),

$$P_n^{(L)}(\eta;z)=\left(\prod_{k=1}^{n}\frac{z-\lambda_{L+k}}{w_{L+k}}\right)\mathfrak{F}\left(\left\{\frac{\gamma_{L+k}^2}{z-\lambda_{L+k}}\right\}_{k=1}^{n}\right),\quad n\in\mathbb{Z}_+.\tag{55}$$

Alternatively, these polynomials can be expressed in terms of Coulomb wave functions.

Proposition 7. *For $n\in\mathbb{Z}_+$ and $\rho\neq 0$ one has*

$$R_n^{(L)}(\eta;\rho)=\frac{\sqrt{(L+1)^2+\eta^2}}{L+1}\sqrt{\frac{2L+2n+3}{2L+3}}\left(F_L(\eta,\rho)G_{L+n+1}(\eta,\rho)-F_{L+n+1}(\eta,\rho)G_L(\eta,\rho)\right).$$

Proof. To verify this identity it suffices to check that the RHS fulfills the same recurrence relation as $R_n^{(L)}(\eta,\rho)$ does, while sharing the same initial conditions. The RHS actually meets the first requirement as it follows from the known recurrence relations for Coulomb wave functions, see [1, Eq. 14.2.3]. The initial condition is a consequence of the Wronskian formula (42). □

For the computations to follow it is useful to note that the weights w_n and the normalization constants $C_L(\eta)$, as defined in (2) and (45), respectively, are related by the equation

$$\prod_{k=0}^{n-1}w_{L+k}=\sqrt{\frac{2L+2n+1}{2L+1}}\frac{C_{L+n}(\eta)}{C_L(\eta)},\quad n=0,1,2,\ldots.\tag{56}$$

Proposition 8. *For the above indicated range of parameters and $\rho\neq 0$,*

$$\lim_{n\to\infty}\sqrt{(2L+3)(2L+2n+1)}C_{L+n}(\eta)\rho^{L+n}R_{n-1}^{(L)}(\eta;\rho)=\sqrt{1+\frac{\eta^2}{(L+1)^2}}F_L(\eta,\rho).\tag{57}$$

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Proof. Referring to (54) and (55), $R_n^{(L)}(\eta;\rho)$ can be expressed in terms of the function $\mathfrak{F}$. The sequence whose truncation stands in the argument of $\mathfrak{F}$ on the RHS of (55) belongs to the domain D defined in (5) – meaning that the second equation in (8) can be applied. Concerning the remaining terms occurring on the LHS of (57), one readily computes, with the aid of (56), that

$$\lim_{n\to\infty}\sqrt{(2L+3)(2L+2n+1)}C_{L+n}(\eta)\rho^{L+n}\left(\prod_{k=1}^{n-1}\frac{\rho^{-1}-\lambda_{k+L}}{w_{k+L}}\right)$$
$$=\frac{\sqrt{(L+1)^2+\eta^2}}{L+1}C_L(\eta)\rho^{L+1}\prod_{k=1}^{\infty}\left(1+\frac{\eta\rho}{(L+k)(L+k+1)}\right).$$

Recalling (46) and (43), the result immediately follows. □

Remark 9. Note that the polynomials $R_n^{(L)}(\eta;\rho)$ can be regarded as a generalization of the Lommel polynomials $R_{n,\nu}(x)$. Actually, if $\eta=0$ then the Jacobi matrix $J_{\nu-1/2}$ is determined by the sequences

$$\lambda_n=0,\qquad w_n=1/\big(2\sqrt{(\nu+n+1)(\nu+n+2)}\,\big).$$

Thus the recurrence (53) reduces to (35) for $\eta=0$ and $L=\nu-1/2$. More precisely, one finds that $P_n^{(\nu-1/2)}(0;z)$ coincides with the polynomial $P_n(2z)$ from Section 4.1 for all n. In view of (39) this means

$$R_n^{(\nu-1/2)}(0;\rho)=\sqrt{\frac{\nu+n+1}{\nu+1}}R_{n,\nu+1}(\rho)\tag{58}$$

for $n\in\mathbb{Z}_+$, $\rho\in\mathbb{C}\setminus\{0\}$ and $\nu>-1$. In addition we remark that, for the same values of parameters, (57) yields Hurwitz' limit formula (see §9.65 in [22])

$$\lim_{n\to\infty}\frac{(\rho/2)^{\nu+n}}{\Gamma(\nu+n+1)}R_{n,\nu+1}(\rho)=J_\nu(\rho).$$

A more explicit description of the polynomials $P_n^{(L)}(\eta;z)$ can be derived. Let us write

$$P_n^{(L)}(\eta;z)=\sum_{k=0}^{n}c_k(n,L,\eta)z^{n-k}.\tag{59}$$

Proposition 10. *Let $\{Q_k(n,L;\eta);\ k\in\mathbb{Z}_+\}$ be a sequence of monic polynomials in the variable η defined by the recurrence*

$$Q_{k+1}(n,L;\eta)=\eta Q_k(n,L;\eta)-h_k(n,L)Q_{k-1}(n,L;\eta)\quad \text{for } k\in\mathbb{Z}_+,\tag{60}$$

with the initial conditions $Q_{-1}(n,L;\eta)=0$, $Q_0(n,L;\eta)=1$, where

$$h_k(n,L)=\frac{k(2L+k+1)(2n-k+2)(2L+2n-k+3)}{4(2n-2k+1)(2n-2k+3)},\quad k\in\mathbb{Z}_+.$$

Then the coefficients $c_k(n,L,\eta)$ defined in (59) fulfill

$$c_k(n,L,\eta)=\frac{\sqrt{2L+2n+3}}{(L+1)\sqrt{2L+3}}\left|\frac{\Gamma(L+2+i\eta)}{\Gamma(L+n+2+i\eta)}\right|\frac{\Gamma(2n-k+2)\Gamma(2L+2n-k+3)}{\Gamma(2n-2k+2)\Gamma(2L+k+2)}\frac{2^{-n+k-1}}{k!}Q_k(n,L;\eta)\tag{61}$$

for $k=0,1,2,\ldots,n$.

For the proof we shall need an auxiliary identity. Note that, if convenient, $Q_k(n, L; \eta)$ can be treated as a polynomial in η with coefficients belonging to the field of rational functions in the variables n, L.

Lemma 11. *The polynomials $Q_k(n, L; \eta)$ defined in Proposition 10 fulfill*

$$Q_k(n, L; \eta) - \alpha_k(n, L)Q_k(n-1, L; \eta) - \beta_k(n, L)\eta Q_{k-1}(n-1, L; \eta) = 0 \tag{62}$$

for $k = 0, 1, 2, \dots,$ where

$$\alpha_k(n, L) = \frac{2(2n-2k+1)(L+n+1)}{(2n-k+1)(2L+2n-k+2)}, \qquad \beta_k(n, L) = \frac{k(2L+k+1)}{(2n-k+1)(2L+2n-k+2)}.$$

Proof. It is easy to verify that $\alpha_k(n, L)$, $\beta_k(n, L)$ fulfill the following identities

$$\alpha_k(n, L) + \beta_k(n, L) = 1, \tag{63}$$

$$\alpha_{k+1}(n, L)h_k(n-1, L) - \alpha_{k-1}(n, L)h_k(n, L) = 0, \tag{64}$$

$$\beta_k(n, L)h_{k-1}(n-1, L) - \beta_{k-1}(n, L)h_k(n, L) = 0. \tag{65}$$

In order to show (62) one can proceed by induction in k. The case $k = 0$ means $\alpha_0(n, L) = 1$ which is obviously true. Furthermore, $Q_1(n, L; \eta) = \eta$, meaning that the case $k = 1$ is a consequence of (63), with $k = 1$. Suppose $k \geq 2$ and further suppose that the identity is true for $k - 1$ and $k - 2$. Applying (60) both to $Q_k(n, L; \eta)$ and $Q_k(n-1, L; \eta)$ and using (63), (64) one can show the LHS of (62) to be equal to

$$-h_{k-1}(n, L)\big(Q_{k-2}(n, L; \eta) - \alpha_{k-2}(n, L)Q_{k-2}(n-1, L; \eta)\big) + \eta\big(Q_{k-1}(n, L; \eta) - Q_{k-1}(n-1, L; \eta)\big).$$

In the next step, we apply the induction hypothesis both to $Q_{k-1}(n, L; \eta)$ and $Q_{k-2}(n, L; \eta)$ in the above expression. It equals, up to a common factor η,

$$-\beta_{k-1}(n, L)Q_{k-1}(n-1, L; \eta) + \beta_{k-1}(n, L)\eta Q_{k-2}(n-1, L; \eta) - h_{k-1}(n, L)\beta_{k-2}(n, L)Q_{k-3}(n-1, L; \eta).$$

Finally, making once more use of (60), this time for the term $Q_{k-1}(n-1, L; \eta)$, one can prove the last expression to be equal to

$$\big(h_{k-2}(n-1, L)\beta_{k-1}(n, L) - h_{k-1}(n, L)\beta_{k-2}(n, L)\big)Q_{k-3}(n-1, L; \eta) = 0$$

as it follows from (65). □

Proof of Proposition 10. Write down the polynomials $P_n^{(L)}(\eta; z)$ in the form of (59) and substitute the RHS of (61) for the coefficients $c_k(n, L, \eta)$. By substituting the resulting expression for the polynomials in (53), one finds that the recurrence relation for the sequence $\{P_n^{(L)}(\eta; z)\}$ is satisfied if and only if the terms $Q_k(n, L; \eta)$ from the substitution fulfill

$$\begin{aligned} a_k(n, L)Q_k(n, L; \eta) - b_k(n, L)Q_k(n-1, L; \eta) - c_k(n, L)\eta Q_{k-1}(n-1, L; \eta) \\ + d_k(n, L, \eta)Q_{k-2}(n-2, L; \eta) = 0 \end{aligned} \tag{66}$$

for $k, n \in \mathbb{N}$, $n \geq k$, where we again put $Q_{-1}(n, L; \eta) = 0$, $Q_0(n, L; \eta) = 1$, and

$$a_k(n,L) = \frac{(2n-k)(2n-k+1)(2L+2n-k+1)(2L+2n-k+2)}{(2L+k)(2L+k+1)(L+n+1)},$$

$$b_k(n,L) = \frac{4(2L+2n+1)(n-k)(2n-2k+1)}{(2L+k)(2L+k+1)},$$

$$c_k(n,L) = \frac{k(2L+2n+1)(2n-k)(2L+2n-k+1)}{(2L+k)(L+n)(L+n+1)},$$

$$d_k(n,L,\eta) = \frac{(k-1)k(\eta^2+(L+n)^2)}{L+n}.$$

Note that for $k=0$ one has

$$a_0(n,L)Q_0(n,L,\eta) - b_0(n,L)Q_0(n-1,L,\eta) - c_0(n,L)\eta Q_{-1}(n-1,L,\eta) = a_0(n,L) - b_0(n,L) = 0.$$

It can be observed that relation (66) provides the terms $Q_k(n,L;\eta)$ with an unambiguous specification. For this fact to be visible, one can proceed by induction in k. Suppose $k>0$ and all terms $Q_j(n,L;\eta)$ are already known for $j<k$, $n\geq j$. Putting $n=k$ in (66) one can express $Q_k(k,L;\eta)$ in terms of $Q_{k-1}(k-1,L;\eta)$ and $Q_{k-2}(k-2,L;\eta)$ since $b_k(k,L)=0$. Then, treating k as being fixed and n as a variable, one can interpret (66) as a first order difference equation in the index n, with a right hand side, for an unknown sequence $\{Q_k(n,L;\eta);\ n\geq k\}$. The initial condition for $n=k$ is now known as well as the right hand side and so the difference equation can be solved unambiguously.

To prove this proposition it suffices to verify that if $\{Q_k(n,L;\eta)\}$ is a sequence of monic polynomials in the variable η defined by the recurrence (60) then it obeys, too, the relation (66). To this end, one may apply repeatedly the rule (62) to bring the LHS of (66) to the form

$$e_0(n,L)Q_k(n-2,L;\eta) + e_1(n,L)\eta Q_{k-1}(n-2,L;\eta) + e_2(n,L,\eta)Q_{k-2}(n-2,L;\eta) \tag{67}$$

where

$$\begin{aligned}e_0(n,L) &= a_k(n,L)\alpha_k(n-1,L)\alpha_k(n,L) - b_k(n,L)\alpha_k(n-1,L),\\ e_1(n,L) &= a_k(n,L)\alpha_{k-1}(n-1,L)\beta_k(n,L) + a_k(n,L)\alpha_k(n,L)\beta_k(n-1,L)\\ &\quad - b_k(n,L)\beta_k(n-1,L) - c_k(n,L)\alpha_{k-1}(n-1,L),\end{aligned}$$

and

$$e_2(n,L,\eta) = \big(a_k(n,L)\beta_{k-1}(n-1,L)\beta_k(n,L) - c_k(n,L)\beta_{k-1}(n-1,L)\big)\eta^2 + d_k(n,L,\eta).$$

Direct evaluation then yields

$$e_1(n,L)/e_0(n,L) = -1, \qquad e_2(n,L,\eta)/e_0(n,L) = h_{k-1}(n-2,L).$$

Referring to the defining relation (60), this proves (67) to be equal to zero indeed. □

Remark 12. Let us shortly discuss what Proposition 10 tells us in the particular case when $\eta=0$, $L=\nu-1/2$. The recurrence (60) can be easily solved for $\eta=0$. One has $Q_{2k+1}(n,L;0)=0$ and

$$Q_{2k}(n,L;0) = (-1)^k \frac{(2k)!(n-k)!(2n-4k+1)!}{k!(n-2k)!(2n-2k+1)!}\,\frac{\Gamma(L+k+1)\Gamma(L+n+2)}{\Gamma(L+1)\Gamma(L+n-k+2)}$$

for $k=0,1,2,\ldots$. Whence $c_{2k+1}(n,\nu-1/2,0)=0$ and

Please cite this article in press as: F. Štampach, P. Šťovíček, Orthogonal polynomials associated with Coulomb wave functions, J. Math. Anal. Appl. (2014), http://dx.doi.org/10.1016/j.jmaa.2014.04.049

$$c_{2k}\left(n,\nu-\frac{1}{2},0\right)=\sqrt{\frac{\nu+n+1}{\nu+1}}(-1)^k2^{n-2k}\binom{n-k}{k}\frac{\Gamma(\nu+n-k+1)}{\Gamma(\nu+k+1)}.$$

Recalling (58), the explicit expression (33) for the Lommel polynomials can be rederived.

Let us mention two more formulas. The first one is quite substantial and shows that the polynomials $R_n^{(L)}(\eta,\rho)$ play the same role for Coulomb wave functions as Lommel polynomials do for Bessel functions. It follows from the abstract identity (10) where we specialize $d=n$,

$$x_k=\frac{\gamma_{L+k-1}^2}{\rho^{-1}-\lambda_{L+k-1}}, \tag{68}$$

and again make use of (56). Thus we get

$$\begin{aligned}&R_n^{(L-1)}(\eta,\rho)F_L(\eta,\rho)-\frac{L+1}{L}\sqrt{\frac{2L+3}{2L+1}}\frac{\sqrt{\eta^2+L^2}}{\sqrt{\eta^2+(L+1)^2}}R_{n-1}^{(L)}(\eta,\rho)F_{L-1}(\eta,\rho)\\&=\sqrt{\frac{2L+2n+1}{2L+1}}F_{L+n}(\eta,\rho),\end{aligned} \tag{69}$$

where $n\in\mathbb{Z}_+$, $0\neq L>-1/2$, $\eta\in\mathbb{R}$ and $\rho\neq 0$. Moreover, referring to (48) and (58), one observes that relation (36) is a particular case of (69) if one lets $\eta=0$ and $L=\nu-1/2$.

Similarly, the announced second identity can be derived from (9) by making the same choice as the one in (68) but writing z instead of ρ^{-1}. Recalling (55) one finds that

$$P_n^{(L-1)}(\eta;z)P_{n+s}^{(L)}(\eta;z)-P_{n+s+1}^{(L-1)}(\eta;z)P_{n-1}^{(L)}(\eta;z)=\frac{w_L}{w_{L+n}}P_s^{(L+n)}(\eta;z)$$

holds for all $n,s\in\mathbb{Z}_+$.

We conclude this subsection by describing the measure of orthogonality for the generalized Lommel polynomials. To this end, we need an auxiliary result concerning the zeros of the function $\phi_L(\eta,\cdot)$. It is obtained by applying Theorem 6 to the sequences w and λ defined in (2). From (46) we know that $\phi_L(\eta,\rho)=\mathcal{G}_{J_L}(\rho)$ and we note that, obviously, $J_{L+1}=J_L^{(1)}$. Thus we arrive at a proposition stated below.

Proposition 13. *Let $-1\neq L>-3/2$ if $\eta\in\mathbb{R}\setminus\{0\}$, and $L>-3/2$ if $\eta=0$. Then the zeros of the function $\phi_L(\eta,\cdot)$ form a countable subset of $\mathbb{R}\setminus\{0\}$ with no finite accumulation points. Moreover, the zeros of $\phi_L(\eta,.)$ are all simple, the functions $\phi_L(\eta,\cdot)$ and $\phi_{L+1}(\eta,\cdot)$ have no common zeros, and the zeros of the same sign of $\phi_L(\eta,\cdot)$ and $\phi_{L+1}(\eta,\cdot)$ mutually separate each other.*

Let us arrange the zeros of $\phi_L(\eta,\cdot)$ into a sequence $\rho_{L,n}$, $n\in\mathbb{N}$ (not indicating the dependence on η explicitly). According to Proposition 13 we can carry this out in such a way that $0<|\rho_{L,1}|\leq|\rho_{L,2}|\leq|\rho_{L,3}|\leq\ldots$. Thus we have

$$\{\rho_{L,n};\ n\in\mathbb{N}\}=\{\rho\in\mathbb{R};\ \phi_L(\eta,\rho)=0\}=\{\rho\in\mathbb{R}\setminus\{0\};\ F_L(\eta,\rho)=0\}. \tag{70}$$

Theorem 14. *Let $-1\neq L>-3/2$ if $\eta\in\mathbb{R}\setminus\{0\}$, and $L>-3/2$ if $\eta=0$. Then the orthogonality relation*

$$\sum_{k=1}^{\infty}\rho_{L,k}^{-2}R_n^{(L)}(\eta;\rho_{L,k})R_m^{(L)}(\eta;\rho_{L,k})=\frac{(L+1)^2+\eta^2}{(2L+3)(L+1)^2}\delta_{mn} \tag{71}$$

holds for all $m,n\in\mathbb{Z}_+$.

Proof. According to Theorem 3, we have the orthogonality relation

$$\int_{\mathbb{R}} P_m^{(L)}(\eta;\rho)P_n^{(L)}(\eta;\rho)\,\mathrm{d}\mu(\rho)=\delta_{mn}$$

where $\mathrm{d}\mu$ is supported on the set $\{\rho_{L,n}^{-1};\ n\in\mathbb{N}\}\cup\{0\}$. Applying formula (27) combined with (46) and (52) one finds that

$$\mu(\rho_{L,k}^{-1})-\mu(\rho_{L,k}^{-1}-0)=-\rho_{L,k}^{-1}\frac{\phi_{L+1}(\eta,\rho_{L,k})}{\partial_\rho\phi_L(\eta,\rho_{L,k})}=\frac{(2L+3)(L+1)^2}{(L+1)^2+\eta^2}\rho_{L,k}^{-2}.$$

We claim that 0 is a point of continuity of μ. Indeed, let us denote by Λ_k the magnitude of the jump of μ at $\rho_{L,k}^{-1}$ if $k\in\mathbb{N}$, and at 0 if $k=0$. Then, since $d\mu$ is a probability measure, one has

$$1=\sum_{k=0}^{\infty}\Lambda_k=\Lambda_0+\frac{(2L+3)(L+1)^2}{(L+1)^2+\eta^2}\sum_{k=1}^{\infty}\rho_{L,k}^{-2}=\Lambda_0+\frac{(2L+3)(L+1)^2}{(L+1)^2+\eta^2}\|J_L\|_2^2 \tag{72}$$

where $\|J_L\|_2$ stands for the Hilbert–Schmidt norm of J_L. This norm, however, can be computed directly,

$$\|J_L\|_2^2=\sum_{n=1}^{\infty}\lambda_{L+n}^2+2\sum_{n=1}^{\infty}w_{L+n}^2=\frac{(L+1)^2+\eta^2}{(2L+3)(L+1)^2}.$$

Comparing this equality to (72) one finds that $\Lambda_0=0$. To conclude the proof it suffices to recall (54). □

In the course of the proof of Theorem 14 we have shown that 0 is a point of continuity of μ. It follows that 0 is not an eigenvalue of the compact operator J_L.

Corollary 15. *Let $-1\neq L>-3/2$ if $\eta\in\mathbb{R}\setminus\{0\}$, and $L>-3/2$ if $\eta=0$. Then the operator J_L is invertible.*

Remark 16. Again, letting $\eta=0$ and $L=\nu-1/2$ in (71) and recalling (58), one readily verifies that (40) is a special case of (71).

Remark 17. In the same way as above one can define a sequence of OPs associated with the function $\partial_\rho F_L(\eta,\rho)$ or, more generally, with $\alpha F_L(\eta,\rho)+\rho\partial_\rho F_L(\eta,\rho)$ for a real α. Since our results in this respect are not complete yet and since they are notably less elegant than those for the function $F_L(\eta,\rho)$, we confine ourselves to this short comment for now and leave this problem open for further research.

Let us just consider the particular case of the function $\partial_\rho F_L(\eta,\rho)$ and call the corresponding sequence of OPs $\{\tilde{P}_n^{(L)}(\eta;z)\}_{n=0}^{\infty}$. It is defined by a recurrence analogous to (53) but now the coefficients in the relation are matrix entries of $\tilde{J}_L$ rather than those of J_L. The initial conditions are the same, and one has to restrict the range of parameters to the values $L>-1/2$ and $\eta\in\mathbb{R}$. Let us denote $\tilde{R}_n^{(L)}(\eta;\rho):=\tilde{P}_n^{(L)}(\eta;\rho^{-1})$ for $\rho\neq0$, $n\in\mathbb{Z}_+$. The zeros of the function $\rho\mapsto\partial_\rho F_L(\eta,\rho)$ can be shown to be all real and simple and to form a countable set with no finite accumulation points. One can arrange the zeros into a sequence which we call $\{\tilde{\rho}_{L,n};\ n\in\mathbb{N}\}$. Then the corresponding orthogonality measure $\mathrm{d}\mu$ is again supported on the set $\{\tilde{\rho}_{L,k}^{-1};\ k\in\mathbb{N}\}\cup\{0\}$. It is possible to directly compute the magnitude Λ_k of the jump at $\tilde{\rho}_{L,k}^{-1}$ of the piece-wise constant function μ with the result

$$\Lambda_k=\frac{L+1}{\tilde{\rho}_{L,k}^2-2\eta\tilde{\rho}_{L,k}-L(L+1)}.$$

We propose that μ has no jump at the point 0 or, equivalently, that the Jacobi matrix $\tilde{J}_L$ is invertible, but we have no proof for this hypothesis yet.

6. The spectral zeta function associated with $F_L(\eta, \rho)$

Let us recall that the spectral zeta function of a positive definite operator A with a discrete spectrum whose inverse A^{-1} belongs to the p-th Schatten class is defined as

$$\zeta^{(A)}(s) := \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} = \operatorname{Tr} A^{-s}, \quad \operatorname{Re} s \geq p,$$

where $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ are the eigenvalues of A. The zeta function can be used to approximately compute the ground state energy of A, i.e. the lowest eigenvalue λ_1. This approach is known as Euler's method (initially applied to the first positive zero of the Bessel function J_0) which is based on the inequalities

$$\zeta^{(A)}(s)^{-1/s} < \lambda_1 < \frac{\zeta^{(A)}(s)}{\zeta^{(A)}(s+1)}, \quad s \geq p. \tag{73}$$

In fact, the inequalities in (73) become equalities in the limit $s \to \infty$.

In this section we describe recursive rules for the zeta function associated with the regular Coulomb wave function. The procedure can be applied, however, to a wider class of special functions. For example, this approach can also be applied to Bessel functions, resulting in the well known convolution formulas for the Rayleigh function [11]. Not surprisingly, the recurrences derived below can be viewed as a generalization of these previously known results.

Recall also that the regularized determinant,

$$\det_2(1+A) := \det\big((1+A)\exp(-A)\big),$$

is well defined if A is a Hilbert–Schmidt operator, i.e. belonging to the second Schatten class, on a separable Hilbert space. Moreover, the regularized determinant is continuous in the Hilbert–Schmidt norm [18, Theorem 9.2].

Referring to (1), (46) and (70), we start from the identity

$$\begin{aligned}\det_2(1-\rho J_{L,n}) &= \exp(\rho \operatorname{Tr} J_{L,n}) \det(1-\rho J_{L,n}) \\ &= \exp\left(\frac{\eta\rho}{L+n+1} - \frac{\eta\rho}{L+1}\right)\left(\prod_{k=1}^{n}(1-\rho\lambda_{k+L})\right)\mathfrak{F}\left(\left\{\frac{\gamma_{L+k}^2}{\lambda_{L+k}-\rho^{-1}}\right\}_{k=1}^{n}\right)\end{aligned}$$

where $J_{L,n}$ stands for the $n \times n$ truncation of J_L. The formula can be verified straightforwardly by mathematical induction in n with the aid of the rule (7). Now, sending n to infinity and using (46), we get

$$\det_2(1-\rho J_L) = \exp\left(-\frac{\eta\rho}{L+1}\right)\phi_L(\eta,\rho). \tag{74}$$

On the other hand, we have the Hadamard product formula

$$\det_2(1-\rho J_L) = \prod_{n=1}^{\infty}\left(1-\frac{\rho}{\rho_{L,n}}\right)\mathrm{e}^{\rho/\rho_{L,n}}, \tag{75}$$

see [18, Theorem 9.2]. Combining (74) and (75) one arrives at the Hadamard infinite product expansion of $\phi_L(\eta, .)$,

$$\phi_L(\eta,\rho) = \exp\left(\frac{\eta\rho}{L+1}\right)\prod_{n=1}^{\infty}\left(1-\frac{\rho}{\rho_{L,n}}\right)\mathrm{e}^{\rho/\rho_{L,n}}. \tag{76}$$

Let us define

$$\zeta_L(k) := \sum_{n=1}^{\infty} \frac{1}{\rho_{L,n}^k}, \quad k \geq 2.$$

In view of (76), we can expand the logarithm of $\phi_L(\eta,\rho)$ into a power series,

$$\ln \phi_L(\eta,\rho) = \frac{\eta\rho}{L+1} - \sum_{n=1}^{\infty}\sum_{k=2}^{\infty} \frac{1}{k}\left(\frac{\rho}{\rho_{L,n}}\right)^k,$$

whenever $\rho \in \mathbb{C}$, $|\rho| < |\rho_{L,1}|$. Whence

$$\frac{\partial_\rho \phi_L(\eta,\rho)}{\phi_L(\eta,\rho)} = \frac{\eta}{L+1} - \sum_{k=1}^{\infty} \zeta_L(k+1)\rho^k.$$

Comparing this equality to (52) one finds that

$$\sum_{k=0}^{\infty} \zeta_L(k+2)\rho^k = \frac{1}{2L+3}\left(1 + \frac{\eta^2}{(L+1)^2}\right)\frac{\phi_{L+1}(\eta,\rho)}{\phi_L(\eta,\rho)} \quad \text{for } |\rho| < |\rho_{L,1}|. \tag{77}$$

In this way, the values of $\zeta_L(k)$ for $k \in \mathbb{N}$, $k \geq 2$, can be obtained by an inspection of the Taylor series of the RHS in (77).

However, an apparently more efficient tool to compute the values of the zeta function would be a recurrence formula. To find it, one has to differentiate Eq. (77) with respect to ρ and use both formulas (51) and (52). By doing so, one arrives at the equation

$$\begin{aligned}
&(2L+3)\left(1 + \frac{\eta^2}{(L+1)^2}\right)^{-1} \sum_{k=0}^{\infty} k\,\zeta_L(k+2)\rho^k \\
&\quad = (2L+3)\left(1 - \frac{\phi_{L+1}(\eta,\rho)}{\phi_L(\eta,\rho)}\right) - \frac{2\eta\rho}{L+1}\frac{\phi_{L+1}(\eta,\rho)}{\phi_L(\eta,\rho)} + \frac{\rho^2}{2L+3}\left(1 + \frac{\eta^2}{(L+1)^2}\right)\left(\frac{\phi_{L+1}(\eta,\rho)}{\phi_L(\eta,\rho)}\right)^2.
\end{aligned}$$

Using (77) to express $\phi_{L+1}(\eta,\rho)/\phi_L(\eta,\rho)$, one obtains

$$\begin{aligned}
\sum_{k=1}^{\infty} k\zeta_L(k+2)\rho^k &= 1 + \frac{\eta^2}{(L+1)^2} - (2L+3)\sum_{k=0}^{\infty}\zeta_L(k+2)\rho^k \\
&\quad + \frac{2\eta}{L+1}\sum_{k=1}^{\infty}\zeta_L(k+1)\rho^k + \sum_{k=0}^{\infty}\sum_{l=0}^{k}\zeta_L(l+2)\zeta_L(k-l+2)\rho^{k+2}.
\end{aligned}$$

At this point, it suffices to equate coefficients at the same powers of ρ. In particular, for the absolute term we get

$$\zeta_L(2) = \frac{1}{2L+3}\left(1 + \frac{\eta^2}{(L+1)^2}\right). \tag{78}$$

Note that $\zeta_L(2)$ is the square of the Hilbert–Schmidt norm of J_L. The desired recurrence relation reads

$$\zeta_L(k+1) = \frac{1}{2L+k+2}\left(\frac{2\eta}{L+1}\zeta_L(k) + \sum_{l=1}^{k-2}\zeta_L(l+1)\zeta_L(k-l)\right), \quad k = 2,3,4,\ldots. \tag{79}$$

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As described above, bounds on the first (in modulus) zero $\rho_{L,1}$ can be determined with the aid of the zeta function. The operator J_L is not positive, however, and so the bounds should be written as follows

$$\zeta_L(2s)^{-1/s} < \rho_{L,1}^2 < \frac{\zeta_L(2s)}{\zeta_L(2s+2)}, \quad s \geq 1.$$

In the simplest case, for $s = 1$, we get the estimates

$$\frac{(2L+3)(L+1)^2}{(L+1)^2+\eta^2} < \rho_{L.1}^2 < \frac{(2L+3)(2L+5)(L+2)(L+1)^2}{(L+4)\eta^2+(L+2)(L+1)^2}.$$

Let us further examine the particular case when $\eta = 0$ and $L = \nu - 1/2$. In that case, rules (78) and (79) reproduce the well known recurrence relations for the Rayleigh function $\sigma_{2n}(\nu)$, with $n \geq 2$ and $\nu > -1$ [11]. Recall that

$$\sigma_{2n}(\nu) := \sum_{k=1}^{\infty} j_{\nu,k}^{-2n}$$

where $j_{\nu,k}$ denotes the k-th positive zero of the Bessel function J_ν. The recurrence relation reads

$$\sigma_2(\nu) = \frac{1}{4(\nu+1)}, \qquad \sigma_{2n}(\nu) = \frac{1}{n+\nu}\sum_{k=1}^{n-1}\sigma_{2k}(\nu)\sigma_{2n-2k}(\nu) \quad \text{for } n = 2, 3, 4, \ldots.$$

Remark 18. Let us remark that instead of (79) one can derive a recurrence relation in a form which is a linear combination of zeta functions. Rewrite Eq. (77) as

$$\left(1+\frac{\eta^2}{(L+1)^2}\right)\phi_{L+1}(\eta,\rho) = (2L+3)\phi_L(\eta,\rho)\sum_{k=0}^{\infty}\zeta_L(k+2)\rho^k$$

and replace everywhere the function ϕ_L by the power expansion

$$\phi_L(\eta,\rho) = \mathrm{e}^{-i\rho}\sum_{k=0}^{\infty}\frac{(L+1-i\eta)_k}{(2L+2)_k}\frac{(2i\rho)^k}{k!}.$$

After carrying out obvious cancellations and equating coefficients at the same powers of ρ on both sides one arrives at the identity

$$\frac{2[(L+1)^2+\eta^2]}{(L+1)(L+1-i\eta)}\frac{\Gamma(L+2-i\eta+k)}{\Gamma(2L+4+k)k!} = \sum_{l=0}^{k}\frac{\Gamma(L+1-i\eta+k-l)(2i)^{-l}}{\Gamma(2L+2+k-l)(k-l)!}\zeta_L(l+2),$$

which holds for any $k \in \mathbb{Z}_+$, $L > -1$ and $\eta \in \mathbb{R}$.

Remark 19. The orthogonality measure $\mathrm{d}\mu$ for the sequence of OPs $\{P_n^{(L)}(\eta,\rho)\}$, as described in Theorem 14, fulfills

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}\mu(x) = \frac{(2L+3)(L+1)^2}{(L+1)^2+\eta^2}\sum_{k=1}^{\infty}\rho_{L,k}^{-2}f\left(\rho_{L,k}^{-1}\right)$$

for every $f \in C(\mathbb{R})$. Consequently, the moment sequence associated with the measure $\mathrm{d}\mu$ can be expressed in terms of the zeta function,

Please cite this article in press as: F. Štampach, P. Šťovíček, Orthogonal polynomials associated with Coulomb wave functions, J. Math. Anal. Appl. (2014), http://dx.doi.org/10.1016/j.jmaa.2014.04.049

$$m_n := \int_{\mathbb{R}} x^n \, \mathrm{d}\mu(x) = \frac{\zeta_L(n+2)}{\zeta_L(2)}, \quad n \in \mathbb{Z}_+$$

(recall also (78)). In view of formulas (78) and (79), this means that the moment sequence can be evaluated recursively.

Remark 20. This comment extends Remark 17. We note that it is possible to derive formulas analogous to (79) for the spectral zeta function associated with the function $\partial_\rho F_L(\eta, \rho)$ though the resulting recurrence rule is notably more complicated in this case. One may begin, similarly to (75), with the identities

$$\det_2(1 - \rho \tilde{J}_L) = \prod_{n=1}^{\infty} \left(1 - \frac{\rho}{\tilde{\rho}_{L,n}}\right) \mathrm{e}^{\rho/\tilde{\rho}_{L,n}} = \exp\left(-\frac{(L+2)\eta\rho}{(L+1)^2}\right) \left(\phi_L(\eta, \rho) + \frac{\rho}{L+1} \partial_\rho \phi_L(\eta, \rho)\right).$$

Hence for $\psi_L(\eta, \rho) := \phi_L(\eta, \rho) + (\rho/(L+1))\partial_\rho \phi_L(\eta, \rho)$ we have

$$\ln \psi_L(\eta, \rho) = \frac{(L+2)\eta\rho}{(L+1)^2} - \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{k} \left(\frac{\rho}{\tilde{\rho}_{L,n}}\right)^k \tag{80}$$

whenever $\rho \in \mathbb{C}$, $|\rho| < |\tilde{\rho}_{L,1}|$. Let us define

$$\tilde{\zeta}_L(k) := \sum_{n=1}^{\infty} \frac{1}{\tilde{\rho}_{L,n}^k}, \quad k \ge 2.$$

Having arrived at this point, manipulations quite similar to those used in the case of the zeta function associated with $F_L(\eta, \rho)$ can be applied. Differentiating Eq. (80) twice and both times taking into account that $F_L(\eta, \rho)$ solves (41), one arrives, after some tedious but straightforward computation, at the equation

$$\begin{aligned} &\frac{2(\rho - \eta)}{\rho^2 - 2\eta\rho - L(L+1)} \left(-L\rho - \frac{(L+2)\eta\rho^2}{(L+1)^2} + \sum_{k=2}^{\infty} \tilde{\zeta}_L(k)\rho^{k+1}\right) + \left(-L - \frac{(L+2)\eta\rho}{(L+1)^2} + \sum_{k=2}^{\infty} \tilde{\zeta}_L(k)\rho^k\right)^2 \\ &= L^2 + \frac{2(L^2 + L - 1)\eta\rho}{(L+1)^2} - \rho^2 + \sum_{k=2}^{\infty} (k+1)\tilde{\zeta}_L(k)\rho^k. \end{aligned}$$

From here the sought-after recurrence rules can be extracted in a routine way, yet we refrain from writing them down explicitly because of their length and complexity.

Acknowledgments

The authors wish to acknowledge gratefully partial support from grant No. GA13-11058S of the Czech Science Foundation.

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The Hahn-Exton q-Bessel function as the characteristic function of a Jacobi matrix

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Abstract

A family $\mathcal{T}^{(\nu)}$, $\nu \in \mathbb{R}$, of semiinfinite positive Jacobi matrices is introduced with matrix entries taken from the Hahn-Exton q-difference equation. The corresponding matrix operators defined on the linear hull of the canonical basis in $\ell^2(\mathbb{Z}_+)$ are essentially self-adjoint for $|\nu| \geq 1$ and have deficiency indices $(1,1)$ for $|\nu| < 1$. A convenient description of all self-adjoint extensions is obtained and the spectral problem is analyzed in detail. The spectrum is discrete and the characteristic equation on eigenvalues is derived explicitly in all cases. Particularly, the Hahn-Exton q-Bessel function $J_\nu(z;q)$ serves as the characteristic function of the Friedrichs extension. As a direct application one can reproduce, in an alternative way, some basic results about the q-Bessel function due to Koelink and Swarttouw.

Keywords: Jacobi matrix, Hahn-Exton q-Bessel function, self-adjoint extension, spectral problem

2010 Mathematical Subject Classification: 47B36, 33D45, 47A10, 39A70

1 Introduction

There exist three commonly used q-analogues of the Bessel function $J_\nu(z)$. Two of them were introduced by Jackson in the beginning of the 20th century and are mutually closely related, see [6] for a basic overview and original references. Here we shall be concerned with the third analogue usually named after Hahn and Exton. Its most important features like properties of the zeros and the associated Lommel polynomials including orthogonality relations were studied not so long ago [11, 10, 9]. The Hahn-Exton q-Bessel function is defined as follows

$$J_\nu(z;q) \equiv J_\nu^{(3)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\, z^\nu \, {}_1\phi_1(0;q^{\nu+1};q,qz^2). \tag{1}$$

Here ${}_r\phi_s(a_1,\ldots,a_r;b_1,\ldots,b_s;q,z)$ stands for the basic hypergeometric series (see, for instance, [6]). It is of importance that $J_\nu(z;q)$ obeys the Hahn-Exton q-Bessel difference equation

$$J_\nu(qz;q) + q^{-\nu/2}(qz^2 - 1 - q^\nu)J_\nu(q^{1/2}z;q) + J_\nu(z;q) = 0. \tag{2}$$

Using the coefficients from (2) one can introduce a two-parameter family of real symmetric Jacobi matrices

$$\mathcal{T} \equiv \mathcal{T}^{(\nu)} = \begin{pmatrix} \beta_0 & \alpha_0 & & & \\ \alpha_0 & \beta_1 & \alpha_1 & & \\ & \alpha_1 & \beta_2 & \alpha_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{3}$$

depending on $\nu \in \mathbb{R}$ and also on q, $0 < q < 1$. But q is treated below as having been fixed and in most cases is not indicated explicitly. Matrix entries are supposed to be indexed by $m, n = 0, 1, 2, \ldots$. More formally, we put $\mathcal{T}_{n,n} = \beta_n$, $\mathcal{T}_{n,n+1} = \mathcal{T}_{n+1,n} = \alpha_n$ and $\mathcal{T}_{m,n} = 0$ otherwise, where

$$\alpha_n \equiv \alpha_n^{(\nu)} = -q^{-n+(\nu-1)/2},\ \beta_n \equiv \beta_n^{(\nu)} = (1+q^\nu)\, q^{-n},\ n \in \mathbb{Z}_+. \tag{4}$$

In order to keep notations simple we will also suppress the superscript (ν) provided this cannot lead to misunderstanding.

Our main goal in this paper is to provide a detailed analysis of those operators T in $\ell^2 \equiv \ell^2(\mathbb{Z}_+)$ (with $\mathbb{Z}_+$ standing for nonnegative integers) whose matrix in the canonical basis equals $\mathcal{T}$. This example has that interesting feature that it exhibits a transition between the indeterminate and determinate cases depending on ν. In more detail, denote by $\mathbb{C}^\infty$ the linear space of all complex sequences indexed by $\mathbb{Z}_+$ and by $\mathscr{D}$ the subspace of those sequences having at most finitely many nonvanishing entries. One may also say that $\mathscr{D}$ is the linear hull of the canonical basis in ℓ^2. It turns out that the matrix operator induced by $\mathcal{T}$ on the domain $\mathscr{D}$ is essentially self-adjoint in ℓ^2 if and only if $|\nu| \geq 1$. For $|\nu| < 1$ there exists a one-parameter family of self-adjoint extensions.

Another interesting point is a close relationship between the spectral data for these operators T and the Hahn-Exton q-Bessel function. It turns out that, for an appropriate (Friedrichs) self-adjoint extension, $J_\nu(q^{-1/2}\sqrt{x};q)$ serves as the characteristic function of T in the sense that its zero set on $\mathbb{R}_+$ exactly coincides with the spectrum of T. There also exists an explicit formula for corresponding eigenvectors. Moreover, T^{-1} can be shown to be compact. This makes it possible to reproduce, in a quite straightforward but alternative way, some results originally derived in [11, 9].

Finally we remark that recently we have constructed, in [14, 15], a number of examples of Jacobi operators with discrete spectra and characteristic functions explicitly expressed in terms of special functions, a good deal of them comprising various combinations of q-Bessel functions. That construction confines, however, only to a class of Jacobi matrices characterized by a convergence condition imposed on the matrix entries. For this condition is readily seen to be violated in the case of $\mathcal{T}$, as defined in (3) and (4), in the present paper we have to follow another approach whose essential part is a careful asymptotic analysis of formal eigenvectors of $\mathcal{T}$.

2 Self-adjoint operators induced by $\mathcal{T}$

2.1 A $*$-algebra of semiinfinite matrices

Denote by $\mathscr{M}_{\text{fin}}$ the set of all semiinfinite matrices indexed by $\mathbb{Z}_+ \times \mathbb{Z}_+$ such that each row and column of a matrix has only finitely many nonzero entries. For instance, $\mathscr{M}_{\text{fin}}$ comprises all band matrices and so all finite-order difference operators. Notice that $\mathscr{M}_{\text{fin}}$ is naturally endowed with the structure of a $*$-algebra, matrices from $\mathscr{M}_{\text{fin}}$ act linearly on $\mathbb{C}^\infty$ and $\mathscr{D}$ is $\mathscr{M}_{\text{fin}}$-invariant.

Choose $\mathcal{A} \in \mathscr{M}_{\text{fin}}$ and let $\mathcal{A}^{\text{H}}$ stand for its Hermitian adjoint. Let us introduce, in a fully standard manner, operators $\dot{A}$, A_{min} and A_{max} on ℓ^2, all of them being restrictions of $\mathcal{A}$ to appropriate domains. Namely, $\dot{A}$ is the restriction $\mathcal{A}\big|_{\mathscr{D}}$, A_{min} is the closure of $\dot{A}$ and

$$\operatorname{Dom} A_{\text{max}} = \{f \in \ell^2;\, \mathcal{A}f \in \ell^2\}.$$

Clearly, $\dot{A} \subset A_{\text{max}}$. Straightforward arguments based just on systematic application of definitions show that

$$(\dot{A})^* = (A_{\text{min}})^* = A^{\text{H}}_{\text{max}},\ (A_{\text{max}})^* = A^{\text{H}}_{\text{min}}.$$

Hence A_{max} is closed and $A_{\text{min}} \subset A_{\text{max}}$.

Lemma 1. *Suppose $p, w \in \mathbb{C}$ and let $\mathcal{A} \in \mathscr{M}_{fin}$ be defined by*

$$\mathcal{A}_{n,n} = p^n,\ \mathcal{A}_{n+1,n} = -wp^{n+1} \ \text{ for all } n \in \mathbb{Z}_+,\ \mathcal{A}_{m,n} = 0 \text{ otherwise.} \tag{5}$$

Then $A_{min} \neq A_{max}$ if and only if $1/|p| < |w| < 1$, and in that case

$$\operatorname{Dom} A_{min} = \{f \in \operatorname{Dom} A_{max}\,;\, \lim_{n\to\infty} w^{-n} f_n = 0\}.$$

Proof. Choose arbitrary $f \in \operatorname{Dom} A_{\text{max}}$. Then $f \in \operatorname{Dom} A_{\text{min}}$ iff

$$\forall g \in \operatorname{Dom} A^{\text{H}}_{\text{max}},\ 0 = \langle \mathcal{A}^{\text{H}} g, f\rangle - \langle g, \mathcal{A}f\rangle = -\lim_{n\to\infty} \mathcal{A}_{n,n}\, \overline{g_n} f_n. \tag{6}$$

Since both f and g in (6) are supposed to belong to ℓ^2 this condition is obviously fulfilled if $|p| \le 1$. Furthermore, the situation becomes fully transparent for $w = 0$.

In that case the sequences $\{p^n g_n\}$ and $\{p^n f_n\}$ are square summable and (6) is always fulfilled. In the remainder of the proof we assume that $|p| > 1$ and $w \neq 0$.

Consider first the case when $|w| \geq 1$. Relation $\mathcal{A}f = h$ can readily be inverted even in $\mathbb{C}^\infty$ and one finds that

$$p^n f_n = \sum_{k=0}^{n} (pw)^k h_{n-k} = (pw)^n \sum_{k=0}^{n} (pw)^{-k} h_k, \ \forall n.$$

Denote temporarily by $\tilde{h}$ the sequence with $\tilde{h}_n = (\overline{pw})^{-n}$. It is square summable since, by our assumptions, $|pw| > 1$. For $f \in \operatorname{Dom} A_{\max}$ one has $h \in \ell^2$ and

$$f_n = w^n(\langle \tilde{h}, h \rangle - \zeta_n) \ \text{ where } \zeta_n = \sum_{k=n+1}^{\infty} (pw)^{-k} h_k.$$

Assumption $f \in \ell^2$ clearly implies $\langle \tilde{h}, h \rangle = 0$ and then, by the Schwarz inequality,

$$|\mathcal{A}_{n,n} f_n| \leq \|h\| / \sqrt{|pw| - 1}\,, \ \forall n.$$

Whence $\mathcal{A}_{n,n}\,\overline{g_n} f_n \to 0$ as $n \to \infty$ for all $g \in \operatorname{Dom} A^{\mathrm{H}}_{\max}$ and so $f \in \operatorname{Dom} A_{\min}$.

Suppose now that $|w| < 1$. If $\mathcal{A}^{\mathrm{H}} g = h$ in $\mathbb{C}^\infty$ and h is bounded then, as an easy computation shows,

$$(\overline{p})^n\, g_n = \gamma(\overline{w})^{-n} + \sum_{k=0}^{\infty} (\overline{w})^k h_{n+k} \tag{7}$$

for all n and some constant γ. Observe that, by the Schwarz inequality,

$$\left| \sum_{k=0}^{\infty} (\overline{w})^k h_{n+k} \right| \leq \frac{1}{\sqrt{1 - |w|^2}} \left(\sum_{k=n}^{\infty} |h_k|^2 \right)^{1/2}, \tag{8}$$

and this expression tends to zero as n tends to infinity provided $h \in \ell^2$.

In the case when $|pw| \leq 1$ the property $g \in \ell^2$ and $\mathcal{A}^{\mathrm{H}} g = h \in \ell^2$ implies that the constant γ in (7) is zero, and from (8) one infers that $\mathcal{A}_{n,n}\,\overline{g_n} \to 0$ as $n \to \infty$. Thus one finds condition (6) to be always fulfilled meaning that $f \in \operatorname{Dom} A_{\min}$.

If $|pw| > 1$ then the sequence g defined in (7) is square summable whatever $\gamma \in \mathbb{C}$ and $h \in \ell^2$ are. Condition (6) is automatically fulfilled, however, for $\gamma = 0$. Hence (6) can be reduced to the single nontrivial case when we choose $\tilde{g} \in \operatorname{Dom} A^{\mathrm{H}}_{\max}$ with $\tilde{g}_n = (\overline{pw})^{-n}$. Then $\mathcal{A}^{\mathrm{H}} \tilde{g} = 0$ and condition $\langle \tilde{g}, \mathcal{A}f \rangle = 0$ means that $w^{-n} f_n \to 0$ as $n \to \infty$. It remains to show that there exists $f \in \operatorname{Dom} A_{\max}$ not having this property. However the sequence $\tilde{f}$, with $\tilde{f}_n = w^n$, does the job since $\mathcal{A}\tilde{f} = (1, 0, 0, \ldots) \in \ell^2$. $\square$

2.2 Associated orthogonal polynomials, self-adjoint extensions

The tridiagonal matrix $\mathcal{T}$ defined in (3), (4) belongs to $\mathscr{M}_{\mathrm{fin}}$. With $\mathcal{T}$ there is associated a sequence of monic orthogonal polynomials [7], called $\{P_n(x) \equiv P_n^{(\nu)}(x)\}$ and defined by the recurrence

$$P_n(x) = (x - \beta_{n-1}) P_{n-1}(x) - \alpha_{n-2}^{\,2}\, P_{n-2}(x), \ n \geq 1, \tag{9}$$

with $P_{-1}(x)=0$, $P_0(x)=1$. Put

$$\hat{P}_n(x) \equiv \hat{P}_n^{(\nu)}(x) = (-1)^n q^{n(n-\nu)/2} P_n^{(\nu)}(x). \tag{10}$$

Then $(\hat{P}_0(x), \hat{P}_1(x), \hat{P}_2(x), \ldots)$ is a formal eigenvector of $\mathcal{T}$ ($\equiv$ an eigenvector of $\mathcal{T}$ in $\mathbb{C}^\infty$), i.e.

$$(\beta_0 - x)\hat{P}_0(x) + \alpha_0 \hat{P}_1(x) = 0,\ \alpha_{n-1}\hat{P}_{n-1}(x) + (\beta_n - x)\hat{P}_n(x) + \alpha_n \hat{P}_{n+1}(x) = 0 \text{ for } n \geq 1. \tag{11}$$

Observe that $\mathcal{T}^{(-\nu)} = q^{-\nu}\,\mathcal{T}^{(\nu)}$. Since we are primarily interested in spectral properties of $\mathcal{T}^{(\nu)}$ in the Hilbert space ℓ^2 we may restrict ourselves, without loss of generality, to nonnegative values of the parameter ν. The value $\nu = 0$ turns out to be somewhat special and will be discussed separately later in Subsection 3.2. Thus, if not stated otherwise, we assume from now on that $\nu > 0$.

Given $\mathcal{T} \in \mathscr{M}_{\text{fin}}$ we again introduce the operators $\dot{T}$, $T_{\min}$, $T_{\max}$ as explained in Subsection 2.1. Notice that

$$\beta_n = q^{(\nu-1)/2}|\alpha_{n-1}| + q^{-(\nu-1)/2}|\alpha_n|.$$

It follows at once that the operators $\dot{T}$ and consequently $T_{\min}$ are positive. In fact, for any real sequence $\{f_n\} \in \mathscr{D}$ one has

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathcal{T}_{m,n} f_m f_n = |\alpha_{-1}| q^{(\nu-1)/2}\, f_0^2 + \sum_{n=1}^{\infty} |\alpha_{n-1}| (q^{(\nu-1)/4} f_n - q^{-(\nu-1)/4} f_{n-1})^2 \geq 0.$$

This is equivalent to the factorization $\mathcal{T} = \mathcal{A}^{\mathrm{H}}\mathcal{A}$ where the matrix $\mathcal{A} \equiv \mathcal{A}^{(\nu)} \in \mathscr{M}_{\text{fin}}$ is defined by the prescription: $\forall f \in \mathbb{C}^\infty$,

$$(\mathcal{A}f)_0 = |\alpha_{-1}|^{1/2} q^{(\nu-1)/4}\, f_0,\ (\mathcal{A}f)_n = |\alpha_{n-1}|^{1/2}(q^{(\nu-1)/4} f_n - q^{-(\nu-1)/4} f_{n-1}) \text{ for } n \geq 1.$$

That is, $\forall n \geq 0$,

$$\mathcal{A}_{n,n} = |\alpha_{n-1}|^{1/2} q^{(\nu-1)/4} = q^{-(n-\nu)/2},\ \mathcal{A}_{n+1,n} = -|\alpha_n|^{1/2} q^{-(\nu-1)/4} = -q^{-n/2}, \tag{12}$$

and $\mathcal{A}_{m,n} = 0$ otherwise.

Thus $\mathcal{T}$ induces a positive form on the domain $\mathscr{D}$ with values $\langle f, \mathcal{T} f\rangle = \|\mathcal{A}f\|^2$, $\forall f \in \mathscr{D}$. Let us call $\mathfrak{t}$ its closure. Then $\operatorname{Dom}\mathfrak{t} = \operatorname{Dom} A_{\min}$ and $\mathfrak{t}(x) = \|A_{\min}\, x\|^2$, $\forall x \in \operatorname{Dom}\mathfrak{t}$. The positive operator T^{F} associated with $\mathfrak{t}$ according to the representation theorem is the Friedrichs extension of the closed positive operator $T_{\min}$. One has

$$T^{\mathrm{F}} = A_{\min}^{*} A_{\min}.$$

It is known that T^{F} has the smallest form-domain among all self-adjoint extensions of $T_{\min}$ and also that this is the only self-adjoint extension of $T_{\min}$ with its domain contained in $\operatorname{Dom}\mathfrak{t}$, see [8, Chapter VI].

One can apply Lemma 1, with $p = q^{-1/2}$ and $w = q^{(1-\nu)/2}$, to obtain an explicit description of the form-domain of T^{F}. Using still $\mathcal{A}$ defined in (12) one has $\operatorname{Dom}\mathfrak{t} = \{f \in \ell^2;\, \mathcal{A}f \in \ell^2\}$ for $\nu \geq 1$ and

$$\operatorname{Dom}\mathfrak{t} = \{f \in \ell^2;\, \mathcal{A}f \in \ell^2 \text{ and } \lim_{n\to\infty} q^{(\nu-1)n/2} f_n = 0\} \tag{13}$$

for $0 < \nu < 1$.

In [5] one finds a clear explicit description of the domain of the Friedrichs extension of a positive Jacobi matrix which can be applied to our case. To this end, consider the homogeneous three-term recurrence equation

$$\alpha_n Q_{n+1} + \beta_n Q_n + \alpha_{n-1} Q_{n-1} = 0 \tag{14}$$

on $\mathbb{Z}$. It simplifies to a recurrence equation with constant coefficients,

$$q^{(\nu-1)/2} Q_{n+1} - (1+q^\nu) Q_n + q^{(\nu+1)/2} Q_{n-1} = 0. \tag{15}$$

One can distinguish two independent solutions, $\{Q_n^{(1)}\}$ and $\{Q_n^{(2)}\}$, where

$$Q_n^{(1)} = \frac{q^{(1-\nu)n/2} - q^{\nu+(1+\nu)n/2}}{1-q^\nu},\ Q_n^{(2)} = q^{(1+\nu)n/2},\ n \in \mathbb{Z}. \tag{16}$$

Notice that $\{Q_n^{(1)}\}$ satisfies the initial conditions $Q_{-1}^{(1)} = 0$, $Q_0^{(1)} = 1$, and so $Q_n^{(1)} = \hat{P}_n(0)$, $\forall n \geq 0$. On the other hand, $\{Q_n^{(2)}\}$ is always square summable over $\mathbb{Z}_+$, and this is the so-called minimal solution at $+\infty$ since

$$\lim_{n\to+\infty} \frac{Q_n^{(2)}}{Q_n} = 0$$

for every solution $\{Q_n\}$ of (14) which is linearly independent of $\{Q_n^{(2)}\}$. The Wronskian of $Q^{(1)}$ and $Q^{(2)}$ equals

$$W_n(Q^{(1)}, Q^{(2)}) = 1,\ \forall n \in \mathbb{Z},$$

where $W_n(f,g) := \alpha_n(f_n g_{n+1} - g_n f_{n+1})$. Theorem 4 in [5] tells us that

$$\operatorname{Dom} T^{\mathrm{F}} = \{f \in \ell^2;\, \mathcal{T}f \in \ell^2 \text{ and } W_\infty(f, Q^{(2)}) = 0\} \tag{17}$$

where we put

$$W_\infty(f,g) = \lim_{n\to\infty} W_n(f,g)$$

for $f, g \in \mathbb{C}^\infty$ provided the limit exists. It is useful to note, however, that discrete Green's formula implies existence of the limit whenever $f, g \in \operatorname{Dom} T_{\max}$, and then

$$\langle T_{\max} f, g\rangle - \langle f, T_{\max} g\rangle = -W_\infty(\overline{f}, g).$$

We wish to determine all self-adjoint extensions of the closed positive operator $T_{\min}$. This is a standard general fact that the deficiency indices of $T_{\min}$ for any real symmetric Jacobi matrix $\mathcal{T}$ of the form (3), with all α_n's nonzero, are either $(0,0)$

or $(1,1)$. The latter case happens if and only if for some $x \in \mathbb{C}$ all solutions of the second-order difference equation

$$\alpha_n Q_{n+1} + (\beta_n - x)Q_n + \alpha_{n-1}Q_{n-1} = 0 \tag{18}$$

are square summable on $\mathbb{Z}_+$, and in that case this is true for any value of the spectral parameter x (see, for instance, a detailed discussion in Section 2.6 of [16]).

Let us remark that a convenient description of the one-parameter family of all self-adjoint extensions is also available if the deficiency indices are $(1,1)$. Fix $x \in \mathbb{R}$ and any couple $Q^{(1)}$, $Q^{(2)}$ of independent solutions of (18). Then all self-adjoint extensions of $T_{\min}$ are operators $\tilde{T}(\kappa)$ defined on the domains

$$\operatorname{Dom}\tilde{T}(\kappa) = \{f \in \ell^2;\ \mathcal{T}f \in \ell^2 \text{ and } W_\infty(f, Q^{(1)}) = \kappa\, W_\infty(f, Q^{(2)})\}, \tag{19}$$

with $\kappa \in \mathbb{R} \cup \{\infty\}$. Moreover, all of them are mutually different. Of course, $\tilde{T}(\kappa)f = \mathcal{T}f$, $\forall f \in \operatorname{Dom}\tilde{T}(\kappa)$.

In our case we know, for $x = 0$, a couple of solutions of (18) explicitly, cf. (16). From their form it becomes obvious that $T_{\min} = T_{\max}$ is self-adjoint if and only if $\nu \geq 1$. With this choice of $Q^{(1)}$, $Q^{(2)}$ and sticking to notation (19), it is seen from (17) that the Friedrichs extension T^{F} coincides with $\tilde{T}(\infty)$.

Lemma 2. *Suppose* $0 < \nu < 1$. *Then every sequence* $f \in \operatorname{Dom}T_{max}$ *has the asymptotic expansion*

$$f_n = C_1 q^{(1-\nu)n/2} + C_2 q^{(1+\nu)n/2} + o(q^n) \ \text{ as } n \to \infty, \tag{20}$$

where $C_1, C_2 \in \mathbb{C}$ *are some constants.*

Proof. Let $f \in \operatorname{Dom}T_{\max}$. That means $f \in \ell^2$ and $\mathcal{A}^{\mathrm{H}}\mathcal{A}f = h \in \ell^2$ where $\mathcal{A}$ is defined in (5), with $p = q^{-1/2}$, $w = q^{(1-\nu)/2}$ (then $\mathcal{T} = q^\nu \mathcal{A}^{\mathrm{H}}\mathcal{A}$). Denote $g = \mathcal{A}f$. Hence $\mathcal{A}^{\mathrm{H}}g = h$ and, as already observed in the course of the proof of Lemma 1, there exists a constant γ such that

$$g_n = \gamma q^{\nu n/2} + q^{n/2}\sum_{k=0}^{\infty} q^{(1-\nu)k/2} h_{n+k},\ \forall n.$$

Furthermore, the relation $\mathcal{A}f = g$ can be inverted,

$$f_n = q^{(1-\nu)n/2}\sum_{k=0}^{n} q^{\nu k/2} g_k,\ \forall n.$$

Whence

$$\begin{aligned} f_n &= \frac{\gamma}{1-q^\nu}\left(q^{(1-\nu)n/2} - q^{\nu+(1+\nu)n/2}\right) + q^{(1-\nu)n/2}\sum_{k=0}^{n} q^{(1+\nu)k/2}\sum_{j=0}^{\infty} q^{(1-\nu)j/2} h_{k+j} \\ &= C_1\, q^{(1-\nu)n/2} + C_2\, q^{(1+\nu)n/2} + q^n\, \zeta_n \end{aligned}$$

where

$$C_1 = \frac{\gamma}{1-q^\nu} + \sum_{k=0}^{\infty} q^{(1+\nu)k/2} \sum_{j=0}^{\infty} q^{(1-\nu)j/2} h_{k+j},\ C_2 = -\frac{\gamma q^\nu}{1-q^\nu}\,,$$

and

$$\zeta_n = -\sum_{k=1}^{\infty} q^{(1+\nu)k/2} \sum_{j=0}^{\infty} q^{(1-\nu)j/2} h_{n+k+j}.$$

Bearing in mind that $h \in \ell^2$ one concludes with the aid of the Schwarz inequality that $\zeta_n \to 0$ as $n \to \infty$. □

With the knowledge of asymptotic expansions established in Lemma 2 one can formulate a somewhat simpler and more explicit description of self-adjoint extensions of $T_{\min}$.

Proposition 3. *The operator* $T_{min} \equiv T_{min}^{(\nu)}$, *with* $\nu > 0$, *is self-adjoint if and only if* $\nu \geq 1$. *If* $0 < \nu < 1$ *then all mutually different self-adjoint extensions of* T_{min} *are parametrized by* $\kappa \in P^1(\mathbb{R}) \equiv \mathbb{R} \cup \{\infty\}$ *as follows. For* $f \in \operatorname{Dom} T_{max}$ *let* $C_1(f)$, $C_2(f)$ *be the constants from the asymptotic expansion (20), i.e.*

$$C_1(f) = \lim_{n\to\infty} f_n q^{-(1-\nu)n/2}\,,\ C_2(f) = \lim_{n\to\infty} \big(f_n - C_1(f) q^{(1-\nu)n/2}\big) q^{-(1+\nu)n/2}\,.$$

For $\kappa \in P^1(\mathbb{R})$, *a self-adjoint extension* $T(\kappa)$ *of* T_{min} *is a restriction of* T_{max} *to the domain*

$$\operatorname{Dom} T(\kappa) = \{f \in \ell^2;\ \mathcal{T} f \in \ell^2 \text{ and } C_2(f) = \kappa C_1(f)\}. \tag{21}$$

In particular, $T(\infty)$ *equals the Friedrichs extension* T^F.

Proof. Let $0 < \nu < 1$, $\{\zeta_n\}$ be a sequence converging to zero (bounded would be sufficient) and $g^{(1)}, g^{(2)}, h \in \mathbb{C}^\infty$ be the sequences defined by

$$g_n^{(1)} = q^{(1-\nu)n/2},\ g_n^{(2)} = q^{(1+\nu)n/2},\ h_n = q^n \zeta_n,\ \forall n.$$

Hence, referring to (16),

$$Q^{(1)} = \frac{1}{1-q^\nu}\,(g^{(1)} - q^\nu g^{(2)}),\ Q^{(2)} = g^{(2)}.$$

One finds at once that $W_\infty(g^{(1)}, h) = W_\infty(g^{(2)}, h) = 0$ and

$$W_n(g^{(1)}, g^{(2)}) = 1 - q^\nu,\ \forall n.$$

After a simple computation one deduces from (19) that $f \in \operatorname{Dom} T_{\max}$ belongs to $\operatorname{Dom} \tilde{T}(\tilde{\kappa})$ for some $\tilde{\kappa} \in P^1(\mathbb{R})$, i.e. $W_\infty(f, Q^{(1)}) = \tilde{\kappa} W_\infty(f, Q^{(2)})$, if and only if $C_2(f) = \kappa C_1(f)$ with $\kappa = -q^\nu - (1-q^\nu)\tilde{\kappa}$. In other words, $\tilde{T}(\tilde{\kappa}) = T(\kappa)$. Since the mapping

$$P^1(\mathbb{R}) \to P^1(\mathbb{R}) : \tilde{\kappa} \mapsto \kappa = -q^\nu - (1-q^\nu)\tilde{\kappa}$$

is one-to-one, $P^1(\mathbb{R}) \ni \kappa \mapsto T(\kappa)$ is another parametrization of self-adjoint extensions of $T_{\min}$. Particularly, $\tilde{\kappa} = \infty$ maps to $\kappa = \infty$ and so $T(\infty) = T^{\mathrm{F}}$. □

Remark 4. One can also describe $\operatorname{Dom} T_{\min}$. For $\nu \geq 1$ we simply have $T_{\min} = T_{\max} = T^{\mathrm{F}}$. In the case when $0 < \nu < 1$ it has been observed in [5] that a sequence $f \in \operatorname{Dom} T_{\max}$ belongs to $\operatorname{Dom} T_{\min}$ if and only if $W_\infty(f,g) = 0$ for all $g \in \operatorname{Dom} T_{\max}$. But this is equivalent to the requirement $C_1(f) = C_2(f) = 0$. Thus one has

$$\operatorname{Dom} T_{\min} = \{f \in \ell^2;\ \mathcal{T} f \in \ell^2 \text{ and } \lim_{n\to\infty} f_n q^{-(1+\nu)n/2} = 0\}. \tag{22}$$

2.3 The Green function and spectral properties

For $\nu \geq 1$ we shall write shortly $T \equiv T^{(\nu)}$ instead of $T_{\min} = T_{\max} = T^{\mathrm{F}}$. Referring to solutions (16) we claim that the Green function (matrix) of T, if $\nu \geq 1$, or T^{F}, if $0 < \nu < 1$, reads

$$G_{j,k} = \begin{cases} Q_j^{(1)} Q_k^{(2)} & \text{for } j \leq k, \\ Q_k^{(1)} Q_j^{(2)} & \text{for } j > k. \end{cases} \tag{23}$$

Proposition 5. *The matrix $(G_{j,k})$ defined in (23) represents a Hilbert-Schmidt operator $G \equiv G^{(\nu)}$ on ℓ^2 with the Hilbert-Schmidt norm*

$$\|G\|_{HS}^2 = \frac{1+q^{2+\nu}}{(1-q^2)(1-q^{1+\nu})^2(1-q^{2+\nu})}\,. \tag{24}$$

The operator G is positive and one has, $\forall f \in \ell^2$,

$$\langle f, Gf \rangle = \sum_{k=0}^{\infty} q^k \left| \sum_{j=0}^{\infty} q^{(1+\nu)j/2} f_{k+j} \right|^2.$$

Moreover, the inverse G^{-1} exists and equals T, if $\nu \geq 1$, or T^F, if $0 < \nu < 1$.

Proof. As is well known, if $T_{\min}$ is not self-adjoint then the resolvent of any of its self-adjoint extensions is a Hilbert-Schmidt operator [16, Lemma 2.19]. But in our case the resolvent is Hilbert-Schmidt for $\nu \geq 1$ as well, and one can directly compute the Hilbert-Schmidt norm of G for any $\nu > 0$,

$$\begin{aligned} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} G_{j,k}^2 &= \sum_{j=0}^{\infty} (Q_j^{(1)})^2 (Q_j^{(2)})^2 + 2\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty} (Q_j^{(1)})^2 (Q_k^{(2)})^2 \\ &= \frac{1+q^{1+\nu}}{1-q^{1+\nu}} \sum_{j=0}^{\infty} \left(\frac{1-q^{\nu(j+1)}}{1-q^\nu} \right)^2 q^{2j}. \end{aligned}$$

Thus one obtains (24). Hence the Green matrix unambiguously defines a self-adjoint compact operator G on ℓ^2.

Concerning the formula for the quadratic form one has to verify that, for all $m, n \in \mathbb{Z}_+$, $m \leq n$,

$$Q_m^{(1)} Q_n^{(2)} = \sum_{k=0}^{\infty} q^k \left(\sum_{j=0}^{\infty} q^{(1+\nu)j/2} \delta_{m,k+j} \right) \left(\sum_{j=0}^{\infty} q^{(1+\nu)j/2} \delta_{n,k+j} \right).$$

But this can be carried out in a straightforward manner.

A simple computation shows that for any $f \in \mathbb{C}^\infty$ and $n, N \in \mathbb{Z}_+$, $n < N$,

$$Q_n^{(2)} \sum_{k=0}^{n} Q_k^{(1)} (\mathcal{T} f)_k + Q_n^{(1)} \sum_{k=n+1}^{N} Q_k^{(2)} (\mathcal{T} f)_k = f_n - Q_n^{(1)} \alpha_N \big(Q_{N+1}^{(2)} f_N - Q_N^{(2)} f_{N+1} \big).$$

Considering the limit $N \to \infty$ one finds that, for a given $f \in \operatorname{Dom} T_{\max}$, the equality $G\mathcal{T} f = f$ holds iff $W_\infty(f, Q^{(2)}) = 0$. According to (17), this condition determines the domain of the Friedrichs extension T^{F}. Hence $GT^{\mathrm{F}} \subset I$ (the identity operator).

Furthermore, one readily verifies that, for all $f \in \ell^2$, $\mathcal{T} G f = f$. We still have to check that $\operatorname{Ran} G \subset \operatorname{Dom} T^{\mathrm{F}}$. But using the equality $W_\infty(Q^{(1)}, Q^{(2)}) = 1$ one computes, for $f \in \ell^2$ and $n \in \mathbb{Z}_+$,

$$W_n(Gf, Q^{(2)}) = \sum_{k=n+1}^{\infty} Q_k^{(2)} f_k \to 0 \ \text{ as } n \to \infty,$$

since $Q^{(2)} \in \ell^2$. Hence $T^{\mathrm{F}} G = I$. We conclude that $G^{-1} = T^{\mathrm{F}}$. Remember that we have agreed to write $T^{\mathrm{F}} = T$ for $\nu \geq 1$. □

Considering the case $\nu \geq 1$, the fact that the Jacobi operator T is positive and T^{-1} is compact has some well known consequences for its spectral properties. The same conclusions can be made for $0 < \nu < 1$ provided we replace T by T^{F}. And from the general theory of self-adjoint extensions one learns that $T(\kappa)$, for $\kappa \in \mathbb{R}$, has similar properties as T^{F} [18, Theorem 8.18].

Proposition 6. *The spectrum of any of the operators T, if $\nu \geq 1$, or $T(\kappa)$, with arbitrary $\kappa \in P^1(\mathbb{R})$, if $0 < \nu < 1$, is pure point and bounded from below, with all eigenvalues being simple and without finite accumulation points. Moreover, the operator T, for $\nu \geq 1$, or T^F, for $0 < \nu < 1$, is positive definite and one has the following lower bound on the spectrum, i.e. on the smallest eigenvalue $\xi_1 \equiv \xi_1^{(\nu)}$,*

$$\xi_1^{\,2} \geq \frac{(1-q^2)(1-q^{1+\nu})^2(1-q^{2+\nu})}{1+q^{2+\nu}}\,.$$

Proof. This is a simple general fact that all formal eigenvectors of the Jacobi matrix $\mathcal{T}$ are unique up to a multiplier [3]. By Proposition 5, $(T^{\mathrm{F}})^{-1}$ is compact and therefore the spectrum of T^{F} is pure point and with eigenvalues accumulating only at infinity.

For $0 < \nu < 1$, the deficiency indices of $T_{\min}$ are $(1,1)$. Whence, by the general spectral theory, if T^{F} has an empty essential spectrum then the same is true for all other self-adjoint extensions $T(\kappa)$, $\kappa \in \mathbb{R}$. Moreover, there is at most one eigenvalue of $T(\kappa)$ below $\xi_1 := \min \operatorname{spec}(T^{\mathrm{F}})$, see [18, § 8.3]. Referring once more to Proposition 5 one has

$$\min \operatorname{spec}(T^{\mathrm{F}}) = (\max \operatorname{spec}(G))^{-1} \geq \|G\|_{\mathrm{HS}}^{-1}.$$

In view of (24), one obtains the desired estimate on ξ_1. □

2.4 More details on the indeterminate case

In this subsection we confine ourselves to the case $0 < \nu < 1$ and focus on some general spectral properties of the self-adjoint extensions $T(\kappa)$, $\kappa \in P^1(\mathbb{R})$, in addition to those already mentioned in Proposition 6. The spectra of any two different self-adjoint extensions of $T_{\min}$ are known to be disjoint (see, for instance, proof of Theorem 4.2.4 in [3]). Moreover, the eigenvalues of such a couple of self-adjoint extensions interlace (see [12] and references therein, or this can also be deduced from general properties of self-adjoint extensions with deficiency indices $(1, 1)$ [18, § 8.3]). It is useful to note, too, that every $x \in \mathbb{R}$ is an eigenvalue of a unique self-adjoint extension $T(\kappa)$, $\kappa \in P^1(\mathbb{R})$ [13, Theorem 4.11].

For positive symmetric operators there exists another powerful theory of self-adjoint extensions due to Birman, Krein and Vishik based on the analysis of associated quadratic forms. A clear exposition of the theory can be found in [2]. Its application to our case, with deficiency indices $(1, 1)$, is as follows. A crucial role is played by the null space of $T_{\max} = T_{\min}^*$ which we denote by

$$\mathscr{N} := \operatorname{Ker} T_{\max} = \mathbb{C}Q^{(1)}$$

(recall that $Q_n^{(1)} = \hat{P}_n(0)$, $\forall n \in \mathbb{Z}_+$). Let $\mathfrak{t}_\infty = \mathfrak{t}$ be the quadratic form associated with the Friedrichs extension T^{F}. Remember that the domain of $\mathfrak{t}$ has been specified in (13). All other self-adjoint extensions of $T_{\min}$, except of T^{F}, are in one-to-one correspondence with real numbers τ. The corresponding associated quadratic forms $\mathfrak{t}_\tau$, $\tau \in \mathbb{R}$, have all the same domain,

$$\operatorname{Dom} \mathfrak{t}_\tau = \operatorname{Dom} \mathfrak{t}_\infty \dotplus \mathscr{N} \tag{25}$$

(a direct sum), and for $f \in \operatorname{Dom} \mathfrak{t}_\infty$, $\lambda \in \mathbb{C}$, one has

$$\mathfrak{t}_\tau(f + \lambda Q^{(1)}) = \mathfrak{t}_\infty(f) + \tau|\lambda|^2. \tag{26}$$

Our next task is to relate the self-adjoint extensions $T(\kappa)$ described in Proposition 3 to the quadratic forms $\mathfrak{t}_\tau$.

Proposition 7. *The quadratic form associated with a self-adjoint extension $T(\kappa)$, $\kappa \in \mathbb{R}$, is $\mathfrak{t}_\tau$ defined in (25), (26), with $\tau = (\kappa + q^\nu)/(1 - q^\nu)$.*

Proof. Let $\kappa \in \mathbb{R}$ and σ be the real parameter such that $\mathfrak{t}_\sigma$ is the quadratic form associated with $T(\kappa)$. Recall (16). One has $\mathcal{T}Q^{(1)} = 0$ and $(\mathcal{T}Q^{(2)})_n = \delta_{n,0}$, $\forall n \in \mathbb{Z}_+$. According to (17),

$$Q^{(2)} \in \operatorname{Dom} T(\infty) \subset \operatorname{Dom} \mathfrak{t}_\infty.$$

One computes $\mathfrak{t}_\infty(Q^{(2)}) = \langle Q^{(2)}, \mathcal{T}Q^{(2)} \rangle = 1$. Let $\tau = (\kappa + q^\nu)/(1 - q^\nu)$ and

$$h = \tau Q^{(2)} + Q^{(1)} \in \operatorname{Dom} \mathfrak{t}_\infty + \mathbb{C}Q^{(1)} = \operatorname{Dom} \mathfrak{t}_\sigma.$$

Then $(1 - q^\nu)h_n = q^{(1-\nu)n/2} + \kappa q^{(1+\nu)n/2}$, $\forall n \in \mathbb{Z}_+$. Hence, in virtue of (21), $h \in \operatorname{Dom} T(\kappa)$, and, referring to (26),

$$\tau^2 + \sigma = \mathfrak{t}_\sigma(h) = \langle h, T(\kappa)h \rangle = \langle h, \mathcal{T}h \rangle = \tau(\tau + 1).$$

Whence $\sigma = \tau$. □

Now we are ready to describe the announced additional spectral properties of $T(\kappa)$. The terminology and basic results concerning quadratic (sesquilinear) forms used below are taken from Kato [8].

Lemma 8. *Let $\mathscr{S}$ and $\mathscr{B}$ be linear subspaces in a Hilbert space $\mathscr{H}$ such that $\mathscr{S}\cap\mathscr{B} = \{0\}$, and let $\mathfrak{s}$ and $\mathfrak{b}$ be positive quadratic forms on $\mathscr{S}$ and $\mathscr{B}$, respectively. Denote by $\tilde{\mathfrak{s}}$ and $\tilde{\mathfrak{b}}$ the extensions of these forms to $\mathscr{S}+\mathscr{B}$ defined by*

$$\forall\varphi\in\mathscr{S},\,\forall\eta\in\mathscr{B},\;\; \tilde{s}(\varphi+\eta)=\mathfrak{s}(\varphi)\ \text{and}\ \tilde{\mathfrak{b}}(\varphi+\eta)=\mathfrak{b}(\eta),$$

and assume that, for every $\rho\in\mathbb{R}$, the form $\tilde{\mathfrak{s}}+\rho\tilde{\mathfrak{b}}$ is semibounded and closed. Then, for any $\tau\in\mathbb{C}$, the form $\tilde{\mathfrak{s}}+\tau\tilde{\mathfrak{b}}$ is sectorial and closed. In particular, if $\mathscr{S}+\mathscr{B}$ is dense in $\mathscr{H}$ then $\tilde{\mathfrak{s}}+\tau\tilde{\mathfrak{b}}$, $\tau\in\mathbb{C}$, is a holomorphic family of forms of type (a) in the sense of Kato.

Proof. Fix $\tau\in\mathbb{C}$, $\theta\in(\pi/4,\pi/2)$, and choose $\gamma_1,\gamma_2,\gamma_3\in\mathbb{R}$ so that

$$\tilde{\mathfrak{s}}+\operatorname{Re}(\tau)\tilde{\mathfrak{b}}\geq\gamma_1,\;\;(\tan(\theta)-1)\tilde{\mathfrak{s}}+\tan(\theta)\operatorname{Re}(\tau)\tilde{\mathfrak{b}}\geq\gamma_2,\;\;\tilde{\mathfrak{s}}-|\operatorname{Im}(\tau)|\tilde{\mathfrak{b}}\geq\gamma_3.$$

Let $\gamma=\min\{\gamma_1,\cot(\theta)(\gamma_2+\gamma_3)\}$. Then $\operatorname{Re}(\tilde{\mathfrak{s}}+\tau\tilde{\mathfrak{b}})\geq\gamma$ and, for any couple $\varphi\in\mathscr{S}$, $\eta\in\mathscr{B}$,

$$\begin{aligned}|\operatorname{Im}(\tilde{\mathfrak{s}}+\tau\tilde{\mathfrak{b}})(\varphi+\eta)|=|\operatorname{Im}\tau|\mathfrak{b}(\eta)\;&\leq\;\mathfrak{s}(\varphi)-\gamma_3\|\varphi+\eta\|^2\\ &\leq\;\tan(\theta)\big(\mathfrak{s}(\varphi)+\operatorname{Re}(\tau)\mathfrak{b}(\eta)\big)-(\gamma_2+\gamma_3)\|\varphi+\eta\|^2\\ &\leq\;\tan(\theta)\big(\mathfrak{s}(\varphi)+\operatorname{Re}(\tau)\mathfrak{b}(\eta)-\gamma\|\varphi+\eta\|^2\big).\end{aligned}$$

This estimates show that $\tilde{\mathfrak{s}}+\tau\tilde{\mathfrak{b}}$ is sectorial. Finally, a sectorial form is known to be closed if and only if its real part is closed. ☐

Proposition 9. *Let $\{\xi_n(\kappa);\,n\in\mathbb{N}\}$ be the eigenvalues of $T(\kappa)$, $\kappa\in P^1(\mathbb{R})$, ordered increasingly. Then for every $n\in\mathbb{N}$, $\xi_n(\kappa)$ is a real-analytic strictly increasing function on $\mathbb{R}$, and one has, $\forall\kappa\in\mathbb{R}$,*

$$\xi_1(\kappa)<\xi_1(\infty)<\xi_2(\kappa)<\xi_2(\infty)<\xi_3(\kappa)<\xi_3(\infty)<\dots. \tag{27}$$

Moreover,

$$\lim_{\kappa\to-\infty}\xi_1(\kappa)=-\infty,\;\;\lim_{\kappa\to-\infty}\xi_n(\kappa)=\xi_{n-1}(\infty)\ \text{for}\ n\geq2,\;\;\lim_{\kappa\to+\infty}\xi_n(\kappa)=\xi_n(\infty)\ \text{for}\ n\geq1. \tag{28}$$

Proof. The Friedrichs extension of a positive operator is maximal in the form sense among all self-adjoint extensions of that operator [2]. Particularly,

$$\xi_1(\kappa)=\min(\operatorname{spec}T(\kappa))\leq\xi_1(\infty)=\min(\operatorname{spec}T(\infty)).$$

But as already remarked above, the eigenvalues of $T(\kappa)$ and $T(\infty)$ interlace and so we have (27).

Referring to (25), (26), the property $\kappa_1, \kappa_2 \in \mathbb{R}$, $\kappa_1 < \kappa_2$ clearly implies $\mathfrak{t}_{\tau(\kappa_1)} < \mathfrak{t}_{\tau(\kappa_2)}$ where $\tau(\kappa) = (\kappa + q^\nu)/(1 - q^\nu)$. In virtue of Proposition 7 and the min-max principle, $\xi_n(\kappa_1) \leq \xi_n(\kappa_2)$, $\forall n \in \mathbb{N}$. But the spectra of $T(\kappa_1)$ and $T(\kappa_2)$ are disjoint and so the functions $\xi_n(\kappa)$ are strictly increasing on $\mathbb{R}$.

One can admit complex values for the parameter τ in (25), (26). Then, according to Lemma 8, the family of forms $\mathfrak{t}_\tau$, $\tau \in \mathbb{C}$, is of type (a) in the sense of Kato. Referring once more to Proposition 7 one infers from [8, Theorem VII-4.2] that the family of self-adjoint operators $T(\kappa)$, $\kappa \in \mathbb{R}$, extends to a holomorphic family of operators on $\mathbb{C}$. This implies that for any bounded interval $K \subset \mathbb{R}$ there exists an open neighborhood D of K in $\mathbb{C}$ and $\rho \in \mathbb{R}$ sufficiently large so that the resolvents $(T(\kappa) + \rho)^{-1}$, $\kappa \in K$, extend to a holomorphic family of bounded operators on D. In addition we know that, for every fixed $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$, the nth eigenvalue of $T(\kappa)$ is simple and isolated. By the analytic perturbation theory [8, § VII.3], $\xi_n(\kappa)$ is an analytic function on $\mathbb{R}$.

Finally we note that every $x \in \mathbb{R}$ is an eigenvalue of $T(\kappa)$ for some (in fact, unambiguous) $\kappa \in \mathbb{R}$ and so the range $\xi_n(\mathbb{R})$ must exhaust the entire interval either $(-\infty, \xi_1(\infty))$, if $n = 1$, or $(\xi_{n-1}(\infty), \xi_n(\infty))$, if $n > 1$. This clearly means that (28) must hold. ☐

Remark 10. As noted in [2, Theorem 2.15], $\xi_1(\kappa)$ is a concave function.

3 The characteristic function

3.1 A construction of the characteristic function for $\nu > 0$

Recall (9), (10). Observe that the sequence $\{\hat{P}_n(x)\}$ obeys the relation

$$\hat{P}_n(x) = Q_n^{(1)} - xq^{(1-\nu)/2} \sum_{k=0}^{n} Q_{n-k-1}^{(1)} q^k \hat{P}_k(x) \quad \text{for } n \geq -1. \tag{29}$$

This relation already implies that $\hat{P}_{-1}(x) = 0$, $\hat{P}_0(x) = 1$. Notice also that the last term in the sum, with $k = n$, is zero and so (29) is in fact a recurrence for $\{\hat{P}_n(x)\}$. Equation (29) is pretty standard. Nevertheless, one may readily verify it by checking that this recurrence implies the original defining recurrence, i.e. the formal eigenvalue equation (11) which can be rewritten as follows

$$q^{(\nu-1)/2}\hat{P}_{n+1}(x) - (1 + q^\nu)\hat{P}_n(x) + q^{(\nu+1)/2}\hat{P}_{n-1}(x) = -xq^n\hat{P}_n(x), \ \forall n \geq 0. \tag{30}$$

Actually, from (29) one derives that

$$\begin{aligned} q^{(\nu-1)/2}\hat{P}_{n+1}(x) - \hat{P}_n(x) &= q^{(\nu-1)/2}Q_{n+1}^{(1)} - Q_n^{(1)} \\ &\quad -x\sum_{k=0}^{n}\left(Q_{n-k}^{(1)} - q^{(1-\nu)/2}Q_{n-k-1}^{(1)}\right)q^k\hat{P}_k(x) \\ &= q^{\nu+(1+\nu)n/2} - xq^{(1+\nu)n/2}\sum_{k=0}^{n} q^{(1-\nu)k/2}\hat{P}_k(x) \end{aligned}$$

and so

$$q^{(\nu-1)/2}\hat{P}_{n+1}(x)-\hat{P}_n(x)-q^{(\nu+1)/2}\big(q^{(\nu-1)/2}\hat{P}_n(x)-\hat{P}_{n-1}(x)\big)=-xq^n\hat{P}_n(x),$$

as claimed.

Proposition 11. *The sequence of polynomials* $\{q^{(\nu-1)n/2}\hat{P}_n(x)\}$ *converges locally uniformly on* $\mathbb{C}$ *to an entire function* $\Phi(x)\equiv\Phi^{(\nu)}(x;q)$. *Moreover,* $\Phi(x)$ *fulfills*

$$\Phi(x)=\frac{1}{1-q^\nu}\left(1-x\sum_{k=0}^{\infty}q^{(1+\nu)k/2}\hat{P}_k(x)\right) \tag{31}$$

and one has, $\forall n\in\mathbb{Z}$, $n\geq-1$,

$$\hat{P}_n(x)=(1-q^\nu)\Phi(x)Q_n^{(1)}+xQ_n^{(2)}\sum_{k=0}^{n}Q_k^{(1)}\hat{P}_k(x)+xQ_n^{(1)}\sum_{k=n+1}^{\infty}Q_k^{(2)}\hat{P}_k(x). \tag{32}$$

Proof. Denote (temporarily) $H_n(x)=q^{(\nu-1)n/2}\hat{P}_n(x)$, $n\in\mathbb{Z}_+$. Then (29) means that

$$(1-q^\nu)H_n(x)=1-q^{\nu(n+1)}-x\sum_{k=0}^{n-1}(1-q^{\nu(n-k)})q^kH_k(x),\ \ n\in\mathbb{Z}_+. \tag{33}$$

Proceeding by mathematical induction in n one can show that, $\forall n\in\mathbb{Z}_+$,

$$|H_n(x)|\leq\frac{(-a;q)_n}{1-q^\nu}\ \text{ where } a=\frac{|x|}{1-q^\nu}\ \text{ and } (-a;q)_n=\prod_{k=0}^{n-1}(1+q^ka) \tag{34}$$

is the q-Pochhammer symbol. This is obvious for $n=0$. For the the induction step it suffices to notice that (33) implies

$$|H_n(x)|\leq\frac{1}{1-q^\nu}+a\sum_{k=0}^{n-1}q^k|H_k(x)|.$$

Moreover,

$$1+a\sum_{k=0}^{n-1}q^k(-a;q)_k=(-a;q)_n.$$

From the estimate (34) one infers that $\{H_n(x)\}$ is locally uniformly bounded on $\mathbb{C}$. Consequently, from (33) it is seen that the RHS converges as $n\to\infty$ and so $H_n(x)\to\Phi(x)$ pointwise. This leads to identity (31). Furthermore, one can rewrite (29) as follows

$$\begin{aligned}\hat{P}_n(x) &= Q_n^{(1)}-x\sum_{k=0}^{n}\frac{q^{(1-\nu)n/2}q^{(1+\nu)k/2}-q^{(1+\nu)n/2}q^{(1-\nu)k/2}}{1-q^\nu}\hat{P}_k(x)\\ &= Q_n^{(1)}+xQ_n^{(2)}\sum_{k=0}^{n}Q_k^{(1)}\hat{P}_k(x)-xQ_n^{(1)}\sum_{k=0}^{n}Q_k^{(2)}\hat{P}_k(x)\\ &= \left(1-x\sum_{k=0}^{\infty}Q_k^{(2)}\hat{P}_k(x)\right)Q_n^{(1)}+xQ_n^{(2)}\sum_{k=0}^{n}Q_k^{(1)}\hat{P}_k(x)+xQ_n^{(1)}\sum_{k=n+1}^{\infty}Q_k^{(2)}\hat{P}_k(x)\end{aligned}$$

Taking into account (31) one arrives at (32).

Finally, from the locally uniform boundedness and Montel's theorem it follows that the convergence of $\{H_n(x)\}$ is even locally uniform and so $\Phi(x)$ is an entire function. □

It turns out that $\Phi(x)$ may be called the characteristic function of the Jacobi operator T, if $\nu \geq 1$, or the Friedrichs extension T^{F}, if $0 < \nu < 1$.

Lemma 12. *Assume that $\nu \geq 1$. Suppose further that $f \in \mathbb{C}^\infty$, $\{q^{-\sigma_0 n} f_n\}$ is bounded for some $\sigma_0 > -(\nu+1)/2$ and $f = x\tilde{G}f$ for some $x \in \mathbb{R}$ where*

$$(\tilde{G}f)_n := Q_n^{(2)} \sum_{k=0}^{n} Q_k^{(1)} f_k + Q_n^{(1)} \sum_{k=n+1}^{\infty} Q_k^{(2)} f_k,\ n \in \mathbb{Z}_+. \tag{35}$$

Then the sequence $\{q^{-\sigma n} f_n\}$ is bounded for every $\sigma < (\nu+1)/2$. In particular, $f \in \ell^2$.

Proof. Put

$$S = \{\sigma > -(\nu+1)/2;\ \{q^{-\sigma n} f_n\} \in \ell^\infty\},\ \sigma_* = \sup S.$$

Notice that, by the assumptions, $S \neq \emptyset$ and the definition of $\tilde{G}f$ makes good sense. We have to show that $\sigma_* \geq (1+\nu)/2$. Let us assume the contrary.

We claim that if $\sigma \in S$ and $\sigma < (\nu-1)/2$ then $\sigma + 1 \in S$. In particular, $\sigma_* \geq (\nu-1)/2$. In fact, write $f_n = q^{\sigma n} h_n$, $h \in \ell^\infty$. From (35) one derives the estimate

$$|(\tilde{G}f)_n| \leq \frac{\|h\|_\infty}{1-q^\nu} \left(q^{(\nu+1)n/2} \sum_{k=0}^{n-1} q^{(\sigma+(1-\nu)/2)k} + \frac{q^{(\sigma+1)n}}{1-q^{\sigma+(1+\nu)/2}} \right).$$

From here one deduces that there exists a constant $C \geq 0$ such that

$$\forall n,\ |f_n| = |x(\tilde{G}f)_n| \leq C q^{(\sigma+1)n},$$

as claimed.

Choose σ such that $\sigma_* < \sigma < (\nu+1)/2$. Then

$$-\frac{\nu+1}{2} \leq \frac{\nu-1}{2} - 1 \leq \sigma_* - 1 < \sigma - 1 < \frac{\nu-1}{2} \leq \sigma_*$$

and so $\sigma - 1 \in S$. But in that case $\sigma \in S$ as well, a contradiction. □

Proposition 13. *If $\nu \geq 1$, the spectrum of T coincides with the zero set of $\Phi(x)$. If $0 < \nu < 1$ then* $\operatorname{spec} T(\kappa)$*, $\kappa \in P^1(\mathbb{R})$, consists of the roots of the characteristic equation*

$$\kappa\Phi(x) + \Psi(x) = 0 \tag{36}$$

where

$$\Psi(x) \equiv \Psi^{(\nu)}(x;q) = \frac{1}{1-q^\nu} \left(q^\nu - x \sum_{k=0}^{\infty} q^{(1-\nu)k/2} \hat{P}_k(x) \right). \tag{37}$$

In particular, the spectrum of $T^F = T(\infty)$ equals the zero set of $\Phi(x)$.

Proof. From Proposition 6 we already know that the spectrum of T (or $T(\kappa)$) is pure point and with no finite accumulation points. Assume first that $\nu \geq 1$. According to Proposition 3, we are dealing with the determinate case and so x is an eigenvalue of T if and only if the formal eigenvector $\hat{P}(x) = \{\hat{P}_n(x)\}$ is square summable. If $\hat{P}(x) \in \ell^2$ then $q^{(\nu-1)n/2}\hat{P}_n(x) \to 0$ as $n \to \infty$ and so $\Phi(x) = 0$ (see Proposition 11). Conversely, if $\Phi(x) = 0$ then (32) tells us that $\hat{P} = x\tilde{G}\hat{P}$, cf. (35). By Lemma 12, $\hat{P}(x) \in \ell^2$.

Assume now that $0 < \nu < 1$. This the indeterminate case meaning that $\hat{P}(x)$ is square summable for all $x \in \mathbb{C}$. Hence x is an eigenvalue of $T(\kappa)$ iff $\hat{P}(x) \in \operatorname{Dom} T(\kappa)$. Recall that $T(\kappa)$ is defined in Proposition 3. From (32) one derives the asymptotic expansion

$$\hat{P}_n(x) = \Phi(x)(q^{(1-\nu)n/2} - q^{\nu+(1+\nu)n/2}) + xq^{(1+\nu)n/2}\sum_{k=0}^{\infty} Q_k^{(1)}\hat{P}_k(x) + o(q^n) \ \text{ as } n \to \infty.$$

From here it is seen that $\hat{P}(x)$ fulfills the boundary condition in (21) if and only if x solves the equation

$$(\kappa + q^\nu)\Phi(x) - x\langle Q^{(1)}, \hat{P}(x)\rangle = 0.$$

Referring to (16) one finds that $x\langle Q^{(1)}, \hat{P}(x)\rangle = q^\nu\Phi(x) - \Psi(x)$. □

Proposition 14. *For $\nu > 0$ one has*

$$\Phi(x) = \frac{1}{1-q^\nu}\ {}_1\phi_1(0; q^{\nu+1}; q, x) = \frac{(q;q)_\infty}{(q^\nu;q)_\infty}\, q^{\nu/2}x^{-\nu/2}J_\nu(q^{-1/2}\sqrt{x}; q), \tag{38}$$

and for $0 < \nu < 1$,

$$\Psi(x) = \frac{q^\nu}{1-q^\nu}\ {}_1\phi_1(0; q^{1-\nu}; q, q^{-\nu}x) = -\frac{(q;q)_\infty}{(q^{-\nu};q)_\infty}\, q^{-\nu(\nu+1)/2}x^{\nu/2}J_{-\nu}(q^{-(\nu+1)/2}\sqrt{x}; q). \tag{39}$$

If $\Phi(x) = 0$ and so x is an eigenvalue of T, provided $\nu > 0$, or T^F, provided $0 < \nu < 1$, then $x > 0$ and the components of a corresponding eigenvector can be chosen as

$$u_k(x) = q^{k/2}\, J_\nu(q^{k/2}\sqrt{x}; q) = C\, q^{(1+\nu)k/2}\ {}_1\phi_1(0; q^{\nu+1}; q, q^{k+1}x),\ \ k \in \mathbb{Z}_+, \tag{40}$$

where $C = x^{\nu/2}\,(q^{1+\nu};q)_\infty/(q;q)_\infty$.

If $0 < \nu < 1$, $\kappa \in \mathbb{R}$ and $\kappa\Phi(x) + \Psi(x) = 0$ and so x is an eigenvalue of $T(\kappa)$ then the components of a corresponding eigenvector can be chosen as

$$\begin{aligned} u_k(\kappa, x) &= q^{k/2}\left(\kappa J_\nu(q^{k/2}\sqrt{x}; q) - \frac{(q^\nu;q)_\infty}{(q^{-\nu};q)_\infty}\, q^{-\nu(\nu+2)/2}x^\nu J_{-\nu}(q^{(k-\nu)/2}\sqrt{x}; q)\right) \\ &= C\left(\kappa q^{(1+\nu)k/2}\ {}_1\phi_1(0; q^{\nu+1}; q, q^{k+1}x) + q^{(1-\nu)k/2}\ {}_1\phi_1(0; q^{1-\nu}; q, q^{k+1-\nu}x)\right), \end{aligned}$$

with $k \in \mathbb{Z}_+$ (C is the same as above).

Lemma 15. *For every $m \in \mathbb{Z}_+$ and $\sigma > 0$,*

$$\sum_{k=0}^{\infty} q^{(\sigma+(\nu-1)/2)k} \, \frac{d^m \hat{P}_k(0)}{dx^m} = \frac{(-1)^m \, m! \, q^{m\sigma+m(m-1)/2}}{(q^\sigma; q)_{m+1} \, (q^{\sigma+\nu}; q)_{m+1}} \,. \tag{41}$$

Proof. For a given $m \in \mathbb{N}$, one derives from (30) the three-term inhomogeneous recurrence relation

$$q^{(\nu-1)/2} \, \frac{d^m \hat{P}_{n+1}(0)}{dx^m} - (1+q^\nu) \, \frac{d^m \hat{P}_n(0)}{dx^m} + q^{(\nu+1)/2} \, \frac{d^m \hat{P}_{n-1}(0)}{dx^m} = -mq^n \, \frac{d^{m-1} \hat{P}_n(0)}{dx^{m-1}}, \; n \geq 0, \tag{42}$$

with the initial conditions

$$\frac{d^m \hat{P}_{-1}(0)}{dx^m} = 0, \;\; \frac{d^m \hat{P}_0(0)}{dx^m} = \delta_{m,0} \;\; \text{for all } m \geq 0. \tag{43}$$

Recall that, by Proposition 11, the sequence $\{q^{(\nu-1)n/2} \hat{P}_n(x)\}$ converges on $\mathbb{C}$ locally uniformly and hence it is locally uniformly bounded. Combining this observation with Cauchy's integral formula one justifies that, for any $m \in \mathbb{Z}_+$ fixed, the sequence $\{q^{(\nu-1)n/2} d^m \hat{P}_n(0)/dx^m\}$ is bounded as well. Therefore the LHS of (41) is well defined. Let us call it $S_{m,\sigma}$. Using summation in (42) and bearing in mind (43) one derives the recurrence

$$S_{m,\sigma} = -\frac{mq^\sigma}{(1-q^\sigma)\,(1-q^{\sigma+\nu})} \, S_{m-1,\sigma+1} \;\; \text{for } m \geq 1, \sigma > 0.$$

Particularly, for $m = 0$ we know that $\hat{P}_n(0) = Q_n^{(1)}$, $n \in \mathbb{Z}_+$. Whence

$$S_{0,\sigma} = \frac{1}{(1-q^\sigma)\,(1-q^{\sigma+\nu})} \,,$$

cf. (16). A routine application of mathematical induction in m proves (41). $\square$

Proof of Proposition 14. Letting $\sigma = 1$ in (41) and making use of the locally uniform convergence (cf. Proposition 11) one has

$$\frac{1}{m!} \frac{d^m}{dx^m} \sum_{k=0}^{\infty} q^{(\nu+1)k/2} \hat{P}_k(x) \Big|_{x=0} = \frac{(-1)^m q^{m(m+1)/2}}{(q;q)_{m+1} \, (q^{\nu+1}; q)_{m+1}} \,, \;\; \forall m \in \mathbb{Z}_+.$$

Now, since $\Phi(x)$ is analytic it suffices to refer to formula (31) to obtain

$$\Phi(x) = \frac{1}{1-q^\nu} \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n-1)n/2} \, x^n}{(q;q)_n \, (q^{\nu+1}; q)_n} = \frac{1}{1-q^\nu} \; {}_1\phi_1(0; q^{\nu+1}; q, x).$$

Letting $\sigma = 1 - \nu$ in (41), a fully analogous computation can be carried out to evaluate the RHS of (37) thus getting formula (39) for $\Psi(x)$.

From (2) it is seen that the sequences $\{u_k(x);\, k \in \mathbb{Z}\}$ and $\{v_k(x);\, k \in \mathbb{Z}\}$, where

$$u_k(x) = q^{k/2} \, J_\nu(q^{k/2} \sqrt{x}; q) \text{ and } v_k(x) = q^{k/2} \, J_{-\nu}(q^{(k-\nu)/2} \sqrt{x}; q),$$

obey both the difference equation

$$\alpha_k u_{k+1} + \beta_k u_k + \alpha_{k-1} u_{k-1} = x u_k \tag{44}$$

(with α_k, β_k being defined in (4)). In the case of the former sequence, ν can be arbitrary positive, in the case of the latter one we assume that $0 < \nu < 1$. Hence the sequence $(u_0(x), u_1(x), u_2(x), \ldots)$ is a formal eigenvector of the Jacobi matrix $\mathcal{T}$ if and only if $u_{-1}(x) = 0$. A similar observation holds true if we replace $u_k(x)$ by $u_k(\kappa, x)$. In view of Proposition 14, it suffices to notice that $u_{-1}(x)$ is proportional to $\Phi(x)$ and $u_{-1}(\kappa, x)$ to $\kappa\Phi(x) + \Psi(x)$. □

3.2 The case $\nu = 0$

The case $\nu = 0$ is very much the same thing as the case when $0 < \nu < 1$. First of all, this is again an indeterminate case, i.e. $T_{\min}$ is not self-adjoint. On the other hand, there are some differences causing the necessity to modify several formulas, some of them rather substantially. Perhaps the main reason for this is the fact that the characteristic polynomial of the difference equation with constant coefficients, (15), has one double root if $\nu = 0$ while it has two different roots if $0 < \nu$. Here we summarize the basic modifications but without going into details since the arguing remains quite analogous.

For $\nu = 0$ one has $\operatorname{Dom} \mathfrak{t} = \{f \in \ell^2;\ \mathcal{A}f \in \ell^2\}$, and two distinguished solutions of (14) are

$$Q_n^{(1)} = (n+1)q^{n/2},\ Q_n^{(2)} = q^{n/2},\ n \in \mathbb{Z},$$

where again $Q_n^{(1)} = \hat{P}_n(0)$ for $n \geq 0$ and $\{Q_n^{(2)}\}$ is a minimal solution, $W_n(Q^{(1)}, Q^{(2)}) = 1$. The asymptotic expansion of a sequence $f \in \operatorname{Dom} T_{\max}$ reads

$$f_n = \big(C_1\,(n+1) + C_2\big)q^{n/2} + o(q^n) \ \text{ as } n \to \infty,$$

with $C_1, C_2 \in \mathbb{C}$. The one-parameter family of self-adjoint extensions of $T_{\min}$ is again denoted $T(\kappa)$, $\kappa \in P^1(\mathbb{R})$. Definition (21) of $\operatorname{Dom} T(\kappa)$ formally remains the same but the constants $C_1(f)$, $C_2(f)$ in the definition are now determined by the limits

$$C_1(f) = \lim_{n\to\infty} f_n\,(n+1)^{-1}q^{-n/2}\,,\ C_2(f) = \lim_{n\to\infty} \big(f_n - C_1(f)\,(n+1)q^{n/2}\big)q^{-n/2}\,.$$

One still has $T(\infty) = T^{\mathrm{F}}$. Similarly, $f \in \operatorname{Dom} T_{\max}$ belongs to $\operatorname{Dom} T_{\min}$ if and only if $C_1(f) = C_2(f) = 0$ meaning that (22) is true for $\nu = 0$, too. Furthermore, everything what is claimed in Propositions 5 and 6 about the values $0 < \nu < 1$ is true for $\nu = 0$ as well.

Proposition 7 should be modified so that the quadratic form associated with a self-adjoint extension $T(\kappa)$, $\kappa \in \mathbb{R}$, is $\mathfrak{t}_{\kappa+1}$, i.e. for $\nu = 0$ one lets $\tau \equiv \tau(\kappa) = \kappa + 1$. On the other hand, Proposition 9 holds verbatim true also for $\nu = 0$.

Relation (29) is valid for $\nu = 0$ as well but more substantial modifications are needed in Proposition 11. One has

$$\frac{q^{-n/2}}{n+1}\,\hat{P}_n(x) \to \Phi(x) = 1 - x\sum_{k=0}^{\infty} q^{k/2}\hat{P}_k(x) \ \text{ as } n \to \infty,$$

and the convergence is locally uniform on $\mathbb{C}$ for one can estimate

$$\left|\frac{q^{-n/2}}{n+1}\,\hat{P}_n(x)\right| \le \prod_{k=0}^{n}\big(1+(k+1)q^k|x|\big),\ \ n\in\mathbb{Z}_+.$$

Equation (32) should be replaced by

$$\hat{P}_n(x) = \Phi(x)Q_n^{(1)} + xQ_n^{(2)}\sum_{k=0}^{n}Q_k^{(1)}\hat{P}_k(x) + xQ_n^{(1)}\sum_{k=n+1}^{\infty}Q_k^{(2)}\hat{P}_k(x).$$

From here one infers the asymptotic expansion

$$\hat{P}_n(x) = \Phi(x)(n+1)q^{n/2} + xq^{n/2}\sum_{k=0}^{\infty}Q_k^{(1)}\hat{P}_k(x) + o(q^n)\ \text{ as } n\to\infty.$$

One concludes that what is claimed in Proposition 13 about the values $0<\nu<1$ is true for $\nu=0$ as well but instead of (37) one should write

$$\Psi(x) = -x\sum_{k=0}^{\infty}(k+1)q^{k/2}\hat{P}_k(x).$$

Finally let us consider modifications needed in Proposition 14. For $\nu=0$ one has

$$\Phi(x) = {}_1\phi_1(0;q;q,x) = J_0(q^{-1/2}\sqrt{x};q)$$

and

$$\begin{aligned}\Psi(x) &= \frac{\partial}{\partial p}\,{}_2\phi_2(0,q;pq,pq;q,px)\bigg|_{p=1}\\ &= 2q\,\frac{\partial}{\partial p}\,{}_1\phi_1(0;p;q,x)\bigg|_{p=q} + x\,\frac{\partial}{\partial x}\,{}_1\phi_1(0;q;q,x).\end{aligned}$$

Let

$$u_k(x) = q^{(k+1)/2}J_0(q^{-k/2}\sqrt{x};q)$$

and

$$\begin{aligned}v_k(x) &= (k+1)q^{(k+1)/2}\,{}_1\phi_1(0;q;q,q^{k+1}x) + 2q^{(k+3)/2}\,\frac{\partial}{\partial p}\,{}_1\phi_1(0;p;q,q^{k+1}x)\bigg|_{p=q}\\ &\quad + q^{(k+1)/2}x\,\frac{\partial}{\partial x}\,{}_1\phi_1(0;q;q,q^{k+1}x),\end{aligned}$$

$k\in\mathbb{Z}$. Then both sequences $\{u_k(x)\}$ and $\{v_k(x)\}$ solve (44) on $\mathbb{Z}$ and $u_{-1}(x)=\Phi(x)$, $v_{-1}(x)=\Psi(x)$. Consequently, if $\Phi(x)=0$ then components of an eigenvector of $T(\infty)=T^{\mathrm{F}}$ corresponding to the eigenvalue x can be chosen to be $u_k(x)$, $k\in\mathbb{Z}_+$. Similarly, if $\kappa\Phi(x)+\Psi(x)=0$ for some $\kappa\in\mathbb{R}$ then components of an eigenvector of $T(\kappa)$ corresponding to the eigenvalue x can be chosen to be $\kappa u_k(x)+v_k(x)$, $k\in\mathbb{Z}_+$.

4 Some applications to the q-Bessel functions

In this section we are going to only consider the Friedrichs extension if $0<\nu<1$. To simplify the formulations below we will unify the notation and use the same symbol T^{F} for the corresponding self-adjoint Jacobi operator for all values of $\nu>0$, this is to say even in the case when $\nu\geq 1$. Making use of the close relationship between the spectral data for T^{F} and the q-Bessel functions, as asserted in Propositions 13 and 14, we are able to reproduce in an alternative way some results from [11, 9].

Proposition 16 (Koelink, Swarttouw)**.** *Assume that $\nu>0$. The zeros of $z\mapsto J_\nu(z;q)$ are all real (arranged symmetrically with respect to the origin), simple and form an infinite countable set with no finite accumulation points. Let $0<w_1<w_2<w_3<\ldots$ be the positive zeros of $J_\nu(z;q)$. Then the sequences*

$$u(n)=\big(J_\nu(q^{1/2}w_n;q),q^{1/2}J_\nu(qw_n;q),qJ_\nu(q^{3/2}w_n;q),\ldots\big),\ n\in\mathbb{N},\tag{45}$$

form an orthogonal basis in ℓ^2. In particular, the orthogonality relation

$$\sum_{k=0}^{\infty}q^kJ_\nu(q^{(k+1)/2}w_m;q)\,J_\nu(q^{(k+1)/2}w_n;q)=-\frac{q^{-1+\nu/2}}{2w_n}\,J_\nu(q^{1/2}w_n;q)\,\frac{\partial J_\nu(w_n;q)}{\partial z}\,\delta_{m,n}\tag{46}$$

holds for all $m,n\in\mathbb{N}$.

Remark. It is not difficult to show that the proposition remains valid also for $-1<\nu\leq 0$. To this end, one can extend the values $\nu>0$ to $\nu=0$ following the lines sketched in Subsection 3.2 and employ Propositions 13 and 14 while letting $\kappa=0$ in order to treat the values $-1<\nu<0$. But we omit the details. An original proof of this proposition can be found in [11, Section 3].

Proof. All claims, except the simplicity of zeros and the normalization of eigenvectors, follow from the known spectral properties of T^{F}. Namely, T^{F} is positive definite, $(T^{\mathrm{F}})^{-1}$ is compact, $\operatorname{spec}T^{\mathrm{F}}=\{qw_n^2;\ n\in\mathbb{N}\}$ and corresponding eigenvectors are given by formula (40); cf. Propositions 5, 6, 13 and 14.

The remaining properties can be derived, in an entirely standard way, with the aid of discrete Green's formula. Suppose a sequence of differentiable functions $u_n(x)$, $n\in\mathbb{Z}$, obeys the difference equation (44). Then Green's formula implies that, for all $m,n\in\mathbb{Z}$, $m\leq n$,

$$\sum_{k=m}^{n}u_k(x)^2=\alpha_{m-1}\big(u'_{m-1}(x)u_m(x)-u_{m-1}(x)u'_m(x)\big)-\alpha_n\big(u'_n(x)u_{n+1}(x)-u_n(x)u'_{n+1}(x)\big)$$

(with the dash standing for a derivative). We choose $m=0$ and $u_k(x)$ as defined in (40). From definition (1) one immediately infers the asymptotic behavior

$$u_k(x)=C(x)(1+O(q^k))\,q^{(\nu+1)k/2},\ u'_k(x)=C'(x)(1+O(q^k))\,q^{(\nu+1)k/2},\ \text{as }k\to\infty,\tag{47}$$

where $C(x) = x^{\nu/2}\,(q^{1+\nu};q)_\infty/(q;q)_\infty$. It follows that one can send $n\to\infty$ in Green's formula. For $x = qw_n^2$ we have $u_{-1}(x) = 0$ and the formula reduces to the equality

$$\sum_{k=0}^{\infty} q^k J_\nu(q^{(k+1)/2}w_n;q)^2 = -q^{\nu/2}\,J_\nu(q^{1/2}w_n;q)\,\frac{\partial J_\nu(q^{-1/2}\sqrt{x};q)}{\partial x}\bigg|_{x=qw_n^2}.$$

Whence (46). From the asymptotic behavior (47) it is also obvious that $u_k(x) \neq 0$ for sufficiently large k. Necessarily, $\partial J_\nu(w_n;q)/\partial z \neq 0$. □

In addition, one obtains at once an orthogonality relation for the sequence of orthogonal polynomials $\{\hat{P}_n(x)\}$. As is well known from the general theory [3] and Proposition 3, the orthogonality relation is unique if $\nu \geq 1$ and indeterminate if $0 < \nu < 1$. It was originally derived in [9, Theorem 3.6].

Proposition 17 (Koelink)**.** *Assume that* $\nu > 0$ *and let* $\{\hat{P}_n(x)\}$ *be the sequence of orthogonal polynomials defined in (9), (10), and* $0 < w_1 < w_2 < w_3 < \ldots$ *be the positive zeros of* $z \mapsto J_\nu(z;q)$*. Then the orthogonality relation*

$$-2q^{1-\nu/2}\sum_{k=1}^{\infty}\frac{w_k J_\nu(q^{1/2}w_k;q)}{\partial J_\nu(w_k;q)/\partial z}\,\hat{P}_m(qw_k^2)\hat{P}_n(qw_k^2) = \delta_{m,n} \tag{48}$$

holds for all $m,n \in \mathbb{Z}_+$*.*

Proof. Let $u(k)$, $k\in\mathbb{N}$, be the orthogonal basis in ℓ^2 introduced in (45), i.e. we put

$$u(k)_n = q^{n/2}J_\nu(q^{(n+1)/2}w_k;q),\ \ k\in\mathbb{N},\ n\in\mathbb{Z}_+.$$

Notice that the norm $\|u(k)\|$ is known from (46). The vectors $u(k)$ and $\hat{P}(x) = (\hat{P}_0(x),\hat{P}_1(x),\hat{P}_2(x),\ldots)$, with $x = qw_k^2$, are both eigenvectors of T^{F} corresponding to the same eigenvalue. Hence these vectors are linearly dependent and one has

$$q^{n/2}J_\nu(q^{(n+1)/2}w_k;q) = J_\nu(q^{1/2}w_k;q)\hat{P}_n(qw_k^2),\ \ k\in\mathbb{N},\ n\in\mathbb{Z}_+.$$

One concludes that Parseval's equality

$$\sum_{k=1}^{\infty}\frac{u(k)_m u(k)_n}{\|u(k)\|^2} = \delta_{m,n},\ \ m,n\in\mathbb{Z}_+,$$

yields (48). □

Remark 18. To complete the picture let us mention two more results which are known about the Hahn-Exton q-Bessel functions and the associated polynomials. First, denote again by $w_n^{(\nu)} \equiv w_n$, $n\in\mathbb{N}$, the increasingly ordered positive zeros of $J_\nu(z;q)$. In [1] it is proved that if q is sufficiently small, more precisely, if $q^{\nu+1} < (1-q)^2$ then

$$q^{-m/2} > w_m > q^{-m/2}\left(1 - \frac{q^{m+\nu}}{1-q^m}\right),\ \ \forall m\in\mathbb{N}.$$

More generally, in Theorem 2.2 and Remark 2.3 in [4] it is shown that for any q, $0 < q < 1$, one has

$$w_m = q^{-m/2} (1 + O(q^m)) \text{ as } m \to \infty.$$

Second, in [11, 9] one can find an explicit expression for the sequence of orthogonal polynomials $\{\hat{P}_n(x)\}$, namely

$$\hat{P}_n(x) = q^{n/2} \sum_{j=0}^{n} \frac{q^{n(j-\nu/2)}(q^{-n}; q)_j}{(q; q)_j} \, {}_2\phi_1(q^{j-n}, q^{j+1}; q^{-n}; q, q^{-j+\nu}) \, x^j, \; n \in \mathbb{Z}_+.$$

Let us remark that a relative formula in terms of the Al-Salam–Chihara polynomials has been derived in [17, Theorem 2].

Acknowledgments

The authors wish to acknowledge gratefully partial support from grant No. GA13-11058S of the Czech Science Foundation.

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The Nevanlinna parametrization for q-Lommel polynomials in the indeterminate case

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Abstract

The Hamburger moment problem for the q-Lommel polynomials which are related to the Hahn-Exton q-Bessel function is known to be indeterminate for a certain range of parameters. In this paper, the Nevanlinna parametrization for the indeterminate case is provided in an explicit form. This makes it possible to describe all respective N-extremal measures of orthogonality. Moreover, a linear and quadratic recurrence relation are derived for the moment sequence, and the asymptotic behavior of the moments for large powers is revealed with the aid of appropriate estimates.

Keywords: q-Lommel polynomials, Nevanlinna parametrization, measure of orthogonality, moment sequence

2010 Mathematical S1ubject Classification: 42C05, 33C47, 33D45

1 Introduction

The Lommel polynomials represent a class of orthogonal polynomials known from the theory of Bessel functions. Several q-analogues of the Lommel polynomials have been introduced and studied in [12, 11, 13]. One of the three commonly used q-analogues of the Bessel function of the first kind is known as the Hahn-Exton q-Bessel function (sometimes also called the third Jackson q-Bessel function or ${}_1\phi_1$ q-Bessel function). It is defined by the equality

$$J_\nu(z;q) = z^\nu \, \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \, {}_1\phi_1(0;q^{\nu+1};q,qz^2). \tag{1}$$

It is of importance that $J_\nu(z;q)$ satisfies the recurrence relation

$$J_{\nu+1}(z;q) - \left(z + \frac{1-q^\nu}{z}\right) J_\nu(z;q) + J_{\nu-1}(z;q) = 0.$$

By iterating this rule one arrives at the formula

$$J_{\nu+n}(z;q) = h_{n,\nu}(z^{-1};q)J_\nu(z;q) - h_{n-1,\nu+1}(z^{-1};q)J_{\nu-1}(z;q) \tag{2}$$

where $h_{m,\nu}(w;q)$ are polynomials in q^ν and Laurent polynomials in w, see [11] for more details. This is a familiar situation, with equation (2) being analogous to the well known relation between the Lommel polynomials and the Bessel functions, cf. [20, Chapter 9]. Thus the polynomials $h_{m,\nu}(w;q)$ can be referred to as the q-Lommel polynomials.

On one hand, the polynomials $h_{n,\nu}(w;q)$ can be treated as orthogonal Laurent polynomials in the variable w. The corresponding orthogonality relation has been described in [11]. On the other hand, $h_{n,\nu}(w;q)$ are also orthogonal polynomials in the variable q^ν. In Theorem 3.6 and Corollary 3.7 in [13], Koelink described a corresponding measure of orthogonality. It turns out that the measure of orthogonality is supported on the zeros of the Hahn-Exton q-Bessel function considered as a function of the order ν . Moreover, the measure of orthogonality is unique if $w^{-2} \leq q$ or $w^{-2} \geq q^{-1}$. For $q < w^{-2} < q^{-1}$, however, the corresponding Hamburger moment problem is indeterminate and so there exist infinitely many measures of orthogonality. The measure described in [13] represents a Nevanlinna (or N-) extremal solution of the indeterminate Hamburger moment problem and it can be seen to correspond to the Friedrichs extension of the underlying Jacobi matrix operator.

Let us also remark that the q-Lommel polynomials admit another interpretation in the framework of a birth and death process with exponentially growing birth and death rates. More precisely, the birth rate is supposed to be $\lambda_n = w^{-2}q^{-n}$ while the death rate is $\mu_n = q^{-n}$ (or vice versa). See, for example, [9] for more information on the subject.

As already pointed out in [13], it is of interest and in fact a fundamental question to determine all possible measures of orthogonality in terms of the Nevanlinna parametrization. An explicit solution of this problem becomes the main goal of the current paper. To achieve it we heavily rely on the knowledge of the generating function for the q-Lommel polynomials. Having the Nevanlinna parametrization at hand it is straightforward to describe all N-extremal measures of orthogonality. The case when $w = 1$ turns out to be somewhat special and requires additional efforts though no new ideas are in principle needed. To our best knowledge, formulas for this particular case have been omitted in the past research works on the q-Lommel polynomials.

In addition, we pay some attention to the sequence of moments related to the q-Lommel polynomials. By Favard's theorem, the moments are unambiguously determined by the coefficients in the recurrence relation for the q-Lommel polynomials and otherwise they are independent of a particular choice of the measure of orthogonality in the indeterminate case. It does not seem that the moment sequence can be found explicitly. We provide at least a linear and quadratic recurrence relation for it and describe qualitatively its asymptotic behavior for large powers.

Let us note that throughout the whole paper the parameter q is assumed to satisfy $0 < q < 1$. Furthermore, as far as the basic (or q-) hypergeometric series are concerned, as well as other q-symbols and functions, we follow the notation of Gasper and Rahman [8].

2 The Nevanlinna functions for q-Lommel polynomials

2.1 The q-Lommel polynomials

In the current paper we prefer to work directly with the ${}_1\phi_1$ basic hypergeometric function and do not insist on its interpretation as the q-Bessel function in accordance with (1). This leads us to using a somewhat modified notation if compared to that usually used in connection with q-Bessel functions, for instance, in [13]. Moreover, the notation used in this paper may stress some similarity of the Hamburger moment problem for the q-Lommel polynomials with the same problem for the Al-Salam-Carlitz II polynomials. The Hamburger moment problem is actually known to be indeterminate for particular values of parameters in both cases but there are also some substantial differences, see [3, Section 4].

Thus we write $a > 0$ instead of w^{-2} and $x \in \mathbb{C}$ instead of q^ν. The basic recurrence relation we are going to study, defining a sequence of monic orthogonal polynomials $\{F_n(a, q; x)\}_{n=0}^\infty$ (in the variable x and depending on two parameters a and q), reads

$$u_{n+1} = \big(x - (a+1)q^{-n}\big)u_n - aq^{-2n+1}u_{n-1}, \quad n \in \mathbb{Z}_+ \tag{3}$$

($\mathbb{Z}_+$ standing for nonnegative integers). As usual, the initial conditions are imposed in the form $F_{-1}(a, q; x) = 0$ and $F_0(a, q; x) = 1$. In order to be able to compare some results derived below with the already known results on the q-Lommel polynomials let us remark that the q-Lommel polynomials $h_{n,\nu}(w; q)$ introduced in (2) are related to the monic polynomials $F_n(a, q; x)$ by the formula

$$h_{n,\nu}(w; q) = (-1)^n w^n q^{n(n-1)/2} F_n(w^{-2}, q; q^\nu).$$

From (3) one immediately deduces the symmetry property

$$a^n F_n(a^{-1}, q; x) = F_n(a, q; ax), \quad n \in \mathbb{Z}_+.$$

This suggests that one can restrict values of the parameter a to the interval $0 < a < 1$. We usually try, however, to formulate our results for both cases, $a < 1$ and $a > 1$, for the sake of completeness. The case $a = 1$ is somewhat special and should be treated separately.

Letting

$$G_n(a, q; x) = q^{1-n} F_{n-1}(a, q; qx), \quad n \in \mathbb{Z}_+, \tag{4}$$

we get a second linearly independent solution of (3), a sequence of monic polynomials $\{G_n(a,q;x)\}$ fulfilling the initial conditions $G_0(a,q;x)=0$ and $G_1(a,q;x)=1$. Normalizing the monic polynomials $F_n(a,q;x)$ we get an orthonormal polynomial sequence $\{P_n(a,q;x)\}_{n=0}^{\infty}$. Explicitly,

$$P_n(a,q;x)=a^{-n/2}q^{n^2/2}F_n(a,q;x),\quad n\in\mathbb{Z}_+. \tag{5}$$

The polynomials of the second kind, $Q_n(a,q;x)$, are related to the monic polynomials $G_n(a,q;x)$ by a similar equality,

$$Q_n(a,q;x)=a^{-n/2}q^{n^2/2}G_n(a,q;x),\quad n\in\mathbb{Z}_+, \tag{6}$$

and obey the initial conditions $Q_0(a,q;x)=0$, $Q_1(a,q;x)=\sqrt{q/a}$.

Note that polynomials $P_n(a,q;x)$ solve the second-order difference equation

$$\sqrt{a}q^{-n+1/2}v_{n-1}+\big((a+1)q^{-n}-x\big)v_n+\sqrt{a}q^{-n-1/2}v_{n+1}=0,\quad n\in\mathbb{Z}_+,$$

with the initial conditions $P_{-1}(a,q;x)=0$ and $P_0(a,q;x)=1$. Denote by α_n and β_n the coefficients in this difference equation,

$$\alpha_n=a^{1/2}q^{-n-1/2},\ \ \beta_n=(a+1)q^{-n},\quad n\in\mathbb{Z}_+. \tag{7}$$

The difference equation can be interpreted as the formal eigenvalue equation for the Jacobi matrix

$$J=J(a,q)=\begin{pmatrix}\beta_0 & \alpha_0 & & & \\ \alpha_0 & \beta_1 & \alpha_1 & & \\ & \alpha_1 & \beta_2 & \alpha_2 & \\ & & \ddots & \ddots & \ddots\end{pmatrix}. \tag{8}$$

Then $(P_0(x),P_1(x),P_2(x),\ldots)$ is a formal eigenvector (where $P_j(x)\equiv P_j(a,q;x)$). Let us emphasize that J is positive on the subspace in $\ell^2(\mathbb{Z}_+)$ formed by sequences with only finitely many nonzero entries, i.e. on the linear hull of the canonical basis in $\ell^2(\mathbb{Z}_+)$. Actually, it is not difficult to verify that for every $N\in\mathbb{Z}_+$ and $\xi\in\mathbb{R}^{N+1}$,

$$\sum_{n=0}^{N}\beta_n\xi_n^2+2\sum_{n=0}^{N-1}\alpha_n\xi_n\xi_{n+1}=a\xi_0^2+q^{-N}\xi_N^2+\sum_{n=0}^{N-1}q^{-n}\left(\left(\frac{a}{q}\right)^{1/2}\xi_{n+1}+\xi_n\right)^2\geq 0. \tag{9}$$

Recurrence (3) can be solved explicitly in the particular case when $x=0$. One finds that

$$F_n(a,q;0)=(-1)^nq^{-n(n-1)/2}\,\frac{1-a^{n+1}}{1-a},\ \ G_n(a,q;0)=(-1)^{n+1}q^{-n(n-1)/2}\,\frac{1-a^n}{1-a},$$

for $n\in\mathbb{Z}_+$ and $a\neq 1$. Consequently,

$$P_n(a,q;0)=(-1)^nq^{n/2}a^{-n/2}\,\frac{1-a^{n+1}}{1-a},\ \ Q_n(a,q;0)=(-1)^{n+1}q^{n/2}a^{-n/2}\,\frac{1-a^n}{1-a}. \tag{10}$$

The quantities $P_n(1,q;0)$ and $Q_n(1,q;0)$ can be obtained from (10) in the limit $a\to 1$,

$$P_n(1,q;0)=(-1)^nq^{n/2}(n+1),\ \ Q_n(1,q;0)=(-1)^{n+1}q^{n/2}n.$$

2.2 The generating function

A formula for the generating function for the q-Lommel polynomials has been derived in [12, Eq. (4.22)]. Here we reproduce the formula and provide its proof since it is quite crucial for the computations to follow of the Nevanlinna functions A, B, C, and D.

Proposition 1. *Let $a > 0$. The generating function for the polynomials $F_n(a, q; x)$ equals*

$$\sum_{n=0}^{\infty} q^{n(n-1)/2} F_n(a, q; x)(-t)^n = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}(-xt)^k}{(t; q)_{k+1}(at; q)_{k+1}} = \frac{{}_2\phi_2(q, 0; qt, qat; q, xt)}{(1-t)(1-at)} \tag{11}$$

where $|t| < \min(1, a^{-1})$.

Proof. The last equality in (11) is obvious from the definition of the basic hypergeometric function. Suppose a and x being fixed and put

$$V(t) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}(-xt)^k}{(t; q)_{k+1}(at; q)_{k+1}}\,.$$

$V(t)$ is a well defined analytic function for $|t| < \min(1, a^{-1})$ which is readily seen to satisfy the q-difference equation

$$(1-t)(1-at)V(t) = 1 - xtV(qt). \tag{12}$$

Writing the power series expansion of $V(t)$ at $t = 0$ in the form

$$V(t) = \sum_{n=0}^{\infty} u_n q^{n(n-1)/2}(-t)^n$$

and inserting the series into (12) one finds that the coefficients u_n obey the recurrence (3) and the initial conditions $u_0 = 1$, $u_1 = -1 - a + x$. Necessarily, $u_n = F_n(a, q; x)$ for all $n \in \mathbb{Z}_+$. □

In [12, Section 4] and particularly in [13, Eq. (2.6)] there is stated an explicit formula for the polynomials $F_n(a, q; x)$, namely

$$F_n(a, q; x) = (-1)^n q^{-n(n-1)/2} \sum_{j=0}^{n} \frac{q^{jn}(q^{-n}; q)_j}{(q; q)_j}\, {}_2\phi_1(q^{j-n}, q^{j+1}; q^{-n}; q, q^{-j}a)\, x^j.$$

Let us restate this formula as an immediate corollary of Proposition 1.

Corollary 2. *The polynomials $F_n(a, q; x)$, $n \in \mathbb{Z}_+$, can be expressed explicitly as follows*

$$F_n(a, q; x) = (-1)^n q^{-n(n-1)/2} \sum_{j=0}^{n} \frac{(-1)^j q^{j(j-1)/2}}{(q; q)_j^2} \left(\sum_{k=0}^{n-j} (q^{k+1}; q)_j (q^{n-j-k+1}; q)_j\, a^k \right) x^j. \tag{13}$$

Proof. The formula can be derived by equating the coefficients of equal powers of t in (11). To this end, one has to apply the q-binomial formula

$$\frac{1}{(z;q)_k} = {}_1\phi_0(q^k;\,;q;z) = \sum_{n=0}^{\infty} \frac{(q^k;q)_n}{(q;q)_n}\, z^n, \quad |z| < 1,$$

cf. [8, Eq. (II.3)]. □

2.3 The indeterminate case and the Nevanlinna parametrization

We are still assuming that a is positive. In [13, Lemma 3.1] it is proved that the Hamburger moment problem for the orthogonal polynomials $F_n(a,q;x)$ (or $P_n(a,q;x)$) is indeterminate if and only if $q < a < q^{-1}$. This is, however, clear from formulas (10) and from the well known criterion (cf. Addenda and Problems 10. to Chapter 2 in [1]) according to which the Hamburger moment problem is indeterminate iff

$$\sum_{n=0}^{\infty} \left(P_n(a,q;0)^2 + Q_n(a,q;0)^2\right) < \infty.$$

This also means that the Jacobi matrix operator J defined in (8), (7), with $\operatorname{Dom} J$ equal to the linear hull of the canonical basis in $\ell^2(\mathbb{Z}_+)$, is not essentially self-adjoint if and only if a belongs to the interval (q, q^{-1}), and if so then the deficiency indices are $(1,1)$ [1, Chapter 4].

Hence for $q < a < q^{-1}$ there exist infinitely many distinct measures of orthogonality parametrized with the aid of the Nevanlinna functions A, B, C and D,

$$A(z) = z\sum_{n=0}^{\infty} Q_n(0)Q_n(z), \qquad B(z) = -1 + z\sum_{n=0}^{\infty} Q_n(0)P_n(z),$$
$$C(z) = 1 + z\sum_{n=0}^{\infty} P_n(0)Q_n(z), \qquad D(z) = z\sum_{n=0}^{\infty} P_n(0)P_n(z),$$

where P_n and Q_n are the polynomials of the first and second kind, respectively [1, 16]. All these Nevanlinna functions are entire and

$$A(z)D(z) - B(z)C(z) = 1, \quad \forall z \in \mathbb{C}. \tag{14}$$

According to the Nevanlinna theorem, all measures of orthogonality μ_φ for which the set $\{P_n;\, n \in \mathbb{Z}_+\}$ is orthonormal in $L^2(\mathbb{R}, \mathrm{d}\mu_\varphi)$, are in one-to-one correspondence with functions φ belonging to the one-point compactification $\mathcal{P} \cup \{\infty\}$ of the space of Pick functions $\mathcal{P}$. Recall that Pick functions are defined and holomorphic on the open complex halfplane $\operatorname{Im} z > 0$, with values in the closed halfplane $\operatorname{Im} z \geq 0$. The correspondence is established by identifying the Stieltjes transform of the measure μ_φ,

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu_\varphi(x)}{z - x} = \frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}\,, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{15}$$

By a theorem due to M. Riesz, $\{P_n;\, n \in \mathbb{Z}_+\}$ is an orthonormal basis in $L^2(\mathbb{R}, \mathrm{d}\mu_\varphi)$ if and only if $\varphi = t$ is a constant function with $t \in \mathbb{R} \cup \{\infty\}$ [1, Theorem 2.3.3]. Then the measure μ_t is said to be N-extremal. Moreover, the N-extremal measures μ_t are in one-to-one correspondence with self-adjoint extensions T_t of the Jacobi operator J mentioned above. In more detail, if E_t is the spectral measure of T_t and e_0 is the first vector of the canonical basis in $\ell^2(\mathbb{Z}_+)$ then $\mu_t = \langle e_0, E_t(\cdot)e_0\rangle$ [1, Chapter 4]. The operators T_t in the indeterminate case are known to have a compact resolvent. Hence any N-extremal measure μ_t is purely discrete and supported on $\operatorname{spec} T_t$.

On the other hand, referring to (15), the support of μ_t is also known to be equal to the zero set

$$\mathfrak{Z}_t = \{x \in \mathbb{R};\, B(x)t - D(x) = 0\} \tag{16}$$

[1, Section 2.4]. Hence

$$\mu_t = \sum_{x \in \mathfrak{Z}_t} \rho(x)\delta_x \tag{17}$$

where $\rho(x) = \mu_t(\{x\})$ and δ_x is the Dirac measure supported on $\{x\}$. Equation (15), with $\varphi = t$, is nothing but the Mittag-Leffler expansion of the meromorphic function on the right-hand side,

$$\sum_{x \in \mathfrak{Z}_t} \frac{\rho(x)}{z - x} = \frac{A(z)t - C(z)}{B(z)t - D(z)}\,,$$

cf. [1, footnote on p. 55]. From here it can be deduced that

$$\rho(x) = \operatorname{Res}_{z=x} \frac{A(z)t - C(z)}{B(z)t - D(z)} = \frac{A(x)t - C(x)}{B'(x)t - D'(x)} = \frac{1}{B'(x)D(x) - B(x)D'(x)} \tag{18}$$

since, for $x \in \mathfrak{Z}_t$, $t = D(x)/B(x)$.

It should be noted that we are dealing with the Stieltjes case for the matrix operator J is positive on its domain of definition, see (9). This means that, for any choice of parameters from the specified range, there always exists a measure of orthogonality with its support contained in $[0, +\infty)$. In particular, if $a \in (q, q^{-1})$ then at least one of the measures of orthogonality is supported by $[0, +\infty)$. From [7, Lemma 1] it is seen that there exists the limit

$$\lim_{n\to\infty} \frac{P_n(0)}{Q_n(0)} = \alpha \in (-\infty, 0\,].$$

And, as explained in [3, Remark 2.2.2], an N-extremal measure of orthogonality μ_t is supported by $[\,0, \infty)$ iff $t \in [\,\alpha, 0\,]$, the Stieltjes moment is determinate for $\alpha = 0$ and indeterminate for $\alpha < 0$. Let us note that μ_0 is the unique N-extremal measure for which 0 is a mass point.

In our case, making once more use of the explicit form (10), we have

$$\alpha = \lim_{n\to\infty} \frac{P_n(a,q;0)}{Q_n(a,q;0)} = \begin{cases} -1, & \text{if } a \in (0,1], \\ -a, & \text{if } a > 1. \end{cases} \tag{19}$$

Hence the Stieltjes problem is indeterminate for any value $a \in (q, q^{-1})$.

The self-adjoint operator T_α corresponding to the N-extremal measure μ_α is nothing but the Friedrichs extension of J [15, Proposition 3.2]. The parameter α can also be computed in the limit

$$\alpha = \lim_{x\to-\infty} \frac{D(x)}{B(x)}\,,$$

and by inspection of the function $D(x)/B(x)$ one finds that μ_t has exactly one negative mass point if $t \notin [\,\alpha, 0\,]$ including $t=\infty$ [3, Lemma 2.2.1]. It is known, too, that Markov's theorem applies in the indeterminate Stieltjes case meaning that

$$\lim_{n\to\infty} \frac{Q_n(z)}{P_n(z)} = \frac{A(z)\alpha - C(z)}{B(z)\alpha - D(z)}\,,\ z \in \mathbb{C} \setminus \operatorname{supp}(\mu_\alpha) \tag{20}$$

[4, Theorem 2.1]. In addition, in the same case, one has the limit

$$\lim_{n\to\infty} \frac{P_n(z)}{Q_n(0)} = D(z) - B(z)\alpha,\ z \in \mathbb{C}, \tag{21}$$

as derived in [7] and also in [15].

Finally we wish to recall yet another interesting application of the Nevanlinna functions. It is shown in [6] that the reproducing kernel can be expressed in terms of functions $B(z)$ and $D(z)$,

$$K(u,v) := \sum_{n=0}^{\infty} P_n(u)P_n(v) = \frac{B(u)D(v) - D(u)B(v)}{u-v}\,, \tag{22}$$

see also [5, Section 1].

2.4 An explicit form of the Nevanlinna functions

In order to describe conveniently the Nevanlinna parametrization in the studied case we introduce a shorthand notation for particular basic hypergeometric series while not indicating the dependance on q explicitly. We put

$$\varphi_a(z) = {}_1\phi_1(0; qa; q, z),\ \psi_a(z) = {}_1\phi_1(0; qa^{-1}; q, a^{-1}z), \tag{23}$$

and

$$\chi_1(z) = \frac{\partial}{\partial p}\, {}_1\phi_1(0; p; q, z)\Big|_{p=q}.$$

Theorem 3. *Let $1 \neq a \in (q, q^{-1})$. Then the entire functions A, B, C and D from the Nevanlinna parametrization are as follows:*

$$\begin{aligned} A(a,q;z) &= \frac{\varphi_a(qz) - \psi_a(qz)}{1-a}\,,\ B(a,q;z) = \frac{a\psi_a(z) - \varphi_a(z)}{1-a}\,,\\ C(a,q;z) &= \frac{\psi_a(qz) - a\varphi_a(qz)}{1-a}\,,\ D(a,q;z) = \frac{a\big(\varphi_a(z) - \psi_a(z)\big)}{1-a}\,. \end{aligned} \tag{24}$$

For $a = 1$ these functions take the form

$$A(1,q;z) = -2q\,\chi_1(qz) - z\,\frac{\partial}{\partial z}\,\varphi_1(qz),\ \ B(1,q;z) = 2q\,\chi_1(z) + z^2\frac{\partial}{\partial z}\big(z^{-1}\varphi_1(z)\big),$$
$$C(1,q;z) = 2q\,\chi_1(qz) + \frac{\partial}{\partial z}\big(z\varphi_1(qz)\big),\ \ D(1;q,z) = -2q\,\chi_1(z) - z\frac{\partial}{\partial z}\,\varphi_1(z). \qquad (25)$$

Proof. We shall confine ourselves to computing the function A only. The formulas for B, C and D can be derived in a fully analogous manner. Starting from the definition of A and recalling formulas (10) and (6), (4) for $Q_n(a,q;0)$ and $Q_n(a,q;x)$, respectively, one has

$$\begin{aligned} A(a,q;z) &= \frac{qz}{1-a}\sum_{n=1}^{\infty}(-1)^{n+1}(a^{-n}-1)q^{n(n-1)/2}F_{n-1}(a,q;qz) \\ &= \frac{zq}{1-a}\left(a^{-1}\sum_{n=0}^{\infty}q^{n(n-1)/2}F_n(a,q;qz)(-qa^{-1})^n - \sum_{n=0}^{\infty}q^{n(n-1)/2}F_n(a,q;qz)(-q)^n\right). \end{aligned}$$

From comparison of both sums in the last expression with formula (11) for the generating function it becomes clear that the sums can be expressed in terms of basic hypergeometric functions, namely

$$\begin{aligned} A(a,q;z) &= \frac{qz}{1-a}\left(\frac{a^{-1}\,{}_2\phi_2(0,q;q^2a^{-1},q^2;q,q^2a^{-1}z)}{(1-qa^{-1})(1-q)} - \frac{{}_2\phi_2(0,q;q^2,aq^2;q,zq^2)}{(1-q)(1-qa)}\right) \\ &= \frac{1}{1-a}\Big(\big(1-{}_1\phi_1(0;qa^{-1};q,qa^{-1}z)\big) - \big(1-{}_1\phi_1(0;qa;q,qz)\big)\Big). \end{aligned}$$

Thus one arrives at the first equation in (24).

Concerning the particular case $a = 1$, formulas (25) can be derived by applying the limit $a \to 1$ to formulas (24). This is actually possible since Proposition 2.4.1 and Remark 2.4.2 from [3] guarantee that the functions $A(a,q;z)$, $B(a,q;z)$, $C(a,q;z)$, $D(a,q;z)$ depend continuously on $a \in (q,q^{-1})$. In order to be able to apply this theoretical result one has to note that the coefficients in the recurrence (3) depend continuously on a, and to verify that the series $\sum_{n=0}^{\infty}P_n(a,q;0)^2$ and $\sum_{n=0}^{\infty}Q_n(a,q;0)^2$ converge uniformly for a in compact subsets of (q,q^{-1}). But the latter fact is obvious from (10).

For instance, in case of function A one finds that

$$A(1,q;z) = \lim_{a\to 1}\frac{\varphi_a(qz)-\psi_a(qz)}{1-a} = -\frac{\partial}{\partial a}\big(\varphi_a(qz)-\psi_a(qz)\big)\Big|_{a=1}.$$

A straightforward computation yields the first equation in (25), and similarly for the remaining three equations. □

Corollary 4. *The following limits are true:*

$$\lim_{n\to\infty}(-1)^n\left(\frac{a}{q}\right)^{n/2}P_n(a,q;x)=\frac{{}_1\phi_1(0;qa;q,x)}{1-a}\,,\qquad \text{if } q<a<1,$$
$$\lim_{n\to\infty}(-1)^n(qa)^{-n/2}P_n(a,q;x)=\frac{a\;{}_1\phi_1(0;qa^{-1};q,a^{-1}x)}{a-1}\,,\quad \text{if } 1<a<q^{-1},\qquad (26)$$
$$\lim_{n\to\infty}\frac{(-1)^n}{n}\,q^{-n/2}P_n(1,q;x)=\,{}_1\phi_1(0;q;q,x),$$

and

$$\lim_{n\to\infty}\frac{Q_n(z)}{P_n(z)}=\begin{cases}-\dfrac{{}_1\phi_1(0;qa;q,qz)}{{}_1\phi_1(0;qa;q,z)} & \text{for } q<a<1,\\[2ex] -\dfrac{{}_1\phi_1(0;qa^{-1};q,qa^{-1}z)}{a\,{}_1\phi_1(0;qa^{-1};q,a^{-1}z)} & \text{for } 1<a<q^{-1},\\[2ex] -\dfrac{{}_1\phi_1(0;q;q,qz)}{{}_1\phi_1(0;q;q,z)} & \text{for } a=1.\end{cases}\qquad (27)$$

Proof. From (24), (25) and (19) one immediately infers that

$$\begin{aligned}C(a,q;z)-A(a,q;z)\alpha &= \varphi_a(qz)\ \text{ or }\ \psi_a(qz)\ \text{ or }\ \varphi_1(qz),\\ D(a,q;z)-B(a,q;z)\alpha &= -\varphi_a(z)\ \text{ or }\ -a\psi_a(z)\ \text{ or }\ -\varphi_1(z),\end{aligned}$$

depending on whether $q<a<1$ or $1<a<q^{-1}$ or $a=1$. Equations (26) follow from (21) and (10) while equations (27) are a direct consequence of (20). □

Remark 5. The limits (26) can be proved, in an alternative way, by applying Darboux's method to the generating function whose explicit form is given in (11). According to this method, the leading asymptotic term of $q^{n(n-1)/2}F_n(a,q;x)$ is determined by the singularity of the function on the left-hand side in (11) which is located most closely to the origin, cf. [14, Section 8.9]. Proceeding this way one can show that the first limit in (26) is valid even for all $0<a<1$ while the second one is valid for all $a>1$. Let us also note that the limits established in (26) can be interpreted as a q-analogue to Hurwitz's limit formula for the Lommel polynomials. The case $a<1$ has been derived, probably for the first time, in [12, Eq. (4.24)], see also [11, Eq. (3.4)] and [13, Eq. (2.7)], while the case $a>1$ has been treated in [11, Eq. (3.6)].

The following formula for the reproducing kernel can be established.

Corollary 6. *Suppose $q<a<q^{-1}$. Then*

$$K(u,v)=\frac{a\big(\varphi_a(u)\psi_a(v)-\psi_a(u)\varphi_a(v)\big)}{(1-a)(u-v)}$$

if $a\neq 1$, and

$$K(u,v)=\frac{\varphi_1(u)\big(2q\chi_1(v)+v\varphi_1'(v)\big)-\big(2q\chi_1(u)+u\varphi_1'(u)\big)\varphi_1(v)}{u-v}$$

if $a=1$.

Proof. This is a direct consequence of (22) and (24), (25). □

Remark 7. In [18], self-adjoint extensions of the Jacobi matrix J, defined in (7), (8), are described in detail while addressing only the case $q < a < 1$. The self-adjoint extensions, called $T(\kappa)$, are parametrized by $\kappa \in \mathbb{R} \cup \{\infty\}$, with $\kappa = \infty$ corresponding to the Friedrichs extension. The domain $\operatorname{Dom} T(\kappa) \subset \operatorname{Dom} J^*$ is specified by the asymptotic boundary condition: a sequence f from $\operatorname{Dom} J^*$ belongs to $\operatorname{Dom} T(\kappa)$ iff $C_2(f) = \kappa C_1(f)$ where

$$C_1(f) = \lim_{n\to\infty} (-1)^n \left(\frac{a}{q}\right)^{n/2} f_n, \; C_2(f) = \lim_{n\to\infty} \left((-1)^n f_n - C_1(f)\left(\frac{q}{a}\right)^{n/2}\right)(qa)^{-n/2}$$

(the limits can be shown to exist). The eigenvalues of $T(\kappa)$ are exactly the roots of the equation

$$\kappa \;{}_1\phi_1(0; qa; q, x) + a \;{}_1\phi_1(0; qa^{-1}; q, a^{-1}x) = 0.$$

On the other hand, consider the self-adjoint extension T_t corresponding the measure of orthogonality μ_t, with $t \in \mathbb{R} \cup \{\infty\}$ being a Nevanlinna parameter. The eigenvalues of T_t are the mass points from the support of μ_t, i.e. the zeros of the function

$$(1-a)\big(B(a,q;x)t - D(a,q;x)\big) = (t+1)a\,{}_1\phi_1(0; qa^{-1}; q, a^{-1}x) - (t+a)\,{}_1\phi_1(0; qa; q, x),$$

as one infers from (23) and (24). Since a self-adjoint extension is unambiguously determined by its spectrum (see, for instance, proof of Theorem 4.2.4 in [1]) one gets the correspondence $\kappa = -(t+a)/(t+1)$.

2.5 Measures of orthogonality

With the explicit knowledge of the Nevanlinna parametrization established in Theorem 3 it is straightforward to describe all N-extremal solutions.

Proposition 8. *Let $1 \neq a \in (q, q^{-1})$. Then all N-extremal measures $\mu_t = \mu_t(a,q)$ are of the form*

$$\mu_t = \sum_{x\in\mathfrak{Z}_t} \rho(x)\,\delta_x \quad \textit{where} \quad \frac{1}{\rho(x)} = \frac{a}{1-a}\big(\psi_a(x)\varphi_a'(x) - \varphi_a(x)\psi_a'(x)\big), \tag{28}$$

$$\mathfrak{Z}_t = \mathfrak{Z}_t(a,q) = \{x \in \mathbb{R};\; a(t+1)\psi_a(x) - (t+a)\varphi_a(x) = 0\},$$

and δ_x stands for the Dirac measure supported on $\{x\}$.

For $a = 1$, all N-extremal measures $\mu_t = \mu_t(1,q)$ are of the form $\mu_t = \sum_{x\in\mathfrak{Y}_t} \rho(x)\,\delta_x$ where

$$\frac{1}{\rho(x)} = 2q\big(\varphi_1'(x)\chi_1(x) - \varphi_1(x)\chi_1'(x)\big) + x\big(\varphi_1'(x)\big)^2 - \varphi_1(x)\varphi_1'(x) - x\varphi_1(x)\varphi_1''(x).$$

and

$$\mathfrak{Y}_t = \mathfrak{Y}_t(q) = \{x \in \mathbb{R};\; 2q(t+1)\chi_1(x) + (t+1)x\varphi_1'(x) - t\varphi_1(x) = 0\}.$$

Proof. Referring to general formulas (17) and (16), (18), it suffices to apply Theorem 3. □

Lemma 9. *With the notation introduced in (23) it holds true that*

$$\varphi_a(z)\psi_a(qz) - a\psi_a(z)\varphi_a(qz) = 1 - a \quad \text{if} \;\; a \neq 1, \tag{29}$$

and

$$2q\big(\varphi_1(z)\chi_1(qz) - \chi_1(z)\varphi_1(qz)\big) + z\big(q\varphi_1(z)\varphi_1'(qz) - \varphi_1'(z)\varphi_1(qz)\big) + \varphi_1(z)\varphi_1(qz) = 1,$$

for all $z \in \mathbb{C}$.

Proof. These identities follow from (14) and, again, from Theorem 3. □

Let us examine a bit more closely two particular N-extremal measures μ_t described in Proposition 8, with $t = -1$ and $t = -a$. They correspond to the distinguished case $t = \alpha$ if $a \in (q, 1)$ or $a \in (1, q^{-1})$, respectively (cf. (19)). As already mentioned, if $t = \alpha$ then the corresponding self-adjoint extension of the underlying Jacobi matrix is the Friedrichs extension, and the measure μ_t is necessarily a Stieltjes measure. In the case $t = -1$ the orthogonality relation for the orthonormal polynomials $P_n(a, q; x)$ reads

$$-\sum_{k=1}^{\infty} \frac{\varphi_a(q\xi_k)}{\varphi_a'(\xi_k)}\, P_n(a, q; \xi_k) P_m(a, q; \xi_k) = \delta_{mn} \tag{30}$$

where $\{\xi_k;\, k \in \mathbb{N}\}$ are the zeros of the function φ_a. Actually, from (28) and (29) one infers that $\rho(x) = -\varphi_a(q\xi_k)/\varphi_a'(\xi_k)$ if $\varphi_a(\xi_k) = 0$. Similarly, the same orthogonality relation for $t = -a$ reads

$$-\frac{1}{a}\sum_{k=1}^{\infty} \frac{\psi_a(q\eta_k)}{\psi_a'(\eta_k)}\, P_n(a, q; \eta_k) P_m(a, q; \eta_k) = \delta_{mn} \tag{31}$$

where $\{\eta_k;\, k \in \mathbb{N}\}$ are the zeros of the function ψ_a.

Remark 10. The orthogonality relation (30) has been derived already in [13, Theorem 3.6.]. This is the unique orthogonality relation for the polynomials $P_n(a, q; x)$ if $a \in (0, q]$ (the determinate case), and an example of an N-extremal orthogonality relation if $a \in (q, 1)$. Similarly, (31) is the unique orthogonality relation if $a \geq q^{-1}$. Of course, (30) and (31) coincide for $a = 1$.

Remark 11. In [2, Section 1], an explicit expression has been found for the measures of orthogonality μ_φ corresponding to constant Pick functions $\varphi(z) = \beta + i\gamma$, with $\beta \in \mathbb{R}$ and $\gamma > 0$. Let us call these measures $\mu_{\beta,\gamma} = \mu_{\beta,\gamma}(a, q)$. It turns out that $\mu_{\beta,\gamma}$ is an absolutely continuous measure supported on $\mathbb{R}$ with the density

$$\frac{\mathrm{d}\mu_{\beta,\gamma}}{\mathrm{d}x} = \frac{\gamma}{\pi}\Big(\big(\beta B(a, q; x) - D(a, q; x)\big)^2 + \gamma^2 B(a, q; x)^2\Big)^{-1}.$$

In our case, referring to (24), (25), we get the probability density

$$\frac{\mathrm{d}\mu_{\beta,\gamma}}{\mathrm{d}x} = \frac{\gamma\,(1-a)^2}{\pi\Big(\big((\beta+1)a\psi_a(x)-(\beta+a)\varphi_a(x)\big)^2+\gamma^2\big(a\psi_a(x)-\varphi_a(x)\big)^2\Big)}\,,$$

provided $1 \neq a \in (q, q^{-1})$, and

$$\frac{\mathrm{d}\mu_{\beta,\gamma}}{\mathrm{d}x} = \frac{\gamma}{\pi}$$
$$\times\Big(\big(2q(\beta+1)\chi_1(x)-\beta\varphi_1(x)+(\beta+1)x\varphi_1'(x)\big)^2+\gamma^2\big(2q\chi_1(x)-\varphi_1(x)+x\varphi_1'(x)\big)^2\Big)^{-1},$$

provided $a = 1$. Letting $\beta = -1$ or $\beta = -a$ and $\gamma > 0$ arbitrary, one obtains comparatively simple and nice orthogonality relations for the polynomials $P_n(a,q;x)$, namely

$$\int_{\mathbb{R}} \frac{P_m(a,q;x)P_n(a,q;x)}{\gamma\big(a\psi_a(x)-\varphi_a(x)\big)^2+\gamma^{-1}(a-1)^2\varphi_a(x)^2}\,\mathrm{d}x = \frac{\pi}{(a-1)^2}\,\delta_{mn}$$

and

$$\int_{\mathbb{R}} \frac{P_m(a,q;x)P_n(a,q;x)}{\gamma\big(a\psi_a(x)-\varphi_a(x)\big)^2+\gamma^{-1}(a-1)^2a^2\psi_a(x)^2}\,\mathrm{d}x = \frac{\pi}{(a-1)^2}\,\delta_{mn},$$

valid for all $m, n \in \mathbb{Z}_+$ and $a \in (q, q^{-1})$, $a \neq 1$. If $a = 1$, a similar orthogonality relation takes the form

$$\int_{\mathbb{R}} \frac{P_m(1,q;x)P_n(1,q;x)}{\gamma\big(2q\chi_1(x)+x\varphi_1'(x)-\varphi_1(x)\big)^2+\gamma^{-1}\varphi_1(x)^2}\,\mathrm{d}x = \pi\delta_{mn}.$$

3 The moment sequence

3.1 Passing to the determinate case

Let μ be any measure of orthogonality for the orthonormal polynomials $P_n(a,q;x)$ introduced in (5). Denote by

$$m_n(a,q) = \int_{\mathbb{R}} x^n\,\mathrm{d}\mu(x), \quad n \in \mathbb{Z}_+,$$

the corresponding moment sequence. It is clear from Favard's theorem, however, that the moments do not depend on the particular choice of the measure of orthogonality. It is even known that

$$m_n(a,q) = \langle e_0, J(a,q)^n e_0\rangle, \quad n \in \mathbb{Z}_+, \tag{32}$$

where $J(a,q)$ is the Jacobi matrix defined in (7), (8), and e_0 is the first vector of the canonical basis in $\ell^2(\mathbb{Z}_+)$. Whence $m_n(a,q)$ is a polynomial in a and q^{-1}. Consequently, in order to compute the moments one can admit a wider range of parameters than that we were using up to now, namely $0 < q < 1$ and $q < a < q^{-1}$. This observation can be of particular importance for the parameter q since the properties of the matrix operator $J(a, q)$ would change dramatically if q was allowed to take values $q > 1$. We wish to stick, however, to the widely used convention according to which the modulus of q is smaller than 1. This is why we replace the symbol q by p in this section whenever this restriction is relaxed. Concerning the parameter a, it is always supposed to be positive.

Put, for $p > 0$ and $a > 0$,

$$\omega_n(a, p) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_p p^{-k(n-k)} a^k, \quad n \in \mathbb{Z}_+. \tag{33}$$

The meaning of the q-binomial ceofficient in (33) is the standard one, cf. [8, Eq. (I.39)]. Let us remark that $\omega_n(a, p)$ can be expressed in terms of the continuous q-Hermite polynomials $H_n(x; q)$, namely

$$\omega_n(a, p) = {}_2\phi_0(p^n, 0; ; p^{-1}, p^{-n}a) = a^{n/2}\, H_n\Big(\frac{1}{2}\left(a^{1/2} + a^{-1/2}\right); p^{-1}\Big), \tag{34}$$

see [10].

As before, the monic polynomials $F_n(a, p; x)$ are generated by the recurrence (3), with $F_{-1}(a, p; x) = 0$ and $F_0(a, p; x) = 1$ (writing p instead of q). The following proposition is due to Van Assche and is contained in [19, Theorem 2].

Proposition 12. *For $p > 1$ and $x \neq 0$ one has*

$$\lim_{n\to\infty} x^{-n} F_n(a, p; x) = \sum_{k=0}^{\infty} \frac{\omega_k(a, p)}{(p; p)_k} \left(\frac{p}{x}\right)^k.$$

Note that if $p > 1$ then the Jacobi matrix $J(a, p)$ represents a compact (even trace class) operator on $\ell^2(\mathbb{Z}_+)$. In particular, this implies that the Hamburger moment problem is determinate. Several additional useful facts are known in this case which we summarize in the following remark.

Remark 13. In [17, Section 3] it is noted that if $\{\beta_n\}_{n=0}^{\infty}$ is a real sequence belonging to $\ell^1(\mathbb{Z}_+)$, $\{\alpha_n\}_{n=0}^{\infty}$ is a positive sequence belonging to $\ell^2(\mathbb{Z}_+)$ and $\{F_n(x)\}_{n=0}^{\infty}$ is a sequence of monic polynomials defined by the recurrence

$$F_{n+1}(x) = (x - \beta_n)F_n(x) - \alpha_{n-1}^2 F_{n-1}(x), \quad n \geq 0,$$

with $F_0(x) = 1$ and (conventionally) $F_{-1}(x) = 0$, then

$$\lim_{n\to\infty} x^{-n} F_n(x) = \mathcal{G}(x^{-1}) \ \text{ for } \ x \neq 0 \tag{35}$$

where $\mathcal{G}(z)$ is an entire function. Moreover, let μ be the (necessarily unique) measure of orthogonality for the sequence of polynomials $\{F_n(x)\}$. Then the Stieltjes transform of μ reads

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(x)}{1 - zx} = \frac{\tilde{\mathcal{G}}(z)}{\mathcal{G}(z)} \tag{36}$$

where $\tilde{\mathcal{G}}(z)$ is an entire function associated in an analogous manner with the shifted sequences $\{\tilde{\alpha}_n = \alpha_{n+1}\}_{n=0}^{\infty}$, $\{\tilde{\beta}_n = \beta_{n+1}\}_{n=0}^{\infty}$.

Theorem 14. *Let $p > 1$ and $x \neq 0$. Then*

$$\lim_{n\to\infty} x^{-n} F_n(a,p;x) = \mathcal{G}(x^{-1})$$

where

$$\mathcal{G}(z) = (z;p^{-1})_\infty \,{}_1\phi_1(0;z;p^{-1},az) = \sum_{k=0}^{\infty} \frac{\omega_k(a,p)}{(p;p)_k}\,(pz)^k \tag{37}$$

is an entire function obeying the second-order q-difference equation

$$\mathcal{G}(z) - \big(1-(a+1)z\big)\mathcal{G}(p^{-1}z) + ap^{-1}z^2\mathcal{G}(p^{-2}z) = 0. \tag{38}$$

The Stieltjes transform of the (unique) measure of orthogonality μ for the sequence of orthogonal polynomials $\{F_n(a,p;x)\}$ is given by the formula

$$\int_{\mathbb{R}} \frac{d\mu(x)}{1-zx} = \frac{\mathcal{G}(p^{-1}z)}{\mathcal{G}(z)}\,. \tag{39}$$

Proof. In view of Proposition 12, in order to show (37) it suffices to verify only the second equality. But this equality follows from the definition of the basic hypergeometric series and from the well known identity [8, Eq. (II.2)]

$$(z;p^{-1})_\infty = \sum_{n=0}^{\infty} \frac{(pz)^n}{(p;p)_n}\,.$$

Using the power series expansion of $\mathcal{G}(z)$ established in (37) one finds that (38) is equivalent to

$$\omega_k - (a+1)\omega_{k-1} + a\,(1-p^{-k+1})\omega_{k-2} = 0 \quad \text{for} \quad k \geq 2$$

and $\omega_1 - (a+1)\omega_0 = 0$. This is true, indeed, if we take into account (34) and the recurrence relation for the continuous q-Hermite polynomials [10, Eq. (14.26.3)]

$$2xH_k(x;q) = H_{k+1}(x;q) + (1-q^k)H_{k-1}(x;q).$$

Recalling once more (3), the polynomials $F_n(a,p;x)$ solve the recurrence relation

$$u_{n+1} = \big(x-(a+1)p^{-n}\big)u_n - ap^{-2n+1}u_{n-1} \tag{40}$$

while the polynomials $\tilde{F}_n(a,p;x) := p^{-n}F_n(a,q;px)$ obviously obey the recurrence

$$\tilde{u}_{n+1} = \big(x-(a+1)p^{-n-1}\big)\tilde{u}_n - ap^{-2n-1}\tilde{u}_{n-1}. \tag{41}$$

Comparing these two equations one observes that (41) is obtained from (40) just by shifting the index. In other words, the sequences of monic polynomials $\{\tilde{F}_n(a,p;x)\}$ and $\{F_n(a,p;x)\}$ are generated by the same recurrence relation, but the index has to be shifted in the latter case. Hence, referring to Remark 13 and equation (35), one can compute

$$\tilde{\mathcal{G}}(x^{-1}) = \lim_{n\to\infty} x^{-n}\tilde{F}_n(a,p;x) = \lim_{n\to\infty}(px)^{-n}F_n(a,p;px) = \mathcal{G}(p^{-1}x^{-1}).$$

Thus $\tilde{\mathcal{G}}(z) = \mathcal{G}(p^{-1}z)$ and (39) is a particular case of (36). □

3.2 Recurrence relations and asymptotic behavior

From (32) it is seen that $m_n(a,p) \leq \|J(a,p)\|^n$. Moreover, from (39) one deduces that

$$\sum_{n=0}^{\infty} m_n(a,p) z^n = \frac{\mathcal{G}(p^{-1}z)}{\mathcal{G}(z)}\,, \tag{42}$$

and the series is clearly convergent if $p > 1$ and $|z| < \|J(a,p)\|^{-1}$.

Remark 15. Any explicit formula for monic polynomials $F_n(x)$, $n \in \mathbb{Z}_+$, which are members of a sequence of orthogonal polynomials with a measure of orthogonality μ, automatically implies a linear recursion for the corresponding moments. In fact, $F_0(x) = 1$ and so, by orthogonality, $\int_{\mathbb{R}} F_n(x)\mathrm{d}\mu(x) = 0$ for $n \geq 1$. Particularly, in our case, formula (13) implies the relation

$$\sum_{j=0}^{n} \frac{(-1)^j q^{(j-1)j/2}}{(q;q)_j{}^2} \left(\sum_{k=0}^{n-j} (q^{k+1};q)_j (q^{n-j-k+1};q)_j\, a^k \right) m_j(a,q) = 0 \quad \text{for} \quad n \geq 1.$$

Further we derive two more recursions for the moments, a linear and a quadratic one.

Proposition 16. *The moment sequence* $\{m_n(a,q)\}$ *solves the equations* $m_0(a,q) = 1$ *and*

$$m_n(a,q) = \frac{\omega_n(a,q)}{(q;q)_{n-1}} - \sum_{k=1}^{n-1} \frac{q^k \omega_k(a,q)}{(q;q)_k}\, m_{n-k}(a,q), \quad n \in \mathbb{N}. \tag{43}$$

Proof. Equations (42) and (37) imply that

$$\sum_{m=0}^{\infty} \frac{p^m \omega_m(a,p)}{(p;p)_m}\, z^m \sum_{n=0}^{\infty} m_n(a,p) z^n = \sum_{m=0}^{\infty} \frac{\omega_m(a,p)}{(p;p)_m}\, z^m$$

holds for $p > 1$ and z from a neighborhood of 0. Equating the coefficients of equal powers of z one finds that (43) holds true for $q = p > 1$. But for the both sides are rational functions in q the equation remains valid also for $0 < q < 1$. □

Proposition 17. *The moment sequence* $\{m_n(a,q)\}$ *solves the equations* $m_0(a;q) = 1$ *and*

$$m_{n+1}(a,q) = (a+1)\, m_n(a,q) + a \sum_{k=0}^{n-1} q^{-k-1} m_k(a,q)\, m_{n-k-1}(a,q), \quad n \in \mathbb{Z}_+. \tag{44}$$

Proof. Equation (38) can be rewritten as

$$\frac{\mathcal{G}(p^{-1}z)}{\mathcal{G}(z)} \left(1 - (a+1)z - ap^{-1}z^2\, \frac{\mathcal{G}(p^{-2}z)}{\mathcal{G}(p^{-1}z)} \right) = 1$$

and holds true for $p > 1$ and z from a neighborhood of the origin. Substituting the power series expansion (42) one has

$$\left(1 - (a+1)z - ap^{-1}z^2 \sum_{n=0}^{\infty} m_n(a,p)\, p^{-n} z^n\right) \sum_{n=0}^{\infty} m_n(a,p) z^n = 1.$$

Equating the coefficients of equal powers of z one concludes that (44) holds for $q = p > 1$. For the both sides are polynomials in q^{-1} the equation is valid for $0 < q < 1$ as well. □

Our final task is to provide estimates bringing some insight into the asymptotic behavior of the moments for large powers. We still assume that $0 < q < 1$ and $a > 0$. On the other hand, a is not required to be restricted to the interval $q < a < q^{-1}$. Let us note that it has been shown in [3, Lemma 4.9.1] that

$$a^{n/2} q^{-n(n-1)/4} \le \omega_n(a,q) \le (1+a)^n q^{-n^2/4}, \quad n \in \mathbb{Z}_+. \tag{45}$$

Proposition 18. *Let $a > 0$. The moments $m_n(a,q)$ obey the inequalities*

$$m_n(a,q) \le \frac{(1+a)^n}{(q;q)_{n-1}}\, q^{-n^2/4}, \;\; n \in \mathbb{Z}_+, \tag{46}$$

and

$$m_{2n}(a,q) \ge a^n q^{-n^2}, \quad m_{2n+1}(a,q) \ge (a+1) a^n q^{-n(n+1)}, \;\; n \in \mathbb{Z}_+. \tag{47}$$

Proof. It is clear, for instance from (44), that each moment $m_n(a,q)$ is a polynomial in a and q^{-1} with nonnegative integer coefficients. Furthermore, by the very definition (34), $\omega_n(a,q)$ is a polynomial in a of degree n with positive coefficients. From (43) it is seen that

$$m_n(a;q) \le \frac{\omega_n(a,q)}{(q;q)_{n-1}}\,,$$

and then (45) implies (46).

From (44) one infers that

$$m_{2n+1}(a,q) \ge a q^{-2n} m_{2n-1}(a,q), \;\; m_{2n}(a,q) \ge a q^{-2n+1} m_{2n-2}(a,q), \;\; \text{for } n \ge 1.$$

Using these inequalities and proceeding by mathematical induction one can verify (47). □

Acknowledgments

The authors wish to acknowledge gratefully partial support from grant No. GA13-11058S of the Czech Science Foundation.

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